Reflexivity and connectedness

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**Assumption**

*R* is a commutative noetherian ring with identity.

**Definition (Foxby ’72, Vasconcelos ’74)**

A finitely generated *R*-module *C* is **semidualizing** if

\[ R \xrightarrow{\sim} \text{Hom}_R(C, C) \quad \text{and} \quad \text{Ext}^1_R(C, C) = 0. \]

**Fact**

1. *R* is semidualizing.
2. *D* is dualizing iff it is semidualizing and \( \text{id}_R(D) < \infty \).
3. **The semidualizing property is local:** TFAE:
   1. *U* is semidualizing over *R*.
   2. *U*_\(p\) is semidualizing over *R*_\(p\) for all prime *p*.
   3. *U*_\(m\) is semidualizing over *R*_\(m\) for all maximal *m*.
Definition (Foxby ’94, Christensen ’01)

Let $U$ be a finitely generated $R$-module. The **Bass class** with respect to $U$ is the class $\mathcal{B}_U(R)$ of all $R$-modules $M$ such that

$$U \otimes_R \text{Hom}_R(U, M) \xrightarrow{\sim} M$$

and

$$\text{Ext}^1_{R}(U, M) = 0 = \text{Tor}^1_{R}(U, \text{Hom}_R(U, M)).$$

Fact

Let $U$ be a finitely generated $R$-module. TFAE.

1. $U$ is semidualizing.
2. Every $R$-module $N$ with $\text{id}_R(N) < \infty$ is in $\mathcal{B}_U(R)$.
3. $\mathcal{B}_U(R)$ contains a faithfully injective $R$-module.
Fact

Let $U$ be a finitely generated $R$-module.

1. $0 \in \mathcal{B}_U(R)$.
2. If $C$ is semidualizing, then $C \in \mathcal{B}_C(R)$.
3. Membership in $\mathcal{B}_U(R)$ is a local property: TFAE:
   1. $N \in \mathcal{B}_U(R)$.
   2. $N_p \in \mathcal{B}_{U_p}(R_p)$ for all prime $p$.
   3. $N_m \in \mathcal{B}_{U_m}(R_m)$ for all maximal $m$.

Example

If $U$ is a finitely generated $R$-module such that $0 \neq U \in \mathcal{B}_U(R)$, then $U$ may not be semidualizing.
Let $R = k \times k$ and $U = k \times 0$. Then $0 \neq U \in \mathcal{B}_U(R)$, but $U$ is not semidualizing.
Proposition (SSW)

If $R$ is local and $U$ is a finitely generated $R$-module such that $0 \neq U \in \mathcal{B}_U(R)$, then $U$ must be semidualizing.

Sketch of proof

$U \in \mathcal{B}_U(R)$ implies that $\text{Ext}^1_R(U, U) = 0$.

It remains to show that $\chi: R \xrightarrow{\cong} \text{Hom}_R(U, U)$.

$U \otimes_R R$ implies that $\text{Tor}^R(U, \text{Ker}(\chi)) = 0$.

So $\chi$ is 1-1.
Theorem (SSW)

Let $U$ be a finitely generated $R$-module s.t. $0 \neq U \in \mathcal{B}_U(R)$, but $U$ is not semidualizing.

1. There are non-zero $R_1, R_2$ such that $R \cong R_1 \times R_2$.
2. There is a semidualizing $R_1$-module $C_1$ s.t. $U \cong C_1 \times 0$.

Sketch of proof

It suffices to show that $\text{Supp}_R(U)$ is Zariski open in $\text{Spec}(R)$. Since the Bass class condition is local, the Proposition implies

$$\text{Supp}_R(U) = \{ p \in \text{Spec}(R) \mid U_p \text{ is semidualizing for } R_p \}.$$

The exact sequence

$$0 \rightarrow \text{Ker}(\chi) \rightarrow R \rightarrow \text{Hom}_R(U, U) \rightarrow \text{Coker}(\chi) \rightarrow 0$$

implies that

$$\text{Supp}_R(U) = \text{Spec}(R) \setminus (\text{Supp}_R(\text{Ker}(\chi)) \cup \text{Supp}_R(\text{Coker}(\chi)))$$

which is open.
Total reflexivity

Definition
Let $G$ and $U$ be f.g. $R$-modules. Then $G$ is totally $U$-reflexive if

$$G \overset{\cong}{\longrightarrow} \text{Hom}_R(\text{Hom}_R(G, U), U)$$

and

$$\text{Ext}^{\geq 1}_R(G, U) = 0 = \text{Ext}^{\geq 1}_R(\text{Hom}_R(G, U), U).$$

Fact

Let $U$ be a finitely generated $R$-module.

1. $0$ is totally $U$-reflexive.
2. If $C$ is semidualizing, then $C$ is totally $C$-reflexive.
3. Being totally $U$-reflexive is a local property.

Example
Let $R = k \times k$ and $U = k \times 0$. Then $U \neq 0$ is totally $U$-reflexive, but $U$ is not semidualizing.
Proposition (SSW)
If \( R \) is local and \( U \neq 0 \) is a finitely generated \( R \)-module that is totally \( U \)-reflexive, then \( U \) must be semidualizing.

Theorem (SSW)
Let \( U \neq 0 \) be a finitely generated \( R \)-module that is totally \( U \)-reflexive but not semidualizing.

1. There are non-zero \( R_1, R_2 \) such that \( R \cong R_1 \times R_2 \).
2. There is a semidualizing \( R_1 \)-module \( C_1 \) s.t. \( U \cong C_1 \times 0 \).
Theorem (SSW)

Let $U$ be a homologically finite $R$-complex. TFAE:

1. $0 \not\cong U \in \mathcal{B}_U(R)$ and $U$ is not semidualizing.
2. $U$ is derived $U$-reflexive and not semidualizing and $U \not\cong 0$.
3. There are non-zero $R_1, R_2$ such that $R \cong R_1 \times R_2$, and there is a semidualizing $R_1$-complex $C_1$ s.t. $U \cong C_1 \times 0$. 