Path Ideals for Weighted Graphs

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North Dakota State University

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Joint with Bethany Kubik and Chelsey Paulsen

Edge Ideals

Assumption

k is a field, $S = k[x_1, \ldots, x_d]$, and $G = (V, E)$ is a (finite simple) graph with $V = \{x_1, \ldots, x_d\}$.

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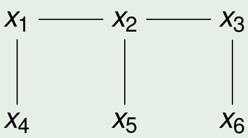

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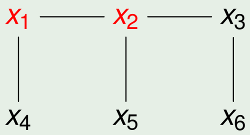

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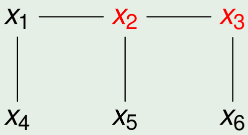

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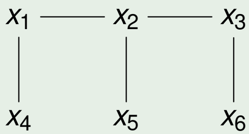

$$I(G) = (x_1x_2, x_2x_3, x_1x_4, x_2x_5, x_3x_6)S.$$

Irreducible Decompositions of Edge Ideals

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A vertex cover of G is a subset $W \subseteq V$ such that for every $x_i x_j \in E$, either $x_i \in W$ or $x_j \in W$.

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$$I(G) = \bigcap_{W} (W)S = \bigcap_{W \ min} (W)S.$$

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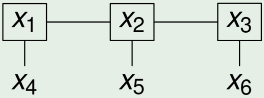

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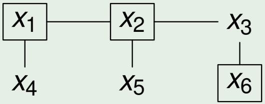

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Cohen-Macaulayness of Trees

Theorem (Villareal 1990)

If T is a tree, then $S/I(T)$ is Cohen-Macaulay if and only if $I(T)$ is unmixed, if and only if T is a suspension of a tree. (Hence, Cohen-Macaulayness of $S/I(T)$ is characteristic-independent.)

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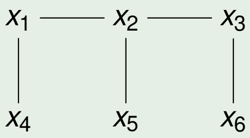

$S/I(T)$ is Cohen-Macaulay.

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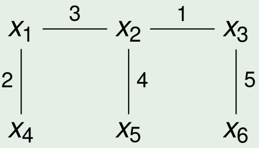

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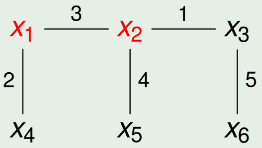

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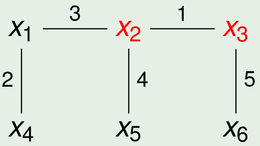

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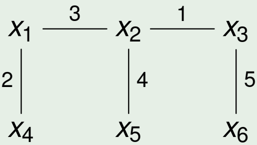

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1. $x_i \in W$ and $\sigma(x_i) \leq \omega(e)$, or
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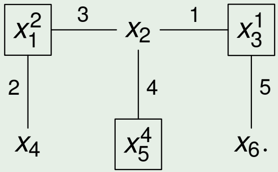

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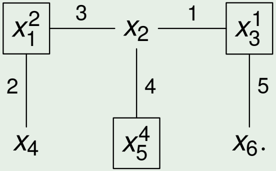

This weighted vertex cover is minimal: no vertices can be unboxed, and no weights (exponents) can be increased.

Decompositions of Weighted Edge Ideals

Theorem (Paulsen-SW '13)

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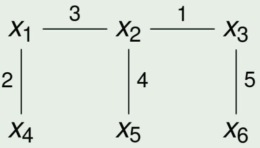

$$\begin{aligned} I(G_\omega) = &(x_1^2, x_2, x_3^5)S \cap (x_1^2, x_2^4, x_3)S \cap (x_1^2, x_2, x_6^5)S \\ &\cap (x_1^2, x_3, x_5^4)S \cap (x_2, x_3^5, x_4^2)S \cap (x_2^3, x_3, x_4^2)S \\ &\cap (x_2, x_4^2, x_6^5)S \cap (x_1^3, x_2^4, x_3, x_4^2)S \cap (x_1^3, x_3, x_4^2, x_5^4)S \end{aligned}$$

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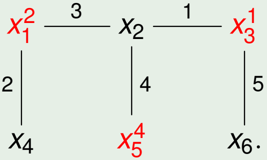

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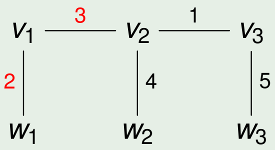

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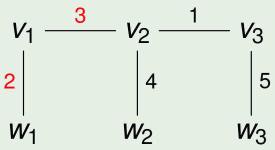

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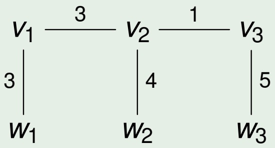

Weighted Path Ideals

Definition (Kubik-SW '13)

Fix an integer $r \geq 1$. The weighted r-path ideal $I_r(G_\omega) \subseteq S$ of a weighted graph G_ω is the ideal of S generated by all monomials

$$x_{i_0}^{\omega(x_{i_0}x_{i_1})} x_{i_1}^{\max(\omega(x_{i_0}x_{i_1}),\omega(x_{i_1}x_{i_2}))} \cdots x_{i_{r-1}}^{\max(\omega(x_{i_{r-2}}x_{i_{r-1}}),\omega(x_{i_{r-1}}x_{i_r}))} x_{i_r}^{\omega(x_{i_{r-1}}x_{i_r})}$$

such that $x_{i_0}x_{i_1}\cdots x_{i_{r-1}}x_{i_r}$ is a path in G.

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Example

1. If $r = 1$, then this is $I(G_\omega)$.
2. If $\omega = 1$ and G is a tree, then this is Conca's $I_r(G)$, generated by all the r-paths in G.

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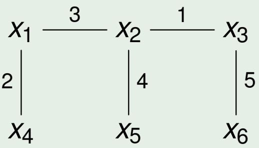

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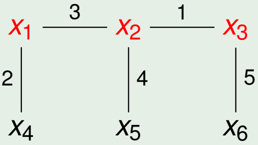

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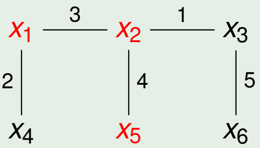

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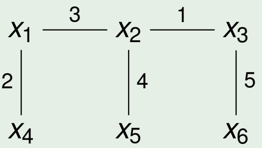

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Let G_ω be a weighted tree with no r-pathless leaves. TFAE:

(i) $I_r(G_\omega)$ *is Cohen-Macaulay;*

(ii) $I_r(G_\omega)$ *is unmixed; and*

(iii) G_ω *is an r-path suspension of a weighted tree Γ_μ s.t. for all $v_iv_j \in E(\Gamma_\mu)$ one has $\omega(v_iv_j) \leq \min\{\omega(v_iy_{i,1}), \omega(v_jy_{j,1})\}$.*

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Let G_ω be a weighted tree with no r-pathless leaves. TFAE:

(i) *$I_r(G_\omega)$ is Cohen-Macaulay;*

(ii) *$I_r(G_\omega)$ is unmixed; and*

(iii) *G_ω is an r-path suspension of a weighted tree Γ_μ s.t. for all $v_i v_j \in E(\Gamma_\mu)$ one has $\omega(v_i v_j) \le \min\{\omega(v_i y_{i,1}), \omega(v_j y_{j,1})\}$.*

Example

$I_2(G_\omega)$