Finite Generation of Ext and ascent of module structures

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Assumption. \((R, \mathfrak{m}, k)\) is a (commutative noetherian) local ring.

Theorem. (Buchweitz, Flenner 2006) If \(M\) is a complete \(R\)-module, then \(\text{Ext}^i_R(F, M) = 0\) for all flat \(R\)-modules \(F\) and all \(i \geq 1\), that is, \(M\) is cotorsion.

Question. What about the converse?

Note. Converse fails without extra assumptions.

Example. If \(I\) is an injective \(R\)-module, then \(\text{Ext}^i_R(F, M) = 0\) for all \(R\)-modules \(F\) and all \(i \geq 1\), but \(I\) is almost never complete.

Question. What about the converse when \(M\) is finitely generated?
Theorem. (Frankild, Sather-Wagstaff 2008) Let \( \alpha \) be a proper ideal of \( R \), and let \( M \) be a finitely generated \( R \)-module. TFAE:

1. \( M \) is \( \alpha \)-adically complete;
2. \( \text{Ext}^i_R(\hat{R}^\alpha, M) = 0 \) for all \( i \geq 1 \); and
3. \( M \) has an \( \hat{R}^\alpha \)-module structure compatible with its \( R \)-module structure via the natural map \( R \to \hat{R}^\alpha \).

Note. Uses big guns: derived local (co)homology.

Note. Originally motivated by questions about G-dimensions.

Question. What about other \( R \)-algebras, like \( R^h \)?
Let $M$ be a finitely generated $R$-module, and let $\varphi : R \to S$ be a flat local ring homomorphism such that $R/m \xrightarrow{\cong} S/mS$.

**Theorem.** (Frankild, Sather-Wagstaff, R. Wiegand 2008) TFAE:

1. $M$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$;
2. $\text{Ext}_R^i(S, M) = 0$ for all $i \geq 1$;
3. $\text{Ext}_R^i(S, M)$ is finitely generated over $R$ for all $i \geq 1$;
4. The map $\text{Hom}_R(S, M) \to M$ taking $\psi$ to $\psi(1)$ is bijective;
5. The map $M \to S \otimes_R M$ taking $m$ to $1 \otimes m$ is bijective; and
6. $S \otimes_R M$ is finitely generated as an $R$-module.

**Note.** Proof is easier, using Koszul complex, though it uses the Amplitude Inequality of Foxby and Iyengar (and Iversen) which is a consequence of the New Intersection Theorem.
Let $M$ be a finitely generated $R$-module, and let $\varphi: R \to S$ be a flat local ring homomorphism such that $R/m \xrightarrow{\cong} S/mS$.

**Lemma.** (Frankild, Sather-Wagstaff, R. Wiegand 2008) The map $\text{Hom}_R(S, M) \to M$ taking $\psi$ to $\psi(1)$ is injective.

**Theorem.** (Christensen, Sather-Wagstaff 2010) If $R$ is Gorenstein and $\text{Ext}^i_R(S, M)$ is finitely generated over $S$ for all $i \geq 1$, then $\text{Ext}^i_R(S, M) = 0$ for all $i \geq 1$ and $M$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$.

**Note.** This is a corollary of a result about G-dimensions.

**Question.** Is the Gorenstein assumption necessary?
Let $M$ be a finitely generated $R$-module, and let $\varphi: R \to S$ be a flat local ring homomorphism such that $R/m \cong S/mS$.

**Theorem.** (Anderson, Coykendall, Sather-Wagstaff 2010) If $\text{Ext}_R^i(S, M)$ satisfies Nakayama’s Lemma (e.g., if $\text{Ext}_R^i(S, M)$ is finitely generated over $S$) for all $i \geq 1$, then $\text{Ext}_R^i(S, M) = 0$ for all $i \geq 1$ and $M$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$.

**Definition.** An $R$-module $N$ satisfies Nakayama’s Lemma if either $N = 0$ or $N/mN \neq 0$.

**Note.** Proof is easier, using only basic properties of the Koszul complex.
Let $M$ be an $R$-module, and let $\varphi : R \to S$ be a flat local ring homomorphism such that $R/m \cong S/mS$.

**Lemma.** (Anderson, Coykendall, Sather-Wagstaff 2010) *If $M$ satisfies Krull’s Intersection Theorem, then the map $\text{Hom}_R(S, M) \to M$ taking $\psi$ to $\psi(1)$ is injective.*

**Definition.** An $R$-module $N$ satisfies *Krull’s Intersection Theorem* if $\bigcap_{i=1}^{\infty} m^i N = 0$.

**Example.** Let $k$ be a field, and consider the rings $R = k[\![X]\!]_{(X)}$ and $S = \hat{R} \cong k[\![X]\!]$. Then $\text{Ext}^i_R(S, R) = 0$ for all $i \neq 1$, but $\text{Ext}^1_R(S, R) \cong \text{Hom}_R(\hat{R}, E) \cong E \oplus k((X))^{(\mu)}$ for some $\mu \neq 0$. 