Local rings of embedding codepth at most 3 have only trivial semidualizing complexes

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**Assumption**

$(R, m, k)$ is a commutative noetherian local ring.

**Definition (Foxby 1972, Grothendieck 1961)**

(a) A finitely generated $R$-module $C$ is **semidualizing** if $R \cong \text{Hom}_R(C, C)$ and $\text{Ext}^1_R(C, C) = 0$.

(b) A semidualizing module of finite inj. dimension is **dualizing**.

(c) $\mathcal{S}_0(R)$ is the set of isomorphism classes of SDMs.

**Fact**

(a) $R$ is a **semidualizing** $R$-module.

(b) $R$ is a dualizing $R$-module if and only if $R$ is Gorenstein.

(c) $R$ has a dualizing module if and only if it is Cohen-Macaulay and a homomorphic image of a Gorenstein ring.
Remark

Semidualizing modules are useful.
(a) Bass series of local ring homomorphisms.
(b) Composition of ring homomorphisms of finite G-dimension.
(c) Growth of Bass numbers of $R$.
(d) Structure of quasi-deformations.

Example

$R_1 = k[[X, Y]]/(X, Y)^2$ with dualizing module $D_1$.
$R_2 = k[[Z, W]]/(Z, W)^2$ with dualizing module $D_2$.

$$R = R_1 \otimes_k R_2 \cong \frac{k[[X, Y, Z, W]]}{(X, Y)^2 + (Z, W)^2}$$

SDMs: $R \otimes_k D_1 \otimes_k R_2 \otimes_k R_1 \otimes_k D_2 \otimes_k D_1 \otimes_k D_2$
Assume that $R$ is artinian with $e = \operatorname{edim}(R)$.

(a) If $e \leq 1$, then $R$ is CI, so the only SDM is $R$.
(b) If $e = 2$, then $R$ is Golod, so the only SDMs are $R$ and $D$.
(c) The previous example has $e = 4$ with four distinct SDMs.

Sketch of proof of (b).

Let $C$ be a semidualizing $R$-module.

Gerko: $\operatorname{Hom}_R(C, D)$ is SDM and Tor-independent with $C$.

D. Jorgensen: $\operatorname{Hom}_R(C, D)$ or $C$ has finite projective dimension.

Therefore, $C \cong D$ or $C \cong R$.

Question

What if $e = 3$?
Theorem (Nasseh-SW)

Assume that \( e = \text{edim}(R) - \text{depth}(R) \leq 3 \). Then \( R \) has at most two SDMs, namely \( R \) and a dualizing module if \( R \) has one.

Sketch of proof

The completion \( \hat{R} \) is a homomorphic image of a regular local ring \( Q \) with \( \text{edim}(Q) = \text{edim}(R) \).

\[
\begin{align*}
R \rightarrow \hat{R} \rightarrow K^{\hat{R}} \cong \hat{R} \otimes_Q K^Q \cong F \otimes_Q K^Q \cong F \otimes_Q k \\
\mathcal{S}_0(R) \hookrightarrow \mathcal{S}_0(\hat{R}) \hookrightarrow \mathcal{S}(K^{\hat{R}}) \twoheadrightarrow \mathcal{S}(F \otimes_Q k)
\end{align*}
\]

\( K^{\hat{R}} \) has the structure of a DG \( \hat{R} \)-algebra.

Auslander-Buchsbaum: The minimal free resolution \( F \) of \( \hat{R} \) over \( Q \) has length \( \leq 3 \).

Buchsbaum-Eisenbud: \( F \) is a DG \( Q \)-algebra.
Example (Koszul complex as DG algebra)

Let $x = x_1, \ldots, x_n \in R$, and set $K = K^R(x)$. The exterior algebra $\wedge R^n \cong \bigoplus_{i=0}^{n} K_i$ is a graded commutative $R$-algebra. With this multiplication, $K$ satisfies the Leibniz rule:

$$\partial^K(ab) = \partial^K(a)b + (-1)^{|a|}a\partial^K(b).$$

Definition

A DG $R$-algebra is a non-negatively graded $R$-complex $A$ such that $A^{\oplus} = \bigoplus_i A_i$ is a graded commutative $R$-algebra and with this multiplication $A$ satisfies the Leibniz rule.

A DG $A$-module is an $R$-complex $M$ such that $\bigoplus_i M_i$ is a graded $A^{\oplus}$-module and $M$ satisfies the Leibniz rule.

Example

$R$ is a DG $R$-algebra. DG $R$-modules are just $R$-complexes.
**Definition (Christensen-SW 2009)**

Let $A$ be a DG $R$-algebraal $A$ homologically finite DG $A$-module $C$ is **semidualizing** if $A \cong \mathbb{R}\text{Hom}_A(C, C)$.

**Example**

"Semidualizing DG $R$-module" = "semidualizing $R$-complex".

**Theorem (Nasseh-SW)**

Let $B$ be a finite-dimensional DG $k$-algebra, and let $W \neq 0$ be a positively graded finite dimensional $k$-vector space. Consider the trivial extension $A = B \ltimes W$. Given two homologically finite DG $A$-modules $M$ and $N$, if $\text{Tor}_0^A(M, N) = 0$, then either $M$ or $N$ has finite projective dimension over $A$. 
\[ |\mathcal{S}(F \otimes_Q k)| \leq 2. \]

Set \( A = F \otimes_Q k \).

If \( R \) is Gorenstein, then so is \( A \), hence \( |\mathcal{S}(A)| = 1 \).

If \( A \cong B \ltimes W \) for a positively graded vector space \( W \neq 0 \), argue as in the Golod case to conclude that \( |\mathcal{S}(A)| \leq 2 \).

Weyman and Avramov-Kustin-Miller: the only case that remains is when there is a positively graded finite dimensional \( k \)-vector space \( V \neq 0 \) such that

\[
A \cong (k \ltimes V) \otimes_k (k \ltimes \Sigma k) \cong (k \ltimes V) \otimes_k K_k(0) \cong K^{k \ltimes V}(0).
\]

\[
|\mathcal{S}(A)| = |\mathcal{S}(K^{k \ltimes V}(0))| = |\mathcal{S}(k \ltimes V)| \leq 2.
\]

By a DG version of Auslander-Ding-Solberg’s lifting theorem.

By the previous theorem.