A local ring has only finitely many semidualizing modules up to isomorphism

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Semidualizing Modules

Assumption

\((R, m, k)\) is a local ring

Definition (Foxby ’72, Vasconcelos ’74)

A finitely generated \(R\)-module is **semidualizing** if \(R \cong \text{Hom}_R(C, C)\) and \(\text{Ext}^i_R(C, C) = 0\) for all \(i \geq 1\).

Example

1. \(R\) is a semidualizing \(R\)-module.
2. \(D\) is dualizing for \(R\) if and only if it is semidualizing for \(R\) and \(\text{id}_R(D) < \infty\).

Notation

\(\mathcal{S}(R) = \{\text{isomorphism classes of semidualizing } R\text{-modules}\}\).
A Conjecture and Partial Solution

**Fact (Base-change)**

If $R \to S$ is a local homomorphism of finite flat dimension, then $\mathcal{G}(R) \hookrightarrow \mathcal{G}(S)$ by $C \mapsto S \otimes_R C$.

**Conjecture (Vasconcelos ’74)**

If $R$ is Cohen-Macaulay, then $\mathcal{G}(R)$ is finite.

**Theorem (Christensen and Sather-Wagstaff ’08)**

If $R$ is Cohen-Macaulay and contains a field, then $\mathcal{G}(R)$ is finite.

**Outline of Proof.**

There is a flat local ring homomorphism $R \to (R', \mathfrak{m}R', \bar{k})$. Let $x \in \mathfrak{m}R'$ be a maximal $R'$-sequence. Then $R'/\langle x \rangle$ is artinian and $\mathcal{G}(R) \hookrightarrow \mathcal{G}(R') \hookrightarrow \mathcal{G}(R'/\langle x \rangle)$. A result of Happel essentially shows that $\mathcal{G}(R'/\langle x \rangle)$ is finite.
**Definition**

A commutative differential graded (DG) $R$-algebra is

1. a graded commutative $R$-algebra $A = \oplus_{i=0}^{\infty} A_i$ with
2. a differential, i.e., a sequence of $R$-linear maps $\partial^A_i : A_i \to A_{i-1}$ such that $\partial^A_i \partial^A_{i+1} = 0$ for all $i$, such that
3. $\partial^A$ satisfies the Leibniz Rule: for all $a_i \in A_i$ and $a_j \in A_j$

$$\partial^A_{i+j}(a_ia_j) = \partial^A_i(a_i)a_j + (-1)^i a_i \partial^A_j(a_j).$$

**Example (The ground ring)**

$R$ is a DG $R$-algebra

**Example (The Koszul complex)**

$K = K^R(x)$ is a DG $R$-algebra for each sequence $x \in R$. 

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Definition

A DG $A$-module is a graded $A$-module $M = \bigoplus_{i=-\infty}^\infty M_i$ with a differential $\partial_i^M : M_i \to M_{i-1}$ that satisfies the Leibniz Rule.

Example (The ground ring)

A DG $R$-module is a bounded below $R$-complex, e.g., a projective resolution of an $R$-module.

Example (The Koszul complex)

$K \otimes_R M$ is a DG $K$-module for each DG $R$-module $M$. 
Semi-free DG Modules

Definition

Let $A$ be a DG $R$-algebra. A DG $A$-module $M$ is **semi-free** if the underlying $A^\natural$-module $M^\natural$ has a graded basis.

Note

The boundedness condition on $M$ is important here.

Example (The ground ring)

A semi-free DG $R$-module is a bounded below complex of free $R$-modules.

Example (The Koszul complex)

$K \otimes_R M$ is a semi-free DG $K$-module for each semi-free DG $R$-module $M$. 

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A semi-free DG $A$-module $C$ is **semidualizing** if it is homologically finite and the natural map $A \to \text{Hom}_A(C, C)$ is a quasi-isomorphism.

**Notation**

$\mathcal{S}_{\text{dg}}(A)$ is the set of shift-quasiisomorphism classes of semidualizing DG $A$-modules.

**Example (The ground ring)**

A projective resolution of a semidualizing $R$-module is a semidualizing DG $R$-module: $\mathcal{S}(R) \hookrightarrow \mathcal{S}_{\text{dg}}(R)$.

**Example (The Koszul complex)**

$K \otimes_R C$ is a semidualizing DG $K$-module for each semidualizing DG $R$-module $C$: $\mathcal{S}_{\text{dg}}(R) \hookrightarrow \mathcal{S}_{\text{dg}}(K)$.
Theorem (Nasseh and Sather-Wagstaff ’12)

The sets $\mathcal{S}(R)$ and $\mathcal{S}_{dg}(R)$ are finite.

Outline of Proof.

It suffices to prove that $\mathcal{S}_{dg}(R)$ is finite since $\mathcal{S}(R) \hookrightarrow \mathcal{S}_{dg}(R)$.

\[
R \to R' \to K \cong R' \otimes_Q \tilde{K} \leftarrow \tilde{A} \otimes_Q \tilde{K} \to \tilde{A} \otimes_Q k
\]

There is a flat local ring homomorphism $R \to (R', mR', \bar{k})$ such that $R'$ is complete.

Let $\mathfrak{x} \in mR'$ be minimal generating sequence and $K = K^{R'}(\mathfrak{x})$.

Let $Q$ be a regular local ring surjecting onto $R'$.

Let $\tilde{\mathfrak{x}} \in Q$ be a lift of $\mathfrak{x}$, and set $\tilde{K} = K^Q(\tilde{\mathfrak{x}})$.

Let $A$ be a DG algebra resolution of $R'$ over $Q$.

$\tilde{K}$ is a minimal $Q$-free resolution of $\bar{k}$.

$A \otimes_Q \bar{k}$ is a finite dimensional DG $\bar{k}$-algebra, and $\mathcal{S}_{dg}(A \otimes_Q \bar{k})$ is finite.