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Complete intersection dimensions for complexes

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Abstract

We extend the notions of complete intersection dimension and lower complete intersection dimension to the category of complexes with finite homology and verify basic properties analogous to those holding for modules. We also discuss the question of the behavior of complete intersection dimension with respect to short exact sequences.

MSC: 13C40; 13D05; 13D25; 13D40; 14H10

1. Introduction

A familiar numerical invariant of a finitely generated module over a Noetherian ring is its projective dimension. The last few decades have seen a number of refinements and extensions of this. One refinement is the notion of Gorenstein dimension, introduced by Auslander and Bridger [2]. More recently, Avramov et al. [6] defined a concept of complete intersection dimension, Gerko [14] forwarded definitions for lower complete intersection dimension and Cohen–Macaulay dimension, and Veliche [21] did the same for upper Gorenstein dimension. The notions of complete intersection dimension and lower complete intersection dimension are the primary focuses of this paper.

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These homological dimensions are well behaved in a number of senses. For example, when M is a finite module over a Noetherian ring R there are inequalities

$$\text{G-dim}_R(M) \leqslant \text{CI}_*\text{-dim}_R(M) \leqslant \text{CI-dim}_R(M) \leqslant \text{pd}_R(M);$$

if one of these dimensions is finite, then it equals those to its left. When R is local each homological dimension satisfies an "AB-formula": if one of the quantities in the displayed formula is finite, then it equals $\text{depth}(R) - \text{depth}_R(M)$. Furthermore, the finiteness of a homological dimension for all finite R-modules characterizes the corresponding ring-theoretic property of R as in the theorem of Auslander, Buchsbaum, and Serre.

In another direction, the projective dimension and Gorenstein dimension have been extended to complexes of R-modules. The projective dimension was systematically developed by Foxby [10–12], and the G-dimension by Yassemi [22] and Christensen [9]. The purpose of this paper is to give a similar extension of complete intersection dimension and lower complete intersection dimension and verify basic properties that one expects to carry over from the situation for modules. This is done in Sections 3 and 5. Also, we prove stability results, Theorems 3.11 and 5.16, that are particular to complexes.

One difficulty with the complete intersection dimension is that we do not know whether it is well behaved with respect to short exact sequences; Section 4 is devoted to this issue. Section 6 consists of a brief discussion of "global" homological dimensions, which can be introduced from the homological dimensions under consideration, like the global dimension of Cartan and Eilenberg [8]. Section 2 is home to a brief catalogue of background material used in the other sections.

2. Background

This section is mostly a summary of standard notions from hyperhomological algebra; the interested reader is directed to [10] for a detailed account. We also include a couple of results that will be important in the sections that follow.

Throughout this paper, all rings are commutative and Noetherian.

A complex of modules over a ring R is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

such that $\partial^X_i \partial^X_{i+1} = 0$ for every integer i. When M is an R-module, identify M with the complex $\cdots \to 0 \to M \to 0 \to \cdots$ concentrated in degree 0.

2.1. A complex X is *bounded below* (resp., *bounded*) if $X_i = 0$ for all $i \ll 0$ (resp., for all $|i| \gg 0$); it is *degreewise finite* if each X_i is a finite R-module; and it is *finite* if it is bounded and degreewise finite. Next, X is *homologically bounded below* (resp., *homologically bounded*) if the homology complex $\text{H}(X)$ is bounded below (resp., bounded); it is *homologically degreewise finite* (resp., *homologically finite*) if $\text{H}(X)$ is degreewise finite (resp., bounded and degreewise finite). The *supremum* and *infimum* of X

are given by the following formulas:

$$\sup(X) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \quad \text{and} \quad \inf(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}.$$

Given an integer n, the nth *suspension* of X is the complex $\Sigma^n X$ with $(\Sigma^n X)_m = X_{m-n}$ and differential $\partial_m^{\Sigma^n X} = (-1)^n \partial_{m-n}^X$ for each m. The kernel and cokernel of ∂_n^X are denoted Z_n^X and C_{n-1}^X, respectively. For any R-module M, one has $C_n^{X \otimes_R M} \cong C_n^X \otimes_R M$, by the right-exactness of $\otimes_R M$. The nth *soft left- and right-truncations* of X are the complexes

$$\tau_{\leqslant n}(X) = \cdots \to 0 \to C_n^X \xrightarrow{\overline{\partial_n^X}} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots,$$
$$\tau_{\geqslant n}(X) = \cdots \xrightarrow{\partial_{n+2}^X} X_{n+1} \xrightarrow{\partial_{n+1}^X} Z_n^X \to 0 \to \cdots,$$

respectively, where $\overline{\partial_n^X}$ is the map induced by ∂_n^X. The nth *hard left- and right-truncations* are the complexes

$$X_{\leqslant n} = \cdots \to 0 \to X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots,$$
$$X_{\geqslant n} = \cdots \xrightarrow{\partial_{n+2}^X} X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \to 0 \to \cdots \ .$$

It is worth noting explicitly that we do not use the machinery of derived categories in this paper. This is for two reasons: we are interested in how the invariants we define behave with respect to short exact sequences, and we use kernels and cokernels of morphisms in our arguments. Instead, we work within the category of complexes of modules.

2.2. Let X, Y be complexes of R-modules. A *morphism* $\sigma : X \to Y$ is a collection of R-module homomorphisms $\sigma_i : X_i \to Y_i$ such that $\partial_i^Y \sigma_i = \sigma_{i-1} \partial_i^X$ for each integer i. A *quasi-isomorphism* is a morphism $\alpha : X \to Y$ such that the map induced on homology $\mathrm{H}(\alpha) \colon \mathrm{H}(X) \to \mathrm{H}(Y)$ is an isomorphism; this is signified by $\alpha : X \xrightarrow{\simeq} Y$. More generally, X and Y are *quasi-isomorphic*, denoted $X \simeq Y$ if there is a finite sequence of quasi-isomorphisms

$$X \xleftarrow{\simeq} X^1 \xrightarrow{\simeq} X^2 \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} Y.$$

If $m \leqslant \inf(X)$ and $n \geqslant \sup X$, then the natural maps $X \to \tau_{\leqslant n}(X)$, $\tau_{\geqslant m}(X) \to X$, and $X_{\geqslant n} \to \Sigma^n C_n^X$ are quasi-isomorphisms. Thus, if $s = \sup(X)$, then $C_s^X \neq 0$.

The homological dimensions studied in this work are descendants of the projective dimension.

2.3. A *projective* (resp., *free*) *resolution* of a homologically bounded below complex X is a bounded below complex $P \simeq X$ of projective (resp., free) R-modules. If X is homologically both degreewise finite and bounded below, then it possesses a degreewise finite free resolution; see [10, (2.6.L)] or apply [20, 3.1.6] to the truncation $\tau_{\geqslant m}(X) \simeq X$ for $m = \inf(X)$. By Avramov and Foxby [4, (1.2.P, 1.4.P)], if $P \simeq X$ is a projective resolution, then there exists a quasi-isomorphism $P \xrightarrow{\simeq} X$.

The *projective dimension of* X is

$$\mathrm{pd}_R(X) = \inf\{\sup\{n \mid P_i \neq 0\} \mid P \text{ is a projective resolution of } X\}.$$

Thus, if $\mathrm{pd}_R(X)$ is finite, then X is homologically both bounded and nonzero. Injective resolutions and the injective dimension $\mathrm{id}_R(X)$ are defined dually.

Given a morphism of complexes $X \to Y$ it can be useful to be able to enlarge X to construct a surjective morphism with the same morphism induced on homology. The next fact [5, (8.4.4, 5)] allows us to do so. See Lemma 2.5 and Proposition 2.6 for applications.

2.4. Given a bounded below degreewise finite complex of R-modules X, there exists a bounded below degreewise finite complex of free R-modules G with $\mathrm{H}(G) = 0$ and a morphism $\varepsilon : G \to X$ such that each ε_i is surjective.

The following is a version of the existence of "strict semifree resolutions".

Lemma 2.5. *Let R be a ring and X a complex of R-modules that is bounded below and degreewise finite. There exists a degreewise finite free resolution $\sigma : P \xrightarrow{\simeq} X$ such that each $\sigma_i : P_i \to X_i$ is surjective.*

Proof. By Roberts [20, 3.1.6] take a degreewise finite free resolution $\alpha : F \xrightarrow{\simeq} X$. Fix a complex G and morphism $\varepsilon : G \to X$ as in 2.4. The complex $P = F \oplus G$ and morphism $\sigma : P \to X$ given by $\sigma_i(f, g) = \alpha_i(f) + \varepsilon_i(g)$ satisfy the conclusions. □

Given a short exact sequence of complexes, it is well known that there exists a short exact sequence on the level of projective resolutions [15, (6.10°)]. It is helpful to know when the projective resolutions can be chosen to be degreewise finite.

Proposition 2.6. *Let R be a ring and $0 \to X \xrightarrow{\eta} Y \xrightarrow{v} Z \to 0$ an exact sequence of complexes of R-modules that are homologically both degreewise finite and bounded below. There exists a commutative diagram of complexes with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \\
 & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\psi} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\lambda} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & X & \xrightarrow{\eta} & Y & \xrightarrow{v} & X & \longrightarrow & 0
\end{array}$$

where each vertical map is a degreewise finite R-projective resolution.

Proof. Let $\alpha : V \xrightarrow{\simeq} Z$ and $\gamma : F \xrightarrow{\simeq} Y$ be degreewise finite R-free resolutions. There exists a morphism $\sigma : F \to V$ such that $v\gamma = \alpha\sigma$, by Avramov and Foxby [4, (1.1.P.1),(1.2.P)]. Since V is bounded below and degreewise finite, fix a complex G and morphism $\varepsilon : G \to V$ as in 2.4. By Avramov et al. [5, (9.8.3.2′),(9.7.1)] there exists a morphism $\rho : G \to Y$ such that $v\rho = \alpha\varepsilon$.

Let $U = F \oplus G$ and define morphisms $\lambda : U \to Y$ and $\theta : U \to V$ by the formulas

$$\lambda_i(f,g) = \gamma_i(f) + \rho_i(g) \quad \text{and} \quad \theta_i(f,g) = \sigma_i(f) + \varepsilon_i(g).$$

It is straightforward to check that $\alpha\theta = v\lambda$. Furthermore, $\lambda : U \to Y$ is a degreewise finite R-free resolution and each θ_i is surjective. Set $T = \mathrm{Ker}(\theta)$ with $\imath : T \to U$ the natural inclusion and $\psi : T \to X$ the morphism induced by λ. Since each sequence $0 \to T_i \to U_i \to V_i \to 0$ is exact with U_i, V_i projective, each T_i is projective. Thus, we have a commutative diagram of the desired form. The 5-lemma applied to the long exact sequences in homology shows that ψ is a quasi-isomorphism. □

Derived Hom and tensor product are ubiquitous tools in the study of complexes.

2.7. Given complexes of R-modules X, Y with X homologically bounded below, then $X \otimes^{\mathbf{L}}_R Y$ and $\mathbf{R}\mathrm{Hom}_R(X,Y)$ denote the complexes $P \otimes_R Y$ and $\mathrm{Hom}_R(P,Y)$, respectively, where $P \simeq X$ is a projective resolution. These complexes are only well defined up to quasi-isomorphism, but this is enough for our applications.

The G-dimension comes to bear directly and indirectly on the study of complete intersection dimension. A nice treatment can be found in [9].

2.8. For a ring R, let $(\text{-})^* = \mathrm{Hom}_R(\text{-}, R)$. A finite R-module M is *totally reflexive over R* if M is reflexive and $\mathrm{Ext}^i_R(M,R) = 0 = \mathrm{Ext}^i_R(M^*,R)$ for all $i > 0$. Each finitely generated projective R-module is totally reflexive over R. A *G-resolution* of a complex X is a bounded below complex $G \simeq X$, such that each G_i is totally reflexive over R. The *G-dimension* of X is

$$\mathrm{G\text{-}dim}_R(X) = \inf\{\sup\{n \mid G_i \neq 0\} \mid G \text{ is a G-resolution of } X\}.$$

By Christensen [9, (2.3.8)], if $\mathrm{G\text{-}dim}_R(X) < \infty$, then one has

$$\mathrm{G\text{-}dim}_R(X) = -\inf(\mathbf{R}\mathrm{Hom}_R(X,R)).$$

The depth of a finite module over a local ring is a familiar invariant. Our definition of depth for complexes is taken from Iyengar [16].

2.9. Let R be a local ring and K the Koszul complex over R on a sequence of generators of length n for the maximal ideal $\mathfrak{m}$ of R. For a complex of R-modules X, the *depth of X* is

$$\mathrm{depth}_R(X) = n - \sup(X \otimes_R K).$$

This is independent of the sequence of generators for $\mathfrak{m}$.

Complexes of finite projective dimension have finite G-dimension, and the finiteness of either of these implies an AB-formula, where “AB” stands for Auslander–Buchsbaum and Auslander–Bridger, cf. [9, (2.3.10,13)].

2.10. For a homologically finite complex X over a ring R, one has an inequality

$$\mathrm{G\text{-}dim}_R(X) \leqslant \mathrm{pd}_R(X)$$

with equality when $\mathrm{pd}_R(X) < \infty$. If R is local and $\text{G-dim}_R(X) < \infty$, then

$$\text{G-dim}_R(X) = \mathrm{depth}(R) - \mathrm{depth}_R(X).$$

The Betti numbers of a complex over a local ring are of particular interest in connection with the complete intersection dimension.

2.11. Let $(R, \mathfrak{m}, k)$ be a local ring and X a homologically bounded below and degreewise finite complex of R-modules. By Roberts [19, (2.2.4)], X has a *minimal* free resolution, that is, a degreewise finite free resolution $F \simeq X$ such that $\partial^F(F) \subseteq \mathfrak{m}F$. As is the case with modules, minimal free resolutions are unique up to isomorphism. The nth *Betti number* of X is

$$\beta_n^R(X) := \mathrm{rank}_R(F_n) = \mathrm{rank}_k \mathrm{H}_n(X \otimes_R^{\mathbf{L}} k).$$

The *Poincaré series* of X is the formal Laurent series

$$P_X^R(t) = \sum_n \beta_n^R(X) t^n.$$

The *complexity* of X, defined by the formula

$$\mathrm{cx}_R(X) = \inf\{c \in \mathbb{N} \mid \text{there exists } \alpha \in \mathbb{R} \text{ such that } \beta_n^R(X) \leqslant \alpha n^{c-1} \text{ for } n \gg 0\}$$

is a measure of the asymptotic size of the minimal free resolution of X. For instance, $\mathrm{cx}_R(X) = 0$ if and only if $\mathrm{pd}_R(X) < \infty$.

The behavior “at infinity” of the sequence of Betti numbers of a complex is almost identical to that of the syzygy modules of the complex.

2.12. Let X be a homologically finite complex of modules over a local ring R, and fix a degreewise finite R-free resolution $P \simeq X$. For $n \geqslant \sup(X)$, it is straightforward to show that the Poincaré series of X and C_n^P are related by the formula

$$P_X^R(t) = t^n P_{C_n^P}^R(t) + t^{\inf(X)} f(t)$$

for some polynomial $f(t) \in \mathbb{Z}[t]$. In particular, it follows that $\mathrm{cx}_R(X) = \mathrm{cx}_R(C_n^P)$.

Certain accounting principles [10, (11.11)] are handy for tracking the behavior of complexity under derived tensor product.

2.13. Let R be a local ring with homologically finite complexes X, Y. There is an equality of Poincaré series

$$P_{X \otimes_R^{\mathbf{L}} Y}^R(t) = P_X^R(t) P_Y^R(t).$$

It follows that, if $\mathrm{H}(Y) \neq 0$, then

$$\mathrm{cx}_R(X) \leqslant \mathrm{cx}_R(X \otimes_R^{\mathbf{L}} Y) \leqslant \mathrm{cx}_R(X) + \mathrm{cx}_R(Y).$$

In particular, if X and Y have finite complexity, then so has $X \otimes_R^{\mathbf{L}} Y$.

3. Complete intersection dimension for complexes

In this section, we introduce the notion of CI-dimension for homologically finite complexes and verify a number of properties which the CI-dimension for modules leads us to expect. For a nonzero finite module, considered as a complex concentrated in degree 0, the definition is the same as that given in [6].

Definition 3.1. Let R be a ring and X a homologically finite complex of R-modules. When R is local, a (*codimension* c) *quasi-deformation* of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$ such that the first map is flat and the second map is surjective with kernel generated by a Q-sequence (of length c). In this situation, let X' denote the complex $X \otimes_R R'$. The *CI-dimension of* X is

$$\text{CI-dim}_R(X) = \inf\{\text{pd}_Q(X') - \text{pd}_Q(R') \mid R \to R' \leftarrow Q \text{ is a quasi-deformation}\}.$$

When R is not necessarily local the *CI-dimension of* X is

$$\text{CI-dim}_R(X) = \sup\{\text{CI-dim}_{R_\mathfrak{m}}(X_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\},$$

where $\text{Max}(R)$ is the set of all maximal ideals of R.

Certain facts are immediate from the definition.

Properties 3.2. Fix a ring R and a homologically finite complex of R-modules X.

3.2.1. $\text{CI-dim}_R(X) \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

3.2.2. $\text{CI-dim}_R(X) = -\infty$ if and only if $X \simeq 0$.

3.2.3. If $X \simeq Y$, then $\text{CI-dim}_R(X) = \text{CI-dim}_R(Y)$.

3.2.4. Each integer n yields $\text{CI-dim}_R(\Sigma^n X) = \text{CI-dim}_R(X) + n$.

The CI-dimension for complexes fits into a hierarchy of homological dimensions like that for modules. Also, over a local ring, an AB-formula is satisfied. This is the analogue of [6, (1.4)] for complexes; since the proof is identical, we omit it here.

Proposition 3.3. *Let R be a ring and X a homologically finite complex of R-modules. There are inequalities*

$$\text{G-dim}_R(X) \leqslant \text{CI-dim}_R(X) \leqslant \text{pd}_R(X);$$

when one of these dimensions is finite it is equal to those on its left. In particular, $\sup(X) \leqslant \text{CI-dim}_R(X)$. *If R is local and* $\text{CI-dim}_R(X) < \infty$, *then*

$$\text{CI-dim}_R(X) = \text{depth}(R) - \text{depth}_R(X).$$

Like the G-dimension and projective dimension, CI-dimension is well behaved with respect to localization. Again, the proof is identical to that of the corresponding result for modules [6, (1.6)]

Proposition 3.4. *Let R be a ring and X a homologically finite complex of R-modules. For every multiplicative subset $S \subset R$, there is an inequality*

$$\text{CI-dim}_{S^{-1}R}(S^{-1}X) \leqslant \text{CI-dim}_R(X).$$

Furthermore,

$$\text{CI-dim}_R(X) = \sup\{\text{CI-dim}_{R_\mathfrak{p}}(X_\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

The following proposition is the expected analogue of the Avramov–Gasharov–Peeva characterization of local complete intersection rings [6, (1.3)]. Recall that a ring R is "locally a complete intersection" if, for every maximal ideal $\mathfrak{m}$ of R, the localization $R_\mathfrak{m}$ is a complete intersection.

Proposition 3.5. *For a ring R with* $\dim(R) < \infty$, *the following are equivalent*:

(a) *R is locally a complete intersection.*
(b) *Each homologically finite complex of R-modules X satisfies*

$$\text{CI-dim}_R(X) \leqslant \dim(R) + \sup(X).$$

(c) *Each maximal ideal $\mathfrak{m} \subset R$ satisfies* $\text{CI-dim}_R(R/\mathfrak{m}) < \infty$.

Proof. (a) ⇒ (b). Let X be a homologically finite complex of R-modules. For each maximal ideal $\mathfrak{m}$ of R, one has $\text{CI-dim}_{R_\mathfrak{m}}(X_\mathfrak{m}) < \infty$; the proof is identical to that of [6, (1.3)]. Furthermore,

$$\text{CI-dim}_{R_\mathfrak{m}}(X_\mathfrak{m}) = \text{depth}(R_\mathfrak{m}) - \text{depth}_{R_\mathfrak{m}}(X_\mathfrak{m}) \leqslant \dim(R_\mathfrak{m}) + \sup(X_\mathfrak{m}),$$

where the equality is by Proposition 3.3 and the inequality is by Foxby and Iyengar [13, (2.7)]. It follows that

$$\begin{aligned}\text{CI-dim}_R(X) &= \sup\{\text{CI-dim}_{R_\mathfrak{m}}(X_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &\leqslant \sup\{\dim(R_\mathfrak{m}) + \sup(X_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &\leqslant \sup\{\dim(R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\} + \sup\{\sup(X_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &= \dim(R) + \sup(X).\end{aligned}$$

(b) ⇒ (c) is trivial.

(c) ⇒ (d). By definition $\text{CI-dim}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) = \text{CI-dim}_R(R/\mathfrak{m}) < \infty$, and so $R_\mathfrak{m}$ is a complete intersection by Avramov et al. [6, (1.3)]. □

The next result is the main tool used to understand the relation between the CI-dimension of a complex X and that of its syzygy modules.

Lemma 3.6. *Let R be a ring and $0 \to X^1 \to X^2 \to X^3 \to 0$ an exact sequence of homologically finite complexes of R-modules. For integers i,j,k such that $\{i,j,k\} = \{1,2,3\}$, there is an inequality*

$$\text{CI-dim}_R(X^k) \leqslant \max\{\text{pd}_R(X^i), \text{CI-dim}_R(X^j)\} + 1.$$

In particular, *if* $\text{pd}_R(X^i)$ *and* $\text{CI-dim}_R(X^j)$ *are finite*, *then* $\text{CI-dim}_R(X^k) < \infty$.

Proof. Assume that $\mathrm{pd}_R(X^i), \text{CI-dim}_R(X^j) < \infty$ and pass to $R_\mathfrak{m}$ to assume that R is local. Let $R \to R' \leftarrow Q$ be a codimension c quasi-deformation such that $\mathrm{pd}_Q((X^j)') < \infty$. It is straightforward to show that

$$\mathrm{pd}_Q((X^k)') \leqslant \max\{\mathrm{pd}_Q((X^i)'), \mathrm{pd}_Q((X^j)')\} + 1 < \infty.$$

The desired conclusion now follows from the equalities $\mathrm{pd}_R(X^i) = \text{CI-dim}_R(X^i)$ and $\text{CI-dim}_R(X^m) = \mathrm{pd}_Q((X^m)') - c$ for $m = 1, 2, 3$. $\square$

Given an exact sequence as in the lemma, it is not known whether one can replace $\mathrm{pd}_R(X^i)$ with $\text{CI-dim}_R(X^i)$, even when each complex is a module concentrated in degree 0. This issue is discussed further in Section 4.

As is the case for modules [6, (1.9)], one can compute the CI-dimension of a complex from that of its syzygies.

Proposition 3.7. *Let X be a homologically finite complex of R-modules. Fix a degreewise finite R-projective resolution $P \simeq X$ and an integer $n \geqslant \sup(X)$.*

(i) *If $C_n^P = 0$, then* $\text{CI-dim}_R(X) = \mathrm{pd}_R(X) < n < \infty$.
(ii) *If $C_n^P \neq 0$, then* $\text{CI-dim}_R(C_n^P) = \max\{0, \text{CI-dim}_R(X) - n\}$.

Proof. Consider the exact sequence of complexes

$$0 \to P_{\leqslant n-1} \to P \to P_{\geqslant n} \to 0 \tag{$\bigstar$}$$

and recall that $P_{\geqslant n} \simeq \Sigma^n C_n^P$. If $C_n^P = 0$, then the morphism $P_{\leqslant n-1} \to P$ is a quasi-isomorphism, and it follows that $\mathrm{pd}_R(X) = \mathrm{pd}_R(P) = \mathrm{pd}_R(P_{\leqslant n-1}) < n$.

If $C_n^P \neq 0$, then $\text{CI-dim}_R(X) < \infty$ if and only if $\text{CI-dim}_R(P_{\geqslant n}) < \infty$ by Lemma 3.6. Since $\text{CI-dim}_R(P_{\geqslant n}) = \text{CI-dim}_R(C_n^P) + n$, the formula holds when $\text{CI-dim}_R(X) = \infty$, so assume that $\text{CI-dim}_R(X) < \infty$. The CI-dimensions of the complexes in ($\bigstar$) agree with their G-dimensions. An analysis of the long exact sequence on homology associated to the exact sequence $\mathbf{R}\mathrm{Hom}((\bigstar), R)$ shows that

$$\begin{aligned} \text{G-dim}_R(C_n^P) + n &= \text{G-dim}_R(P_{\geqslant n}) \\ &= -\inf(\mathrm{Hom}_R(P_{\geqslant n}, R)) \\ &= -\min\{-n, \inf(\mathrm{Hom}_R(P, R))\} \\ &= \max\{n, \text{G-dim}_R(P)\} \end{aligned}$$

and the result now follows from Proposition 3.3. $\square$

Corollary 3.8. *For a homologically finite complex of R-modules X the following conditions are equivalent*:

(a) $\text{CI-dim}_R(X) < \infty$.
(b) *Each degreewise finite R-projective resolution $P \simeq X$ and each $n \geqslant \sup(X)$ yield* $\text{CI-dim}_R(C_n^P) < \infty$.

(c) *Some degreewise finite R-projective resolution* $P \simeq X$ *and some* $n \geqslant \sup(X)$ *yield* $\text{CI-dim}_R(C_n^P) < \infty$.

As a corollary, one sees that a complex of finite CI-dimension has what might be termed a "finite CI-resolution". The converse of this property is related to the behavior of CI-dimension over short exact sequences; see Theorem 4.2.

Corollary 3.9. *If* $\text{CI-dim}_R(X)$ *is finite, then there exists a finite complex of R-modules* $Y \simeq X$ *such that each nonzero* Y_i *has CI-dimension* 0.

Proof. Let $n = \text{CI-dim}_R(X)$ and fix a degreewise finite projective resolution $P \simeq X$. Consider the soft truncation $\tau_{\leqslant n}(P) \simeq X$. Then $\tau_{\leqslant n}(P)_i = 0$ for each $i > n$ and $\tau_{\leqslant n}(P)_i$ is a finitely generated projective for each $i \neq n$. Proposition 3.7 implies that $\tau_{\leqslant n}(P)_n \cong C_n^P$ has CI-dimension 0 so that $\tau_{\leqslant n}(P)$ has the desired form. □

We now use Proposition 3.7 to deduce facts about CI-dimension for complexes directly from the corresponding facts for modules [6, (1.12,13),(4.10),(5.3,6)]. It is worth noting that the results on complexity can be proved using cohomological operators as in [7,23].

Corollary 3.10. *Let X be a homologically finite complex of R-modules.*

(i) *For a faithfully flat ring homomorphism* $R \to S$ *there is an inequality*

$$\text{CI-dim}_R(X) \leqslant \text{CI-dim}_S(X \otimes_R S)$$

with equality when $\text{CI-dim}_S(X \otimes_R S) < \infty$.

(ii) *Let* $\pi : Q \to R$ *be a surjective ring homomorphism with kernel generated by a Q-regular sequence* $\mathbf{x} = x_1, \ldots, x_c$. *There is an inequality*

$$\text{CI-dim}_R(X) \leqslant \text{CI-dim}_Q(X) - c$$

with equality when $\text{CI-dim}_Q(X) < \infty$.

(iii) *Let* $\mathfrak{a} \subset R$ *be an ideal,* R^* *the* $\mathfrak{a}$*-adic completion, and* $X^* = X \otimes_R R^*$. *There is an inequality*

$$\text{CI-dim}_{R^*}(X^*) \leqslant \text{CI-dim}_R(X)$$

with equality when $\mathfrak{a}$ *is contained in the Jacobson radical of R.*

(iv) *If R is local and* $\text{CI-dim}_R(X)$ *finite, then the Poincaré series* $P_X^R(t)$ *is a rational function in* $\mathbb{Z}(t)$, *and* $\text{cx}_R(X)$ *is equal to the order of the pole at* $t = 1$ *of* $P_X^R(t)$; *in particular*, $\text{cx}_R(X) < \infty$.

(v) *If R is local and* $\text{CI-dim}_R(X) < \infty$, *then* $\text{cx}_R(X) \leqslant \text{edim}(R) - \text{depth}(R)$, *and the inequality is strict unless R is a complete intersection.*

Proof. (i) If P is a degreewise finite R-free resolution of X, then $P \otimes_R S$ is a degreewise finite S-free resolution of $X \otimes_R S$, and $C_n^{(P \otimes_R S)} = C_n^P \otimes_R S$ for each integer n. By Avramov et al. [6, (1.13.1)] $\text{CI-dim}_R(C_n^P) \leqslant \text{CI-dim}_S(C_n^P \otimes_R S)$ with equality when $\text{CI-dim}_S(C_n^P \otimes_R S) < \infty$. Applying 3.7 with $n = \sup(X)$ implies the desired result.

(ii) Assume that $\text{CI-dim}_Q(X) < \infty$. For a maximal ideal $\mathfrak{n}$ of Q not containing $\mathbf{x}$, one has $X_{\mathfrak{n}} = 0$. Thus, one reduces to the case where Q and R are local. In this case, apply 3.7 with $n = \sup(X)$ and [6, (1.12.3)] as in (i), to deduce the result.

(iii) This is proved similarly to (i), using [6, (1.13.2)].

(iv) Let P be a minimal free resolution of X and fix an integer $n \geqslant \sup(X)$. By 3.7, $\text{CI-dim}_R(C_n^P) < \infty$. By Avramov et al. [6, (4.10)], [1, (11.1)], the Poincaré series $P_{C_n^P}^R(t)$ is in $\mathbb{Z}(t)$. By Avramov et al. [6, (5.3)], the order of the pole of $P_{C_n^P}^R(t)$ at $t=1$ is exactly $\text{cx}_R(C_n^P)$. As noted in 2.12, one has $P_X^R(t) = t^n P_{C_n^P}^R(t) + t^{\inf(X)} f(t)$ for some polynomial $f(t) \in \mathbb{Z}[t]$. In particular, $P_X^R(t) \in \mathbb{Z}(t)$, the orders of the poles at $t=1$ of $P_X^R(t)$ and $P_{C_n^P}^R(t)$ are equal, and $\text{cx}_R(X) = \text{cx}_R(C_n^P)$.

(v) Use the equality $\text{cx}_R(X) = \text{cx}_R(C_n^P)$ and [6, (5.6)]. □

The final result of this section parallels stability results of Yassemi [22, (2.14,15)] for G-dimension and their generalizations [17, (5.1,7–9)]. It is particular to complexes because, when M and N are finite modules with $\text{pd}_R(N)$ finite, the complexes $M \otimes_R^{\mathbf{L}} N$ and $\mathbf{R}\text{Hom}_R(N, M)$ are generally not concentrated homologically in any single degree. Also, it is easy to construct examples showing that the hypothesis "$\text{pd}_R(P)$ is finite" is necessary: even for two finite modules M, N over a local complete intersection, $M \otimes^{\mathbf{L}} N$ need not be homologically bounded and therefore need not have finite CI-dimension.

Theorem 3.11. *Let R be a ring and X, P homologically finite complexes of R-modules. If $\text{pd}_R(P)$ is finite then*

$$\text{CI-dim}_R(X \otimes_R^{\mathbf{L}} P) = \text{CI-dim}_R(X) + \text{CI-dim}_R(P)$$

and

$$\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(P, X)) = \text{CI-dim}_R(X) - \inf(P).$$

In particular, the CI-dimensions of the complexes X, $X \otimes_R^{\mathbf{L}} P$, and $\mathbf{R}\text{Hom}_R(P, X)$ are simultaneously finite.

Proof. By Yassemi [22, (2.14,15)], it suffices to show that $\text{CI-dim}_R(X)$, $\text{CI-dim}_R(X \otimes_R^{\mathbf{L}} P)$, and $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(P, X))$ are simultaneously finite. Furthermore, it suffices to consider the case where R is local and $\text{H}(X) \neq 0$. It is straightforward to show that $\text{H}(X \otimes_R^{\mathbf{L}} P)$ and $\text{H}(\mathbf{R}\text{Hom}_R(P, X))$ are both nonzero.

For any quasi-deformation $R \to R' \leftarrow Q$ one has $(X' \otimes_{R'}^{\mathbf{L}} P') \simeq (X \otimes_R^{\mathbf{L}} P)'$. Since $\text{pd}_{R'}(P') = \text{pd}_R(P) < \infty$, it follows from [17, (5.8)] that $\text{pd}_Q((X \otimes_R^{\mathbf{L}} P)') = \text{pd}_Q(X') + \text{pd}_{R'}(P')$. In particular, $\text{pd}_Q((X \otimes_R^{\mathbf{L}} P)')$ and $\text{pd}_Q(X')$ are simultaneously finite, and thus the same is true of $\text{CI-dim}_R(X \otimes_R^{\mathbf{L}} P)$ and $\text{CI-dim}_R(X)$.

The tensor-evaluation morphism $X \otimes_R^{\mathbf{L}} \mathbf{R}\text{Hom}_R(P, R) \to \mathbf{R}\text{Hom}_R(P, X)$ is a quasi-isomorphism, because $\text{pd}_R(P) < \infty$ and $\text{H}(P)$ is finite. Since $\mathbf{R}\text{Hom}_R(P, R)$ is homologically finite and $\text{pd}_R(\mathbf{R}\text{Hom}_R(P, R)) < \infty$, the last paragraph implies that $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(P, X))$ is finite if and only if $\text{CI-dim}_R(X)$ is finite. □

4. Exact sequences

In this section, we discuss the behavior of CI-dimension with respect to exact sequences. The primary question is the following.

Question 4.1. Let R be a ring and $0 \to X^1 \to X^2 \to X^3 \to 0$ an exact sequence of homologically finite complexes of R-modules. For integers i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, if $\text{CI-dim}_R(X^i), \text{CI-dim}_R(X^j) < \infty$, must it be that $\text{CI-dim}_R(X^k) < \infty$?

For a ring R, if the answer to Question 4.1 is always "yes", the ring R is said to *satisfy the exact sequence property* (ES). If the answer is always "yes" for exact sequences of finite R-modules, then R *satisfies* (*ES*) *for modules*. Lemma 3.6 implies that one need consider the question in the case where all three complexes have infinite projective dimension.

If R satisfies (ES), then it satisfies (ES) for modules; the converse also holds. In addition, the rings which satisfy (ES) are exactly those rings for which the converse of Corollary 3.9 holds.

Theorem 4.2. *For a ring R, the following conditions are equivalent*:

(a) *R satisfies* (*ES*).
(b) *R satisfies* (*ES*) *for modules.*
(c) *Every finite complex of R-modules X such that* $\text{CI-dim}_R(X_i) < \infty$ *for each integer i satisfies* $\text{CI-dim}_R(X) < \infty$.

Proof. (a) $\Rightarrow$ (c). Fix a finite complex of R-modules X with $\text{CI-dim}_R(X_i) < \infty$ for each integer i. Since X is finite, proceed by induction on the number s of modules X_i that are nonzero. If $s = 0$ or 1, then it is immediate that $\text{CI-dim}_R(X) < \infty$. If $s > 1$, let $t = \sup\{i \mid X_i \neq 0\}$ and consider the exact sequence $0 \to X_{\leqslant t-1} \to X \to \Sigma^t X_t \to 0$. By induction $\text{CI-dim}_R(X_{\leqslant t-1}) < \infty$, and since (ES) holds, one has $\text{CI-dim}_R(X) < \infty$.

(c) $\Rightarrow$ (b). Let $0 \to L \xrightarrow{v} M \xrightarrow{\phi} N \to 0$ be an exact sequence of nonzero finite R-modules and suppose that two of the modules have finite CI-dimension.

Case 1: $\text{CI-dim}_R(L), \text{CI-dim}_R(M) < \infty$. The complex $X = 0 \to L \to M \to 0$ is quasi-isomorphic to N, and thus, $\text{CI-dim}_R(N) = \text{CI-dim}_R(X) < \infty$ by assumption.

Case 2: $\text{CI-dim}_R(M), \text{CI-dim}_R(N) < \infty$. This is similar to Case 1.

Case 3: $\text{CI-dim}_R(L), \text{CI-dim}_R(N) < \infty$. Fix a finitely generated projective R-module P with a surjection $\alpha : P \twoheadrightarrow N$. Lemma 3.6 implies that $K = \text{Ker}(\alpha)$ has $\text{CI-dim}_R(K) < \infty$. Let $\gamma : P \to M$ be a map such that $\alpha = \phi\gamma$; it is straightforward to check that there is an exact sequence

$$0 \to K \to P \oplus L \xrightarrow{(\gamma\ \ v)} M \to 0.$$

Lemma 3.6 implies that $\text{CI-dim}_R(P \oplus L) < \infty$. Since $\text{CI-dim}_R(K) < \infty$, this implies that $\text{CI-dim}_R(M) < \infty$ by Case 1.

(b) $\Rightarrow$ (a). Fix an exact sequence $0 \to X \to Y \to Z \to 0$ of homologically finite complexes of R-modules such that two of the complexes have finite CI-dimension and all three complexes have infinite projective dimension. By Proposition 2.6, there exists a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \\
 & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

where each row is exact and each vertical map is a degreewise finite projective resolution. Replace the original sequence with the top row of this diagram to assume that each complex is bounded below and consists of finitely generated projectives.

For $s = \max\{\sup X, \sup Y, \sup Z\}$ one has an exact sequence

$$0 \to C_s^X \to C_s^Y \to C_s^Z \to 0.$$

Using Proposition 3.7, our assumptions imply that two of the modules in this sequence have finite CI-dimension. Since R satisfies property (ES) for modules, the third module also has finite CI-dimension. Using 3.7 again, it follows that the third complex in the original sequence has finite CI-dimension, as desired. □

If R satisfies (ES) and X is a homologically finite complex whose nonzero homology modules have finite CI-dimension, then X must also have finite CI-dimension. Example 4.4 shows that the converse fails.

Proposition 4.3. *Let R be a ring satisfying (ES) and X a homologically finite complex of R-modules. If* $\text{CI-dim}_R(H_i(X)) < \infty$ *for all i, then* $\text{CI-dim}_R(X) < \infty$.

Proof. Since X is homologically finite, argue by induction on $s = \sup(X) - \inf(X)$. If $s \leqslant 1$, then $X \simeq \Sigma^j H^j(X)$ for some j and so $\text{CI-dim}_R(X) = \text{CI-dim}_R(H^j(X)) + j < \infty$ by 3.2.3 and 3.2.4. When $s > 1$, let $t = \sup(X)$ and consider the exact sequence $0 \to Y \to X \to \tau_{\leqslant t-1}(X) \to 0$. By construction, Y and $\tau_{\leqslant t-1}(X)$ satisfy the induction hypothesis and therefore have finite CI-dimension. As R satisfies (ES), it follows that $\text{CI-dim}_R(X) < \infty$. □

The following is an example of a ring R and a homologically finite complex of R-modules X such that X has finite CI-dimension and each nonzero homology module $H_i(X)$ has infinite CI-dimension. Such a complex must have at least two nonvanishing homology modules, and this example has exactly two of them.

Example 4.4. Let k be a field and $R = k[\![S,T]\!]/(S^2, ST, T^2) = k[\![s,t]\!]$ with maximal ideal $\mathfrak{m} = (s,t)R$. Let $X = (0 \to R \xrightarrow{s} R \to 0)$. Then X has projective dimension 1 and therefore finite CI-dimension. The homology modules are $H_0(X) = R/sR$ and $H_1(X) = \mathfrak{m}$. It is straightforward to verify that each of these modules has infinite complexity and therefore cannot have finite CI-dimension.

5. Lower complete intersection dimension for complexes

In this section, we consider the lower complete intersection dimension, which was introduced for modules in [14] under the name “polynomial complete intersection dimension”. We extend this dimension to the category of homologically finite complexes and present its basic properties. Most of the results in this section have analogues for CI-dimension, and it might seem natural to present the two dimensions in the same section. However, the underlying ideas are rather different, so we consider them separately.

We begin with a more general situation coming from [2, p. 99].

Definition 5.1. For a ring R, a full subcategory $\mathscr{B}$ of the category of finite R-modules is a *resolving subclass* if it satisfies the following:

(1) Every finitely generated projective R-module is in $\mathscr{B}$.
(2) If $0 \to A \to B \to C \to 0$ is an exact sequence of finite R-modules with $C \in \mathscr{B}$, then $A \in \mathscr{B}$ if and only if $B \in \mathscr{B}$.
(3) If A, C are finite R-modules and $B = A \oplus C$ is in $\mathscr{B}$, then $A, C \in \mathscr{B}$.

A *$\mathscr{B}$-resolution* of a homologically finite complex of R-modules X is a bounded below complex $B \simeq X$ with each B_i in $\mathscr{B}$. The *$\mathscr{B}$-dimension* of X is

$$\mathscr{B}\text{-}\dim_R(X) = \inf\{\sup\{i \mid B_i \neq 0\} \mid B \text{ is a } \mathscr{B}\text{-resolution of } X\}.$$

Certain fact follow from the definition.

Properties 5.2. Fix a ring R and a homologically finite complex of R-modules X.

5.2.1. Each degreewise finite projective resolution of X is a $\mathscr{B}$-resolution.

5.2.2. $\mathscr{B}\text{-}\dim_R(X) \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

5.2.3. $\mathscr{B}\text{-}\dim_R(X) = -\infty$ if and only if $X \simeq 0$.

5.2.4. Each integer n yields $\mathscr{B}\text{-}\dim_R(\Sigma^n X) = \mathrm{CI}_*\text{-}\dim_R(X) + n$.

5.2.5. $\sup(X) \leqslant \mathscr{B}\text{-}\dim_R(X)$.

5.3. With the previous sections in mind, let R be a ring and set

$$\mathscr{C} = \{M \mid \mathrm{CI}\text{-}\dim_R(M) = 0\} \cup \{0\}.$$

One might be tempted to consider the $\mathscr{C}$-dimension arising from this choice. However, in the absence of the property (ES), the class $\mathscr{C}$ is not known to be a resolving subclass. When (ES) is satisfied, though, it is straightforward to verify that $\mathscr{C}\text{-}\dim_R(X) = \mathrm{CI}\text{-}\dim_R(X)$ using Propositions 3.7 and 5.6.

The following proposition is a version of [2, (3.12)] for complexes. In the way that Schanuel's lemma allows for the computation of $\mathrm{pd}_R(M)$ from an arbitrary projective resolution of a module M, this result shows that $\mathscr{B}\text{-dim}_R(X)$ can be computed from any $\mathscr{B}$-resolution of X.

Proposition 5.4. *Let R be a ring and $\mathscr{B}$ a resolving subclass of the category of finite R-modules. Consider two complexes of finite R-modules*

$$A = 0 \to A_m \to A_{m-1} \to \cdots \to A_n \to 0,$$

$$B = 0 \to B_{m+j} \to B_{m+j-1} \to \cdots \to B_p \to 0$$

with $j \geqslant 0$ and such that $T \simeq U$ and $A_{m-1}, \ldots, A_n, B_{m+j-1}, \ldots, B_p \in \mathscr{B}$. If A_m is in $\mathscr{B}$ then B_{m+j} is in $\mathscr{B}$.

Proof. When $\mathrm{H}(A) = 0 = \mathrm{H}(B)$, one uses 5.1(2) inductively to show that $\mathrm{Im}(\partial_i^A)$ and $\mathrm{Im}(\partial_i^B)$ are in $\mathscr{B}$ for $i \leqslant m$; in particular, both A_m and B_{m+j} are in $\mathscr{B}$.

In general, it suffices to consider the case $j = 0$. Indeed, since $\sup(B) = \sup(A) \leqslant m$, one has $\tau_{\geqslant m}(B) \simeq B$. By the case $j = 0$, the module $\tau_{\geqslant m}(B)_m = C_m^B$ is in $\mathscr{B}$. Applying the previous paragraph to the exact complex

$$0 \to B_{m+j} \to \cdots \to B_{m+1} \to C_m^B \to 0$$

one concludes that B_{m+j} is in $\mathscr{B}$.

By Lemma 2.5, there exists a degreewise finite free resolution $\sigma : P \xrightarrow{\simeq} A$ such that each σ_i surjective. Since $A \simeq B$, there exists a quasi-isomorphism $\rho : P \xrightarrow{\simeq} B$ by 2.3. Let $P' = \tau_{\leqslant m}(P)$ and consider the canonical quasi-isomorphism $\varepsilon : P \to P'$. Because $A_{m+1} = 0 = B_{m+1}$, it follows that σ and ρ factor through ε. This gives quasi-isomorphisms $\sigma' : P' \xrightarrow{\simeq} A$ and $\rho' : P' \xrightarrow{\simeq} B$ such that each σ_i' is surjective. By construction, the complex $P' = 0 \to P_m' \to P_{m-1} \to \cdots \to P_q \to 0$ has $P_{m-1}, \ldots, P_q \in \mathscr{B}$.

In order to first see that $P_m' \in \mathscr{B}$, set $U = \mathrm{Ker}(\sigma')$, which is homologically zero since σ' is a quasi-isomorphism. For $i < m$, applying 5.1(2) to the exact sequence $0 \to U_i \to P_i \to A_i \to 0$ implies that U_i is in $\mathscr{B}$. Since $\mathrm{H}(U) = 0$ and $U_i = 0$ for $i > m$ and $i < n$, one has $U_m \in \mathscr{B}$. The exact sequence $0 \to U_m \to C_m^P \to A_m \to 0$ implies that P_m' is in $\mathscr{B}$.

To show that $B_m \in \mathscr{B}$, let $V = \mathrm{Cone}(\rho')$ denote the mapping cone of ρ', which is bounded below. Since ρ' is a quasi-isomorphism, $\mathrm{H}(V) = 0$. Since $V_i = B_i \oplus P_{i-1}'$ for each i, it follows from 5.1(2) that $V_i \in \mathscr{B}$ for $i \leqslant m - 1$. As above, one deduces that $\mathrm{Im}(\partial_i^V) \in \mathscr{B}$ for $i \leqslant m$. Furthermore, $V_{m+1} = P_m'$ is in $\mathscr{B}$, so the exact sequence $0 \to V_{m+1} \to V_m \to \mathrm{Im}(\partial_m^V) \to 0$ implies that V_m is in $\mathscr{B}$. As $V_m = B_m \oplus P_{m-1}$, it follows that B_m is in $\mathscr{B}$. □

One can describe $\mathscr{B}\text{-dim}(X)$ in terms of the inclusion of C_n^B in $\mathscr{B}$ for an arbitrary $\mathscr{B}$-resolution $B \simeq X$.

Corollary 5.5. *Every $\mathscr{B}$-resolution B of a homologically finite complex of R-modules X satisfies*

$$\mathscr{B}\text{-dim}_R(X) = \inf\{n \geqslant \sup(X) \mid C_n^B \in \mathscr{B}\}.$$

Proof. Let $t = \mathscr{B}\text{-dim}_R(X)$ and $u = \inf\{n \geqslant \sup(X) \mid C_n^B \in \mathscr{B}\}$. If $t < \infty$, then C_t^B is in $\mathscr{B}$. Indeed, fix a $\mathscr{B}$-resolution $A \simeq X$ with $A_i = 0$ for all $i > t$. Then $\tau_{\leqslant t}(B) \simeq B \simeq X \simeq A$ since $t \geqslant \sup(X)$, so $C_t^B = \tau_{\leqslant t}(B)_t \in \mathscr{B}$ by Proposition 5.4.

Whether or not t is finite, this shows that $t \geqslant u$. If $u = \infty$, then $t = u$. If $u < \infty$, then $\tau_{\leqslant u}(B)$ is a bounded $\mathscr{B}$-resolution of X and so $t \leqslant u$. □

The $\mathscr{B}$-dimension of a complex can be computed from that of the syzygies arising from any $\mathscr{B}$-resolution. Compare this to Proposition 3.7.

Proposition 5.6. *Let R be a ring and $\mathscr{B}$ a resolving subclass of the category of finite R-modules. Fix a $\mathscr{B}$-resolution B of a homologically finite complex of R-modules X and an integer $n \geqslant \sup(X)$.*

(i) *If $C_n^B = 0$, then $\mathscr{B}\text{-dim}_R(X) < n$.*
(ii) *If $C_n^B \neq 0$, then $\text{CI}_*\text{-dim}_R(C_n^B) = \max\{0, \mathscr{B}\text{-dim}_R(X) - n\}$.*

Proof. Since $n \geqslant \sup(X) = \sup(B)$, one has $\tau_{\leqslant n}(B) \simeq B \simeq X$. If $C_n^B = 0$, then $\tau_{\leqslant n}(B)$ is a $\mathscr{B}$-resolution of X with $\tau_{\leqslant n}(B)_i = 0$ for all $i \geqslant n$, and it follows that $\mathscr{B}\text{-dim}_R(X) < n$. Therefore, assume that $C_n^B \neq 0$ and let $t = \mathscr{B}\text{-dim}_R(X)$.

Case 1: $t \leqslant n$. Corollary 5.5 implies that C_t^B is in $\mathscr{B}$. From 5.1(2), it follows that C_n^B is in $\mathscr{B}$, as well. Thus, $\mathscr{B}\text{-dim}_R(C_n^B) = 0$ and the formula holds.

Case 2: $t = \infty$. From Corollary 5.5, it follows that, for all $m \geqslant \sup(X)$, the module C_m^B is not in $\mathscr{B}$. Since the complex $\Sigma^{-n}(B_{\geqslant n})$ is a $\mathscr{B}$-resolution of C_n^B, another application of 5.5 yields $\mathscr{B}\text{-dim}_R(C_n^B) = \infty$, verifying the formula.

Case 3: $\infty > t > n$. Again by Corollary 5.5, the module C_t^B is in $\mathscr{B}$ and C_i^B is not in $\mathscr{B}$ for $i = n, \ldots, t-1$. Therefore, the complex

$$\Sigma^{-n}(\tau_{\leqslant i}(B)) = 0 \to C_i^B \to B_{i-1} \to \cdots \to B_n \to 0$$

is a $\mathscr{B}$-resolution of C_n^B when $i = t$, and is not a $\mathscr{B}$-resolution when $i < t$. By 5.5, $\mathscr{B}\text{-dim}_R(C_n^B) = t - n$ and the formula holds. □

Corollary 5.7. *For a homologically finite complex of R-modules X, the following conditions are equivalent*:

(a) $\mathscr{B}\text{-dim}_R(X) < \infty$.
(b) *Each $\mathscr{B}$-resolution $B \simeq X$ and each $n \geqslant \sup(X)$ yield $\mathscr{B}\text{-dim}_R(C_n^B) < \infty$.*
(c) *Some $\mathscr{B}$-resolution $B \simeq X$ and some $n \geqslant \sup(X)$ yield $\mathscr{B}\text{-dim}_R(C_n^B) < \infty$.*

The $\mathscr{B}$-dimension behaves well with respect to exact sequences of complexes. As discussed in Section 4, this is stronger than what we currently know for CI-dimension. The corresponding statement for CI_*-dimension of modules is [14, (2.8)].

Corollary 5.8. *An exact sequence of homologically finite complexes of R-modules* $0 \to X^1 \to X^2 \to X^3 \to 0$ *and integers* i, j, k *such that* $\{i, j, k\} = \{1, 2, 3\}$ *yield*

$$\mathscr{B}\text{-dim}_R(X^k) \leqslant \max\{\mathscr{B}\text{-dim}_R(X^i), \mathscr{B}\text{-dim}_R(X^j)\} + 1.$$

In particular, if $\mathscr{B}\text{-dim}_R(X^i)$ *and* $\mathscr{B}\text{-dim}_R(X^j)$ *are finite, then* $\mathscr{B}\text{-dim}_R(X^k) < \infty$.

Proof. Almost identical to that of the implication (b) ⇒ (a) in Theorem 4.2; use Corollary 5.7 in place of Proposition 3.7. □

We now specialize the $\mathscr{B}$-dimension to the lower complete intersection dimension. For a nonzero finite module, considered as a complex concentrated in degree 0, the definition is the same as that given in [14, (2.3)].

Definition 5.9. Let R be a ring. The CI_**-class of R*, denoted $\text{CI}_*(R)$, is the collection of totally reflexive R-modules T such that, for every maximal ideal $\mathfrak{m}$ of R, the localized module $T_\mathfrak{m}$ has finite complexity over $R_\mathfrak{m}$. Thus, a finite module T is in $\text{CI}_*(R)$ if and only if, for every maximal ideal $\mathfrak{m}$ of R, the $R_\mathfrak{m}$-module $T_\mathfrak{m}$ is totally reflexive and has finite complexity.

From [9, (1.1.10,11)], [3, (4.2.4)] it follows that $\text{CI}_*(R)$ is a resolving subclass of the category of finite R-modules. The resulting homological dimension $\text{CI}_*\text{-dim}_R$ is the *lower complete intersection dimension*.

Of course, the results stated for $\mathscr{B}$-dimension hold for CI_*-dimension. We continue with properties specifically for the CI_*-dimension. The first of these states that like CI-dimension (3.4) the CI_*-dimension of a complex does not increase after localizing and is determined locally. The result for finite modules is [14, (2.11)].

Proposition 5.10. *Let R be a ring and X a homologically finite complex of R-modules. For every multiplicative subset* $S \subset R$ *there is an inequality*

$$\text{CI}_*\text{-dim}_{S^{-1}R}(S^{-1}X) \leqslant \text{CI}_*\text{-dim}_R(X).$$

Furthermore, there are equalities

$$\text{CI}_*\text{-dim}_R(X) = \sup\{\text{CI}_*\text{-dim}_{R_\mathfrak{m}}(X_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}$$

$$= \sup\{\text{CI}_*\text{-dim}_{R_\mathfrak{p}}(X_\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

Proof. The inequality follows readily; use [14, (2.11)] to show that a CI_*-resolution of X over R localizes to a CI_*-resolution of $S^{-1}X$ over $S^{-1}R$.

For the other formulas, set $v = \sup\{\text{CI}_*\text{-dim}_{R_\mathfrak{p}}(X_\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\}$. It follows from the inequality above that we need only verify that $\text{CI}_*\text{-dim}_R(X) \leqslant v$. To this end, assume that $v < \infty$. Fix a CI_*-resolution $U \simeq X$ over R and note that

$$v \geqslant \sup\{\sup(X_\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\} = \sup(X).$$

For every $\mathfrak{p}$, the complex $U_\mathfrak{p}$ is a CI_*-resolution of $X_\mathfrak{p}$ over $R_\mathfrak{p}$ and $C_u^{U_\mathfrak{p}} \cong (C_u^U)_\mathfrak{p}$. By Corollary 5.5, the module $(C_u^U)_\mathfrak{p}$ is in $\text{CI}_*(R_\mathfrak{p})$ for all $\mathfrak{p}$. By definition, $C_u^U \in \text{CI}_*(R)$, so that $\text{CI}_*\text{-dim}_R(X) \leqslant u$. □

The following result explains the position of CI_*-dimension in the hierarchy of homological dimensions and shows that complexes of finite CI_*-dimension over a local ring satisfy an AB-formula. That this holds for finite modules is in [14, (2.6,7)]. It is important to note that each of the given inequalities can be strict. For the first and third inequalities, this is straightforward. For the second inequality, this is due to Veliche [21, Main Theorem (4)].

Proposition 5.11. *Let R be a ring and X a homologically finite complex of R-modules. There are inequalities*

$$\text{G-dim}_R(X) \leqslant \mathrm{CI}_*\text{-dim}_R(X) \leqslant \text{CI-dim}_R(X) \leqslant \mathrm{pd}_R(X);$$

when one of these dimensions is finite it is equal to those on its left. If R is local and $\mathrm{CI}_*\text{-dim}_R(X) < \infty$, then $\mathrm{CI}_*\text{-dim}_R(X) = \mathrm{depth}(R) - \mathrm{depth}_R(X)$.

Proof. By Proposition 5.10, it suffices to consider the case when R is local. The third inequality is in Proposition 3.3.

The first inequality holds because every CI_*-resolution of X is a G-resolution. When $\mathrm{CI}_*\text{-dim}_R(X) < \infty$, let $T \simeq X$ be a CI_*-resolution. For every $n \geqslant \sup(X)$, one has $\mathrm{CI}_*\text{-dim}_R(C_n^T) < \infty$ by Proposition 5.6. The AB-formulas 2.10 and [14, (2.7)] imply the equality $\text{G-dim}_R(C_n^T) = \mathrm{CI}_*\text{-dim}_R(C_n^T)$ and it follows that C_n^T is in $\mathrm{CI}_*(R)$ if and only if it is totally reflexive. Corollary 5.5 and the corresponding equality for G-dimension [9, (2.3.7)],

$$\text{G-dim}_R(X) = \inf\{n \geqslant \sup(X) \mid C_n^T \text{ is totally reflexive}\}$$

imply that $\mathrm{CI}_*\text{-dim}_R(X) = \text{G-dim}_R(X)$. From the AB-formula 2.10 it follows that this equals $\mathrm{depth}(R) - \mathrm{depth}_R(X)$.

For the second inequality, assume that $\text{CI-dim}_R(X) < \infty$. Using the AB-formula, it suffices to show that $\mathrm{CI}_*\text{-dim}_R(X) < \infty$. Let $F \simeq X$ be a degreewise finite free resolution. By 3.7, one has $\text{CI-dim}_R(C_q^F) \leqslant 0$ for $q \gg 0$. Thus, $\mathrm{CI}_*\text{-dim}_R(C_q^F) \leqslant 0$ by Gerko [14, (2.6)], i.e., $C_q^F \in \mathrm{CI}_*(R)$, and 5.5 implies that $\mathrm{CI}_*\text{-dim}_R(X) < \infty$. $\square$

The next result is the analogue of Proposition 3.5 for CI_*-dimension. The local case for modules is given in [14, (2.5)].

Proposition 5.12. *For a ring R with $\dim(R) < \infty$ the following are equivalent*:

(a) *R is locally a complete intersection.*
(b) *Each homologically finite complex of R-modules X satisfies*

$$\mathrm{CI}_*\text{-dim}_R(X) \leqslant \dim(R) + \sup(X).$$

(c) *Each maximal ideal $\mathfrak{m} \subset R$ satisfies $\mathrm{CI}_*\text{-dim}_R(R/\mathfrak{m}) < \infty$.*

Proof. (a) $\Rightarrow$ (b). For a homologically finite complex of R-modules X, one has

$$\mathrm{CI}_*\text{-dim}_R(X) \leqslant \text{CI-dim}_R(X) \leqslant \dim(R) + \sup(X) < \infty,$$

where the first inequality is by Proposition 5.11 and the second is by Proposition 3.5.

(b) ⇒ (c) is trivial.

(c) ⇒ (a). One has $\mathrm{CI}_*\text{-}\dim_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) = \mathrm{CI}_*\text{-}\dim_R(R/\mathfrak{m})$ for each $\mathfrak{m}$ by Proposition 5.10, and so $R_\mathfrak{m}$ is a complete intersection by [14, (2.5)]. □

The complexes of finite CI_*-dimension are exactly those that behave as a whole like the modules in the CI_*-class.

Theorem 5.13. *A homologically finite complex X over a ring R has finite CI_*-dimension if and only if $\mathrm{G}\text{-}\dim_R(X)$ is finite and $\mathrm{cx}_{R_\mathfrak{m}}(X_\mathfrak{m})$ is finite for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Let $P \simeq X$ be a degreewise finite R-projective resolution.

Assume first that $p = \mathrm{CI}_*\text{-}\dim_R(X) < \infty$. Then $\mathrm{G}\text{-}\dim_R(X) < \infty$ by Proposition 5.11. The module $C_p^{P_\mathfrak{m}} \cong (C_p^P)_\mathfrak{m}$ is in $\mathrm{CI}_*(R_\mathfrak{m})$ by Proposition 5.10 and Corollary 5.5. The result now follows because $\mathrm{cx}_{R_\mathfrak{m}}(X_\mathfrak{m}) = \mathrm{cx}_{R_\mathfrak{m}}((C_p^P)_\mathfrak{m}) < \infty$.

Assume now that $g = \mathrm{G}\text{-}\dim_R(X) < \infty$ and $\mathrm{cx}_{R_\mathfrak{m}}(X_\mathfrak{m}) < \infty$ for all maximal ideals $\mathfrak{m}$ of R. The module C_g^P is totally reflexive over R by Christensen [9, (2.3.7)]. For all $\mathfrak{m}$, one has $\mathrm{cx}_{R_\mathfrak{m}}((C_g^P)_\mathfrak{m}) = \mathrm{cx}_{R_\mathfrak{m}}(X_\mathfrak{m}) < \infty$. Hence, C_g^P is in $\mathrm{CI}_*(R)$ and it follows that $\mathrm{CI}_*\text{-}\dim_R(X) < \infty$. □

A souped-up version of Corollary 3.10 (i) is satisfied by CI_*-dimension.

Proposition 5.14. *Let $\varphi : R \to S$ be a flat ring homomorphism and X a homologically finite complex of R-modules. There is an inequality*

$$\mathrm{CI}_*\text{-}\dim_S(X \otimes_R S) \leqslant \mathrm{CI}_*\text{-}\dim_R(X)$$

with equality when φ is faithfully flat.

Proof. For any $M \in \mathrm{CI}_*(R)$, it follows from flatness that $M \otimes_R S$ is in $\mathrm{CI}_*(S)$. Thus, a CI_*-resolution of X over R base-changes to a CI_*-resolution of $X \otimes_R S$ over S, and hence the inequality holds.

When φ is faithfully flat and M is a finite R-module, it follows readily that M is in $\mathrm{CI}_*(R)$ if and only if $M \otimes_R S$ is in $\mathrm{CI}_*(S)$. To show that $\mathrm{CI}_*\text{-}\dim_S(X \otimes_R S) = \mathrm{CI}_*\text{-}\dim_R(X)$, fix a CI_*-resolution $U \simeq X$ over R. Then $U \otimes_R S$ is a CI_*-resolution of $X \otimes_R S$ over S, and $C_n^{U \otimes_R S} \cong C_n^U \otimes_R S$ for each integer n. Furthermore, $\sup(X \otimes_R S) = \sup(X)$, so one has

$$\begin{aligned}\mathrm{CI}_*\text{-}\dim_S(X \otimes_R S) &= \inf\{n \geqslant \sup(X \otimes_R S) \mid C_n^U \otimes_R S \in \mathrm{CI}_*(S)\}\\ &= \inf\{n \geqslant \sup(X) \mid C_n^U \in \mathrm{CI}_*(R)\}\\ &= \mathrm{CI}_*\text{-}\dim_R(X),\end{aligned}$$

where the first and third equalities are by Corollary 5.5.

The following is a version of Corollary 3.10 (ii) for CI_*-dimension.

Proposition 5.15. *Let $Q \to R$ be a surjective ring homomorphism with kernel generated by a Q-regular sequence of length c. Every homologically finite complex of R-modules X satisfies*

$$\mathrm{CI}_*\text{-}\dim_Q(X) = \mathrm{CI}_*\text{-}\dim_R(X) + c.$$

In particular, $\mathrm{CI}_*\text{-}\dim_Q(X)$ *is finite if and only if* $\mathrm{CI}_*\text{-}\dim_R(X)$ *is finite*.

Proof. By Proposition 5.10, it suffices to consider the case where Q and R are local. By Christensen [9, (2.3.12)], 5.11, and 5.13, one needs only show that $\mathrm{cx}_R(X)$ and $\mathrm{cx}_Q(X)$ are simultaneously finite. Assume that $\mathrm{H}(X)$ is nonzero and fix a degreewise finite free resolution $P \simeq X$ and an integer $n \geqslant \sup(X)$. The complex $P_{\leqslant n-1}$ has finite projective dimension over R, and thus also over Q. The exact sequence $0 \to P_{\geqslant n} \to P \to P_{\leqslant n-1} \to 0$ implies that

$$\mathrm{cx}_Q(X) = \mathrm{cx}_Q(P) = \mathrm{cx}_Q(P_{\geqslant n}) = \mathrm{cx}_Q(C_n^P)$$

and similarly, $\mathrm{cx}_R(X) = \mathrm{cx}_R(C_n^P)$. Thus, it suffices to consider the case where X is a module. This case is in [6, (5.2.4)]. □

The final result of this section is the analogue of Theorem 3.11 for CI_*-dimension.

Theorem 5.16. *Let R be a ring and X, P homologically finite complexes of R-modules. If* $\mathrm{pd}_R(P)$ *is finite*, *then*

$$\mathrm{CI}_*\text{-}\dim_R(X \otimes_R^{\mathbf{L}} P) = \mathrm{CI}_*\text{-}\dim_R(X) + \mathrm{CI}_*\text{-}\dim_R(P)$$

and

$$\mathrm{CI}_*\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(P,X)) = \mathrm{CI}_*\text{-}\dim_R(X) - \inf(P).$$

In particular, *the* CI_**-dimensions of the complexes* X, $X \otimes_R^{\mathbf{L}} P$, *and* $\mathbf{R}\mathrm{Hom}_R(P,X)$ *are simultaneously finite*.

Proof. As in the proof of Theorem 3.11, it suffices to show that the complexes X, $X \otimes_R^{\mathbf{L}} P$, and $\mathbf{R}\mathrm{Hom}_R(P,X)$ have finite CI_*-dimensions simultaneously when R is local. By Iyengar and Sather-Wagstaff [17, (5.1,7)], the G-dimensions of the complexes X, $X \otimes_R^{\mathbf{L}} P$, and $\mathbf{R}\mathrm{Hom}_R(P,X)$ are simultaneously finite, so it suffices to show that

$$\mathrm{cx}_R(X \otimes_R^{\mathbf{L}} P) = \mathrm{cx}_R(X) = \mathrm{cx}_R(\mathbf{R}\mathrm{Hom}_R(P,X)).$$

The first equality follows from 2.13. This also implies the second equality because of the isomorphism $\mathbf{R}\mathrm{Hom}_R(P,X) \simeq X \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(P,R)$ and since $\mathbf{R}\mathrm{Hom}_R(P,R)$ is homologically finite and $\mathrm{pd}_R(\mathbf{R}\mathrm{Hom}_R(P,R))$ is finite. □

6. Global homological dimensions

We use the homological dimensions discussed in the previous sections to define global homological dimensions of rings similar to the global dimension of [8]. The primary focus is the CI-dimension. The first proposition of this section motivates our definition of the global CI-dimension of a ring R. Similar results hold for CI_*- and G-dimension.

Proposition 6.1. *For a ring R and an integer n, the following are equivalent*:

(a) *Each homologically finite complex of R-modules X satisfies*

$$\text{CI-dim}_R(X) \leqslant n + \sup(X).$$

(b) *Each finite R-module M satisfies* $\text{CI-dim}_R(M) \leqslant n$.

Proof. The implication (a) ⇒ (b) is clear. For the other implication, assume (b) holds and fix a homologically finite complex of R-modules X. Set $s = \sup(X)$, and let $P \simeq X$ be a degreewise finite projective resolution. Then $\text{CI-dim}_R(C_s^P) \leqslant n$, by assumption, and Proposition 3.7 implies that $\text{CI-dim}_R(X) - s \leqslant n$. □

Definition 6.2. For a ring R, the *global CI-dimension of R* is

$$\text{glCI-dim}(R) := \inf\{n \in \mathbb{Z} \mid \text{CI-dim}_R(M) \leqslant n \quad \forall \text{ finite } R\text{-modules } M\}.$$

The above proposition implies that this is equal to

$$\inf\{n \in \mathbb{Z} \mid \text{CI-dim}_R(X) \leqslant n + \sup(X) \quad \forall \text{ homologically finite } R\text{-complexes } X\}.$$

In a similar manner, one can define the global CI_*-dimension and global G-dimension. Each of these quantities is in $\mathbb{N} \cup \{\infty\}$.

The hierarchy of global homological dimensions follows from Proposition 5.11.

Proposition 6.3. *For a ring R, there are (in)equalities*

$$\text{glG-dim}(R) \leqslant \text{glCI}_*\text{-dim}(R) \leqslant \text{glCI-dim}(R) \leqslant \text{gl-dim}(R);$$

when one of these dimensions is finite it is equal to those on its left.

Like the CI-dimension, the global CI-dimension is determined locally.

Proposition 6.4. *For a ring R, there are (in)equalities*

$$\begin{aligned} \dim(R) &\leqslant \sup\{\text{CI-dim}_R(R/\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\} \\ &= \sup\{\text{glCI-dim}(R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\} \\ &= \text{glCI-dim}(R) \end{aligned}$$

with equality in the first spot when $\text{glCI-dim}(R) < \infty$.

Proof. Set

$$u = \sup\{\text{CI-dim}_R(R/\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\},$$

$$v = \sup\{\text{glCI-dim}(R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\},$$

$$w = \text{glCI-dim}(R).$$

To verify the inequality $\dim(R) \leqslant u$, assume that u is finite. For each maximal ideal $\mathfrak{m}$, one has $\text{CI-dim}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) = \text{CI-dim}_R(R/\mathfrak{m}) < \infty$. By Proposition 3.5, each $R_\mathfrak{m}$ is a complete intersection, and it follows that

$$\begin{aligned}\dim(R) &= \sup\{\dim(R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &= \sup\{\text{depth}(R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &= \sup\{\text{CI-dim}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &= u.\end{aligned}$$

Next, we verify the inequalities $u \leqslant v \leqslant w \leqslant u$. That $u \leqslant v$ comes from the inequality $\text{CI-dim}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) \leqslant \text{glCI-dim}(R_\mathfrak{m})$. That $v \leqslant w$ is also straightforward: every finite $R_\mathfrak{m}$-module is of the form $M_\mathfrak{m}$ for some finite R-module M and $\text{CI-dim}_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant \text{CI-dim}_R(M)$, so $\text{glCI-dim}(R_\mathfrak{m}) \leqslant \text{glCI-dim}(R)$.

For the final inequality, assume that $u < \infty$. Then R is locally a complete intersection, as above. When M is a finite R-module, one has

$$\begin{aligned}\text{CI-dim}_R(M) &= \sup\{\text{CI-dim}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &= \sup\{\text{depth}(R_\mathfrak{m}) - \text{depth}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &\leqslant \sup\{\text{depth}(R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &= \sup\{\text{CI-dim}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\}\\ &= w.\end{aligned}$$

By definition, it follows that $\text{glCI-dim}(R) \leqslant w$. □

In the same way that the regular rings are characterized as the rings of finite global dimension, the local complete intersection rings of finite Krull dimension are exactly the rings of finite global complete intersection dimension.

Theorem 6.5. *For a ring R, the following conditions are equivalent*:

(a) $\text{glCI-dim}(R) = \dim(R) < \infty$;
(b) $\text{glCI-dim}(R) < \infty$;
(c) $\text{glCI}_*\text{-dim}(R) < \infty$;
(d) *R is locally a complete intersection and* $\dim(R) < \infty$.

Proof. (a) ⇒ (b) is trivial.

(b) ⇒ (c) follows from Proposition 6.3.

(c) ⇒ (d). Since $\text{glCI}_*\text{-dim}(R_{\mathfrak{m}}) \leqslant \text{glCI}_*\text{-dim}(R) < \infty$, Proposition 6.4 implies that R is locally a complete intersection. Arguing as in Proposition 6.4 one sees that $\dim(R) \leqslant \text{glCI}_*\text{-dim}(R) < \infty$.

(d) ⇒ (a) Proposition 3.5 implies that $\text{glCI-dim}(R) \leqslant \dim(R) < \infty$. By Proposition 6.4, $\text{glCI-dim}(R) = \dim(R)$.

Corollary 6.6. *Every ring R* satisfies $\text{glCI}_*\text{-dim}(R) = \text{glCI-dim}(R)$.

Nagata [18, A1. Example 1] constructed a ring that is locally regular with infinite global dimension. This shows that the implication "locally CI ⇒ $\text{glCI-dim}(R) < \infty$" does not hold without the additional hypothesis "$\dim(R) < \infty$".

The final result of this paper is a version of Theorem 6.5 for G-dimension.

Theorem 6.7. *For a ring R, the following conditions are equivalent*:

(a) $\text{glG-dim}(R) = \dim(R) < \infty$;
(b) $\text{glG-dim}(R) < \infty$;
(c) *R is locally Gorenstein and* $\dim(R) < \infty$.
(d) $\text{id}_R(R) = \dim(R) < \infty$.
(e) $\text{id}_R(R) < \infty$.

Proof. The equivalence of (a)–(c) is verified as in Theorem 6.5. The implication (d) ⇒ (e) is trivial. For the other equivalences, recall the following fact [19, (3.5)]: If I is a minimal R-injective resolution for R and $\mathfrak{m}$ is a maximal ideal of R, the localized complex $I_{\mathfrak{m}}$ is a minimal injective resolution of $R_{\mathfrak{m}}$.

(c) ⇒ (d). Let I be a minimal injective R-resolution of R; then

$$\begin{aligned} \text{id}_R(R) &= \sup(I) \\ &= \sup\{\sup(I_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{Max}(R)\} \\ &= \sup\{\dim(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{Max}(R)\} \\ &= \dim(R). \end{aligned}$$

(e) ⇒ (c). The chain of inequalities

$$\dim(R_{\mathfrak{m}}) \leqslant \text{id}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}) \leqslant \text{id}_R(R) < \infty$$

implies that R is locally Gorenstein and $\dim(R) \leqslant \text{id}(R) < \infty$. □

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