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DETECTING COMPLETENESS FROM EXT-VANISHING

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ABSTRACT. Motivated by work of C. U. Jensen, R.-O. Buchweitz, and H. Flenner, we prove the following result. Let R be a commutative noetherian ring and $\mathfrak{a}$ an ideal in the Jacobson radical of R. Let $\widehat{\mathfrak{Ra}}$ be the $\mathfrak{a}$-adic completion of R. If M is a finitely generated R-module such that $\mathrm{Ext}^i_R(M) = 0$ for all $i \neq 0$, then M is $\mathfrak{a}$-adically complete.

INTRODUCTION

A result of Jensen [13, (8.1)] characterizes the completeness property of a semilocal ring in terms of Ext-vanishing: If R is a commutative noetherian ring, then it is a finite product of complete local rings if and only if $\mathrm{Ext}^i_R(B, M) = 0$ for $i \neq 0$ whenever B is flat and M is finitely generated over R. In their investigation of Hochschild homology, Buchweitz and Flenner [3, (2.3)] recover one implication of the local case of this result: Let R be a ring and $\mathfrak{m} \subset R$ a maximal ideal; if M is an $\mathfrak{m}$-adically complete R-module, then $\mathrm{Ext}^i_R(B, M) = 0$ for all $i \neq 0$ and each flat R-module B; see also [8, (3.7)] for the local case.

In this paper, we investigate converses to the Buchweitz-Flenner result: If M is an R-module such that $\mathrm{Ext}^i_R(B, M) = 0$ for all $i \neq 0$ and each flat R-module B, must M be $\mathfrak{m}$-adically complete? One readily sees that this need not be the case when M is not finitely generated. If R is a local domain with $\dim(R) > 0$ and M is the quotient field of R, then M is not $\mathfrak{m}$-adically complete. However, M is injective so $\mathrm{Ext}^i_R(B, M) = 0$ for all $i \neq 0$ and each R-module B.

The following result is proved in 3.1. When M finitely generated, it shows that the completeness of M can be ascertained from the vanishing of the Ext-modules against a single flat module, namely $\widehat{R}$.

Theorem A. *Let R be a commutative noetherian ring and $\mathfrak{a}$ an ideal in the Jacobson radical of R. Let $\widehat{R}^{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of R and let M be a finitely generated R-module. The following conditions are equivalent:*

(i) *M is $\mathfrak{a}$-adically complete.*
(ii) $\mathrm{Ext}^i_R(\widehat{R}^{\mathfrak{a}}, M) = 0$ *for all $i \neq 0$.*

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(iii) $\operatorname{Ext}^i_R(\widehat{R}^{\mathfrak{a}}, M) = 0$ *for all* $i = 1, \ldots, \dim_R(M)$.

As a consequence of this theorem we obtain the following two results. The first is proved in 3.3, and the second is contained in Corollary 3.9.

Theorem B. *The ring* R *is* $\mathfrak{a}$*-adically complete if and only if the completion* $\widehat{R}^{\mathfrak{a}}$ *is module-finite over* R.

Theorem C. *Let* M, N *be finitely generated* R*-modules and* t *an integer such that* $\operatorname{Ext}^i_R(N, M) = 0$ *for each* $i < t$. *If* $\operatorname{Ext}^i_R(\widehat{N}^{\mathfrak{a}}, M) = 0$ *for each* $i \neq t$, *then* $\operatorname{Ext}^i_R(N, M) = 0$ *for each* $i \neq t$ *and* $\operatorname{Ext}^t_R(N, M)$ *is* $\mathfrak{a}$*-adically complete.*

To prove these results, we employ a combination of classical module-theory and derived category techniques. Preliminary module-theoretic results are presented in Section 1. Requisite derived category notions are discussed in Section 2.

1. Analytic conductor submodules

Throughout this work, R *is a commutative noetherian ring and* $\mathfrak{a}$ *is an ideal contained in the Jacobson radical of* R.

Lemma 1.1. *If* M *is a finitely generated* R*-module, then* M *admits a unique maximal* $\mathfrak{a}$*-adically complete submodule* $\mathrm{C}^{\mathfrak{a}}_M$.

Proof. Let $\mathbf{C}^{\mathfrak{a}}(M)$ denote the collection of $\mathfrak{a}$-adically complete submodules of M which is nonempty because it contains the zero submodule. Since M is noetherian, this collection contains maximal elements, each of which is finitely generated. Let $N, N' \in \mathbf{C}^{\mathfrak{a}}(M)$ be maximal elements and suppose that $N \neq N'$. By maximality, one has $N \not\subseteq N'$ and so $N \subsetneq N + N'$. In particular, $N + N'$ is not $\mathfrak{a}$-adically complete. However, the module $N \oplus N'$ is finitely generated and $\mathfrak{a}$-adically complete. Hence, the homomorphic image $N + N'$ of $N \oplus N'$ is $\mathfrak{a}$-adically complete, a contradiction. Thus, $N = N'$, and the maximal element of $\mathbf{C}^{\mathfrak{a}}(M)$ is unique. □

The submodule $\mathrm{C}^{\mathfrak{a}}_M$ is the *analytic conductor* of M with respect to $\mathfrak{a}$. It is the largest R-submodule of M that is also an $\widehat{R}^{\mathfrak{a}}$-module. Before presenting an important property of $\mathrm{C}^{\mathfrak{a}}_M$ for this work, we introduce some frequently used maps.

1.2. Let M be an R-module. The map $g^{\mathfrak{a}}_M \colon \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \to M$ is given by $g^{\mathfrak{a}}_M(\varphi) = \varphi(1)$, and $\varepsilon^{\mathfrak{a}}_M \colon M \to \widehat{M}^{\mathfrak{a}}$ is the natural inclusion. Assume now that M is finitely generated, so that $\mathrm{C}^{\mathfrak{a}}_M$ is defined. Let $i^{\mathfrak{a}}_M \colon \mathrm{C}^{\mathfrak{a}}_M \to M$ denote the natural inclusion. The map $f^{\mathfrak{a}}_M \colon \mathrm{C}^{\mathfrak{a}}_M \to \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$ is given by $f^{\mathfrak{a}}_M(m)(r) = rm$.

The next result yields a well-defined map $k^{\mathfrak{a}}_M \colon \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \to \mathrm{C}^{\mathfrak{a}}_M$, given by $k^{\mathfrak{a}}_M(\varphi) = \varphi(1)$, such that $g^{\mathfrak{a}}_M = i^{\mathfrak{a}}_M k^{\mathfrak{a}}_M$.

Lemma 1.3. *If* M *is a finitely generated* R*-module, then the natural inclusion* $\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, i^{\mathfrak{a}}_M) \colon \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \mathrm{C}^{\mathfrak{a}}_M) \to \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$ *is bijective.*

Proof. By left-exactness of $\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, -)$ the given map is injective. To see that this map is surjective, fix $\varphi \in \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$; it suffices to show $\operatorname{Im}(\varphi) \subseteq \mathrm{C}^{\mathfrak{a}}_M$. The image $\operatorname{Im}(\varphi)$ is finitely generated over R and a homomorphic image of the $\mathfrak{a}$-adically complete R-module $\widehat{R}^{\mathfrak{a}}$. Hence, $\operatorname{Im}(\varphi)$ is $\mathfrak{a}$-adically complete, and the desired conclusion follows from Lemma 1.1. □

2. Derived local homology and cohomology

We work in the derived category $\mathsf{D}(R)$ of complexes of R-modules, indexed homologically. References on the subject include [9, 11]. A complex X is *homologically bounded to the right* if $\mathrm{H}_i(X) = 0$ for all $i \ll 0$; it is *homologically degreewise finite* if $\mathrm{H}_i(X)$ is finitely generated for each i; it is *homologically finite* if $\bigoplus_i \mathrm{H}_i(X)$ is finitely generated; and it is *homologically concentrated in degree s* if $\mathrm{H}_i(X) = 0$ for all $i \neq s$. Isomorphisms in $\mathsf{D}(R)$ are identified by the symbol $\simeq$, as are quasiisomorphisms in the category of complexes. For $X, Y \in \mathsf{D}(R)$ set $\inf(X)$ and $\sup(X)$ to be the infimum and supremum, respectively, of the set $\{n \in \mathbf{Z} \mid \mathrm{H}_n(X) \neq 0\}$. Let $X \otimes^{\mathbf{L}}_R Y$ and $\mathbf{R}\mathrm{Hom}_R(X, Y)$ denote the left-derived tensor product and right-derived homomorphism complexes, respectively.

The left-derived local homology and right-derived local cohomology functors with support in an ideal $\mathfrak{a}$ are denote $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$, respectively; see [1, 10]. These are computed as follows. If $P \xrightarrow{\simeq} X \xrightarrow{\simeq} J$ are K-projective and K-injective resolutions, respectively, as in [2, 16], then

$$\begin{aligned} \Lambda^{\mathfrak{a}}(-) &= \lim\nolimits_n (R/\mathfrak{a}^n \otimes_R -), & \Gamma_{\mathfrak{a}}(-) &= \operatorname{colim}\nolimits_n \mathrm{Hom}_R(R/\mathfrak{a}^n, -), \\ \mathbf{L}\Lambda^{\mathfrak{a}}(X) &= \Lambda^{\mathfrak{a}}(P), & \mathbf{R}\Gamma_{\mathfrak{a}}(X) &= \Gamma_{\mathfrak{a}}(J). \end{aligned}$$

Note that the functor $\Gamma_{\mathfrak{a}}(-)$ is left-exact while $\Lambda^{\mathfrak{a}}(-)$ is neither left- nor right-exact.

2.1. Here is a catalog of properties of $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ that we will utilize.

(a) There are natural transformations of functors on $\mathsf{D}(R)$ [1, (0.3)*],

$$\mathbf{R}\Gamma_{\mathfrak{a}}(-) \xrightarrow{\gamma} 1_{\mathsf{D}(R)}(-) \xrightarrow{\nu} \mathbf{L}\Lambda^{\mathfrak{a}}(-).$$

(b) The following are equivalences of functors on $\mathsf{D}(R)$ [1, Cor. to (0.3)*]:

$$\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(-)) \xrightarrow{\mathbf{L}\Lambda^{\mathfrak{a}}(\gamma)} \mathbf{L}\Lambda^{\mathfrak{a}}(-) \qquad \text{and} \qquad \mathbf{R}\Gamma_{\mathfrak{a}}(-) \xrightarrow{\mathbf{R}\Gamma_{\mathfrak{a}}(\nu)} \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(-)).$$

(c) One has natural equivalences of functors on $\mathsf{D}(R)$ ([1, (0.3)] and [14, (3.1.2)]):

$$\mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), -) \qquad \text{and} \qquad \mathbf{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R -.$$

(d) (Adjointness) There is a natural equivalence of bifunctors on $\mathsf{D}(R)$,

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(-), -) \xrightarrow[\simeq]{\theta} \mathbf{R}\mathrm{Hom}_R(-, \mathbf{L}\Lambda^{\mathfrak{a}}(-)),$$

such that, for all complexes X and Y the next diagram commutes [1, (0.3)].

$$\begin{array}{ccc} \mathbf{R}\mathrm{Hom}_R(X, Y) & & \\ {\scriptstyle \mathbf{R}\mathrm{Hom}_R(\gamma_X, Y)}\downarrow & \searrow^{\mathbf{R}\mathrm{Hom}_R(-, \nu_Y)} & \\ \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X), Y) & \xrightarrow[\simeq]{\theta_{XY}} & \mathbf{R}\mathrm{Hom}_R(X, \mathbf{L}\Lambda^{\mathfrak{a}}(Y)) \end{array}$$

In particular, the morphism $\mathbf{R}\mathrm{Hom}_R(\gamma_X, Y)$ is an isomorphism in $\mathsf{D}(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(-, \nu_Y)$ is so.

(e) One has a natural equivalence of functors on the full subcategory of $\mathsf{D}(R)$ of complexes that are homologically finite and bounded to the right [8, (2.8)],

$$\mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq - \bigotimes_R \widehat{R}^{\mathfrak{a}}.$$

(f) Parts (b)–(c) yield equivalences of (bi)functors on $\mathsf{D}(R)$; see, e.g., [4, (A.4.22)]:

$$\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(-),$$
$$\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(-,-)) \simeq \mathbf{R}\mathrm{Hom}_R(-,\mathbf{L}\Lambda^{\mathfrak{a}}(-)).$$

(g) If X is homologically bounded to the right, then it admits a K-projective resolution $P \xrightarrow{\simeq} X$ such that $X_i = 0$ for each $i \leqslant \inf(X)$, and so

$$\inf(\mathbf{L}\Lambda^{\mathfrak{a}}(X)) = \inf(\Lambda^{\mathfrak{a}}(P)) \geqslant \inf(P) = \inf(X).$$

We now verify facts about $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ for the sequel. Fix $M \in \mathsf{D}(R)$ with K-injective resolution $M \xrightarrow{\simeq} J$. The map $g^{\mathfrak{a}}_J\colon \mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, J) \to J$ given by $\varphi \mapsto \varphi(1)$ describes a well-defined morphism $h^{\mathfrak{a}}_M\colon \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \to M$ in $\mathsf{D}(R)$.

Lemma 2.2. *If M is an R-complex, then the induced morphisms*

$$\mathbf{L}\Lambda^{\mathfrak{a}}(h^{\mathfrak{a}}_M)\colon \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) \to \mathbf{L}\Lambda^{\mathfrak{a}}(M),$$
$$\mathbf{R}\Gamma_{\mathfrak{a}}(h^{\mathfrak{a}}_M)\colon \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) \to \mathbf{R}\Gamma_{\mathfrak{a}}(M)$$

are isomorphisms in $\mathsf{D}(R)$. In particular, if $\mathbf{L}\Lambda^{\mathfrak{a}}(M) \not\simeq 0$ or $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \not\simeq 0$, then $\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \not\simeq 0$.

Proof. For the first isomorphism, it suffices to check that the morphism

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) \xrightarrow{\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), h^{\mathfrak{a}}_M)} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), M)$$

is an isomorphism in $\mathsf{D}(R)$; see 2.1(c). In the commutative diagram

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) & \xrightarrow[\simeq]{(1)} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}, M) \\
{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), h^{\mathfrak{a}}_M)} \downarrow & & \downarrow {\scriptstyle \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R \varepsilon^{\mathfrak{a}}_R, M)} \\
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), M) & \xleftarrow{\simeq} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R R, M),
\end{array}$$

(1) is adjunction and $\varepsilon^{\mathfrak{a}}_R\colon R \to \widehat{R}^{\mathfrak{a}}$ is the natural inclusion. Since $\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes_R \varepsilon^{\mathfrak{a}}_R$ is an isomorphism by 2.1(f), the same is true of $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R \varepsilon^{\mathfrak{a}}_R, M)$. The diagram implies that $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), h^{\mathfrak{a}}_M)$ is an isomorphism.

For the second isomorphism, use the equivalence of 2.1(b) to see that the vertical maps in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc}
\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) & \xrightarrow{\mathbf{R}\Gamma_{\mathfrak{a}}(h^{\mathfrak{a}}_M)} & \mathbf{R}\Gamma_{\mathfrak{a}}(M) \\
{\scriptstyle \mathbf{R}\Gamma_{\mathfrak{a}}(\nu_{\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)})} \downarrow \simeq & & \simeq \downarrow {\scriptstyle \mathbf{R}\Gamma_{\mathfrak{a}}(\nu_M)} \\
\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M))) & \xrightarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(h^{\mathfrak{a}}_M))} & \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(M)).
\end{array}$$

The morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(h^{\mathfrak{a}}_M))$ is an isomorphism in $\mathsf{D}(R)$ because we have shown that $\mathbf{L}\Lambda^{\mathfrak{a}}(h^{\mathfrak{a}}_M)$ is so. The diagram shows that $\mathbf{R}\Gamma_{\mathfrak{a}}(h^{\mathfrak{a}}_M)$ is an isomorphism as well.

The final statement follows from the additivity of $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. $\square$

Lemma 2.3. *If M, N are homologically finite R-complexes, then the complex $X = \mathbf{R}\mathrm{Hom}_R(N, M)$ is homologically degreewise finite and $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq X \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}$. In particular, one has $\inf(\mathbf{L}\Lambda^{\mathfrak{a}}(X)) = \inf(X)$ and $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(X)) = \sup(X)$.*

Proof. The finiteness of each $\mathrm{H}_i(X)$ is standard. A verification of the isomorphism is essentially in [6, Proof of (5.9)]. The flatness of $R \to \widehat{R}^{\mathfrak{a}}$ implies $\mathrm{H}_i(\mathbf{L}\Lambda^{\mathfrak{a}}(X)) \cong \mathrm{H}_i(X) \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}$, and the equalities follow from the faithful flatness of $R \to \widehat{R}^{\mathfrak{a}}$. □

We next prove a vanishing result akin to [3, (2.3)]. Note that M is not assumed to be finitely generated.

Proposition 2.4. *Let M be an R-module such that the morphism $\nu_M \colon M \to \mathbf{L}\Lambda^{\mathfrak{a}}(M)$ is an isomorphism in $\mathsf{D}(R)$. Then $\mathrm{Ext}^i_R(\widehat{R}^{\mathfrak{a}}, M) = 0$ for each $i \neq 0$, and the evaluation map $g^{\mathfrak{a}}_M \colon \mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \to M$ is an isomorphism.*

Proof. Because the morphism $\nu_M \colon M \to \mathbf{L}\Lambda^{\mathfrak{a}}(M)$ is an isomorphism in $\mathsf{D}(R)$, the same is true of $\mathbf{R}\mathrm{Hom}_R(X, \nu_M) \colon \mathbf{R}\mathrm{Hom}_R(X, M) \to \mathbf{R}\mathrm{Hom}_R(X, \mathbf{L}\Lambda^{\mathfrak{a}}(M))$ for each R-complex X. From 2.1(d) it follows that the morphism

$$\mathbf{R}\mathrm{Hom}_R(\gamma_X, M) \colon \mathbf{R}\mathrm{Hom}_R(X, M) \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X), M)$$

is an isomorphism in $\mathsf{D}(R)$.

The naturality of γ provides the following commutative diagram in $\mathsf{D}(R)$.

$$\begin{array}{ccc}
\mathbf{R}\Gamma_{\mathfrak{a}}(R) & \xrightarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}}(\nu_R)} & \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(R)) \\
{\scriptstyle \gamma_R}\downarrow & & \downarrow{\scriptstyle \gamma_{\mathbf{L}\Lambda^{\mathfrak{a}}(R)}} \\
R & \xrightarrow{\nu_R} & \mathbf{L}\Lambda^{\mathfrak{a}}(R)
\end{array}$$

An application of $\mathbf{R}\mathrm{Hom}_R(-, M)$ yields the following commutative diagram in $\mathsf{D}(R)$:

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(R), M) & \xrightarrow{\mathbf{R}\mathrm{Hom}_R(\nu_R, M)} & \mathbf{R}\mathrm{Hom}_R(R, M) \\
{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\gamma_{\mathbf{L}\Lambda^{\mathfrak{a}}(R)}, M)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\gamma_R, M)} \\
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(R)), M) & \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(\nu_R), M)} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), M),
\end{array}$$

where the vertical morphisms are isomorphisms because of the argument of the previous paragraph. Hence, the morphism $\mathbf{R}\mathrm{Hom}_R(\nu_R, M)$ is also an isomorphism.

Next consider the commutative triangle

$$\begin{array}{ccc}
R & & \\
{\scriptstyle \nu_R}\downarrow & \searrow{\scriptstyle \varepsilon^{\mathfrak{a}}_R} & \\
\mathbf{L}\Lambda^{\mathfrak{a}}(R) & \xrightarrow[\simeq]{\kappa} & \widehat{R}^{\mathfrak{a}},
\end{array}$$

where κ is obtained by taking degree 0 homology; see, e.g., 2.1(e). Apply $\mathbf{R}\mathrm{Hom}_R(-, M)$ to produce the following commutative diagram in $\mathsf{D}(R)$:

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) & & \\
{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\kappa, M)}\downarrow{\scriptstyle \simeq} & \searrow{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\varepsilon^{\mathfrak{a}}_R, M)} & \\
\mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(R), M) & \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}_R(\nu_R, M)} & \mathbf{R}\mathrm{Hom}_R(R, M),
\end{array}$$

which implies that $\mathbf{R}\mathrm{Hom}_R(\varepsilon^{\mathfrak{a}}_R, M)$ is an isomorphism in $\mathsf{D}(R)$.

In the final commutative diagram,

$$\begin{array}{ccc} \mathbf{R}\mathrm{Hom}_R(R,M) & & \\ {\scriptstyle \mathbf{R}\mathrm{Hom}_R(\varepsilon_R^{\mathfrak{a}},M)}\downarrow\simeq & \overset{\xi}{\underset{\simeq}{\searrow}} & \\ \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},M) & \xrightarrow[h_M^{\mathfrak{a}}]{} & M, \end{array}$$

the morphism ξ is the natural evaluation isomorphism. The diagram shows that $h_M^{\mathfrak{a}}$ is an isomorphism in $\mathsf{D}(R)$. Since M is a module, this implies $\mathrm{Ext}_R^i(\widehat{R}^{\mathfrak{a}},M)=0$ for each $i\neq 0$ and further that the induced map $\mathrm{H}_0(h_M^{\mathfrak{a}})\colon \mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},M)\to M$ is bijective. The definitions yield an equality $\mathrm{H}_0(h_M^{\mathfrak{a}})=g_M^{\mathfrak{a}}$, completing the proof. □

Remark 2.5. If M is an R-module such that $M\cong\widehat{M}^{\mathfrak{a}}$, then $M\simeq\mathbf{L}\Lambda^{\mathfrak{a}}(M)$. Indeed, the isomorphism $M\cong\widehat{M}^{\mathfrak{a}}$ shows that M is an $\widehat{R}^{\mathfrak{a}}$-module. Let P be an $\widehat{R}^{\mathfrak{a}}$-free resolution of M. Then P is an R-flat resolution of M consisting of $\mathfrak{a}$-adically complete modules. Thus, one has $\mathbf{L}\Lambda^{\mathfrak{a}}(M)\simeq\Lambda^{\mathfrak{a}}(P)\cong P\simeq M$.

We are now in a position to give a useful alternate description of the analytic conductor submodule $\mathrm{C}_M^{\mathfrak{a}}$; see 1.2 for the definitions of the maps.

Proposition 2.6. *Let M be a finitely generated R-module. The homomorphisms $f_M^{\mathfrak{a}}\colon \mathrm{C}_M^{\mathfrak{a}}\to\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},M)$ and $k_M^{\mathfrak{a}}\colon\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},M)\to\mathrm{C}_M^{\mathfrak{a}}$ are inverse isomorphisms. In particular, $\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},M)$ is finitely generated over R.*

Proof. One checks from the definitions that the composition $k_M^{\mathfrak{a}}f_M^{\mathfrak{a}}$ is the identity on $\mathrm{C}_M^{\mathfrak{a}}$. Hence, the first conclusion will be verified once we show that $k_M^{\mathfrak{a}}$ is bijective; the second conclusion will then follow, as $\mathrm{C}_M^{\mathfrak{a}}$ is finitely generated over R.

The module $\mathrm{C}_M^{\mathfrak{a}}$ is $\mathfrak{a}$-adically complete, so Proposition 2.4 implies that the evaluation map $g_{\mathrm{C}_M^{\mathfrak{a}}}^{\mathfrak{a}}\colon\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},\mathrm{C}_M^{\mathfrak{a}})\to\mathrm{C}_M^{\mathfrak{a}}$ is bijective. By Lemma 1.3 the map $\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},i_M^{\mathfrak{a}})\colon\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},\mathrm{C}_M^{\mathfrak{a}})\to\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},M)$ is an isomorphism. In particular, the composition $k_M^{\mathfrak{a}}=g_{\mathrm{C}_M^{\mathfrak{a}}}^{\mathfrak{a}}\circ\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},i_M^{\mathfrak{a}})^{-1}$ is bijective, as desired. □

3. Detecting completeness

3.1. *Proof of Theorem* A. The implication (i) $\Longrightarrow$ (ii) follows from Proposition 2.4 and Remark 2.5, and (ii) $\Longrightarrow$ (iii) is trivial.

For the implication (iii) $\Longrightarrow$ (i), set $S=R/\mathrm{Ann}_R(M)$. A result of Gruson and Raynaud [15, Seconde Partie, Thm. (3.2.6)], and Jensen [12, Prop. 6] provides the following bound on the projective dimension of $\widehat{S}^{\mathfrak{a}}$ as an S-module:

$$(*)\qquad \mathrm{pd}_S(\widehat{S}^{\mathfrak{a}})\leqslant\dim(S)=\dim_R(M).$$

Consider the following sequence of isomorphisms in $\mathsf{D}(R)$:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},M)&\simeq\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}},\mathbf{R}\mathrm{Hom}_S(S,M))\\&\simeq\mathbf{R}\mathrm{Hom}_S(\widehat{R}^{\mathfrak{a}}\otimes_R^{\mathbf{L}}S,M)\\&\simeq\mathbf{R}\mathrm{Hom}_S(\widehat{S}^{\mathfrak{a}},M).\end{aligned}$$

The first isomorphism follows from the fact that M is naturally an S-module. The second is adjunction, and the third is standard as S is finitely generated over R. Combining $(*)$ with the displayed isomorphisms, the assumption $\mathrm{Ext}_R^i(\widehat{R}^{\mathfrak{a}},M)=0$ for all $i=1,\dots,\dim_R(M)$ implies $\mathrm{Ext}_R^i(\widehat{R}^{\mathfrak{a}},M)=0$ for all $i\neq 0$.

It follows that the natural map $\lambda\colon \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \to \mathbf{R}\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$ is an isomorphism in $\mathsf{D}(R)$. Proposition 2.6 implies that the composition $\lambda \circ f_M^{\mathfrak{a}}\colon \mathrm{C}_M^{\mathfrak{a}} \to \mathbf{R}\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$ is also an isomorphism in $\mathsf{D}(R)$. Because M is finitely generated, the natural morphism $\mu\colon \mathbf{L}\Lambda^{\mathfrak{a}}(M) \to \widehat{M}^{\mathfrak{a}}$ is also an isomorphism in $\mathsf{D}(R)$. These data yield the following commutative diagram.

$$\begin{array}{ccccc}
\mathrm{C}_M^{\mathfrak{a}} & \xrightarrow[\simeq]{\nu_{\mathrm{C}_M^{\mathfrak{a}}}} & \mathbf{L}\Lambda^{\mathfrak{a}}(\mathrm{C}_M^{\mathfrak{a}}) & \xrightarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}}(\lambda\circ f_M^{\mathfrak{a}})} & \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) \\
{\scriptstyle i_M^{\mathfrak{a}}}\big\downarrow & & & & \simeq\big\downarrow{\scriptstyle \mathbf{L}\Lambda^{\mathfrak{a}}(h_M^{\mathfrak{a}})} \\
M & \xrightarrow{\varepsilon_M^{\mathfrak{a}}} & \widehat{M}^{\mathfrak{a}} & \xleftarrow[\simeq]{\mu} & \mathbf{L}\Lambda^{\mathfrak{a}}(M)
\end{array}$$

One sees that the composition of natural maps $\mathrm{C}_M^{\mathfrak{a}} \xrightarrow{i_M^{\mathfrak{a}}} M \xrightarrow{\varepsilon_M^{\mathfrak{a}}} \widehat{M}^{\mathfrak{a}}$ is bijective. Since $\varepsilon_M^{\mathfrak{a}}$ is also injective, the result now follows. $\square$

Remark 3.2. As the referee indicated, one can interpret Theorem A as a statement about the $\mathfrak{a}$-adic completeness of $R/\operatorname{Ann}_R(M)$ because M is $\mathfrak{a}$-adically complete if and only if $R/\operatorname{Ann}_R(M)$ is $\mathfrak{a}$-adically complete. For the sake of completeness, we include a sketch of the proof.

For one implication, assume that M is $\mathfrak{a}$-adically complete. For each prime $\mathfrak{p} \in \operatorname{Ass}_R(M)$, the injection $R/\mathfrak{p} \hookrightarrow M$ and the completeness of M imply that $R/\mathfrak{p}$ is $\mathfrak{a}$-adically complete. In particular, this is true for each minimal prime $\mathfrak{p}$ containing $\operatorname{Ann}_R(M)$, and it follows that the same is true for each nonminimal prime $\mathfrak{p}$ containing $\operatorname{Ann}_R(M)$. A prime filtration argument applied to $R/\operatorname{Ann}_R(M)$ shows that $R/\operatorname{Ann}_R(M)$ is $\mathfrak{a}$-adically complete.

Conversely, if $R/\operatorname{Ann}_R(M)$ is $\mathfrak{a}$-adically complete, then there exists an integer r and a surjection $(R/\operatorname{Ann}_R(M))^r \twoheadrightarrow M$, and it follows that M is $\mathfrak{a}$-adically complete.

From this fact, one easily deduces the following: When N is a second finitely generated R-module, if M is $\mathfrak{a}$-adically complete and $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$, then N is $\mathfrak{a}$-adically complete.

3.3. *Proof of Theorem* B. One implication is trivial. For the other, assume that $\widehat{R}^{\mathfrak{a}}$ is module-finite over R. As $\widehat{R}^{\mathfrak{a}}$ is flat and module-finite over R, it is projective, and so $\operatorname{Ext}_R^i(\widehat{R}^{\mathfrak{a}}, R) = 0$ for each $i \neq 0$. The completeness of R follows from Theorem A. $\square$

The next example shows that the nontrivial implication in Corollary B fails if $\mathfrak{a}$ is not assumed to be in the Jacobson radical of R.

Example 3.4. Let k be a field and set $R = k \times k$ and $\mathfrak{b} = k \times 0$. The Jacobson radical of R is 0. One readily checks that $\widehat{R}^{\mathfrak{b}} \cong 0 \times k$, showing that R is not $\mathfrak{b}$-adically complete even though $\widehat{R}^{\mathfrak{b}}$ is module-finite over R.

Theorem A provides the converse to [3, (2.3)] when R is local and M is finitely generated. This is the implication (iii) $\Longrightarrow$ (i) in the next result. The implication (i) $\Longrightarrow$ (ii) is in [8, (3.7)] or [3, (2.3)], while the implication (ii) $\Longrightarrow$ (iii) is trivial.

Corollary 3.5. *Let $(R, \mathfrak{m})$ be a local ring. For a finitely generated R-module M the following conditions are equivalent:*

(i) *M is $\mathfrak{m}$-adically complete.*

(ii) *For each flat R-module B and each $i \neq 0$, one has* $\operatorname{Ext}_R^i(B, M) = 0$.

(iii) *For each $i \neq 0$, one has* $\operatorname{Ext}^i_R(\widehat{R}^{\mathfrak{m}}, M) = 0$. □

With Theorem A and Corollary 3.5 in mind, one may ask what the finitely generated complete R-modules look like, say, when R is not complete. Examples include the modules of finite length. We observe next that one can have complete R-modules of infinite length.

Example 3.6. Let $(S, \mathfrak{n})$ be a non-Artinian complete local ring. Set $R = S[X]_{(\mathfrak{n},X)}$ with maximal ideal $\mathfrak{m} = (\mathfrak{n}, X)R$. The ring R is not $\mathfrak{m}$-adically complete, while the module $R/(X)R \cong S$ is $\mathfrak{m}$-adically complete and has infinite length.

A finitely generated R-module C is semidualizing if $R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(C, C)$.

Corollary 3.7. *If C is a semidualizing R-module such that* $\operatorname{Ext}^i_R(\widehat{R}^{\mathfrak{a}}, C) = 0$ *for all $i \neq 0$, then R is $\mathfrak{a}$-adically complete.*

Proof. Theorem A implies that C is $\mathfrak{a}$-adically complete and hence $C \simeq C \otimes_R \widehat{R}^{\mathfrak{a}} \simeq C \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}$. By [5, (5.8)] the complex $C \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}$ is $\widehat{R}^{\mathfrak{a}}$-semidualizing. This provides (1) in the following sequence while (4) and (5) are by hypothesis:

$$\begin{aligned}
\widehat{R}^{\mathfrak{a}} &\overset{(1)}{\simeq} \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(C \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}, C \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}) \\
&\overset{(2)}{\simeq} \mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}, C \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}) \\
&\overset{(3)}{\simeq} \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}) \\
&\overset{(4)}{\simeq} \mathbf{R}\mathrm{Hom}_R(C, C) \\
&\overset{(5)}{\simeq} R.
\end{aligned}$$

(2) is adjunction [5, (1.5.2)] and (3) is standard [5, (1.5.5)]. □

Here is a version of Theorem A for complexes.

Proposition 3.8. *Let M be a homologically degreewise finite R-complex such that* $\inf(\mathbf{L}\Lambda^{\mathfrak{a}}(M)) = \inf(M)$ *and* $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(M)) = \sup(M)$, *e.g., if M is homologically finite. Fix an integer $s \geqslant \sup(M)$. If $\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$ is homologically concentrated in degree s, then so is M, and the module $\mathrm{H}_s(M)$ is $\mathfrak{a}$-adically complete.*

Proof. Assume $M \not\simeq 0$. Then $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(M)) = \sup(M) > -\infty$, and Lemma 2.2 implies $\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \not\simeq 0$. Our hypotheses provide (1) and (3) in the sequence

$$s \overset{(1)}{\geqslant} \sup(M) \overset{(2)}{\geqslant} \sup(\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) \overset{(3)}{=} s$$

and (2) is from [7, (2.1)]; this implies $s = \sup(M)$. Since $\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$ is homologically concentrated in degree s, one has $\Sigma^s \operatorname{Ext}^{-s}_R(\widehat{R}^{\mathfrak{a}}, M) \simeq \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$, providing the first of the following isomorphisms:

$$\mathbf{L}\Lambda^{\mathfrak{a}}(\Sigma^s \operatorname{Ext}^{-s}_R(\widehat{R}^{\mathfrak{a}}, M)) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(M),$$

while the second one is from Lemma 2.2. This provides (5) in the following sequence:

$$\inf(M) \overset{(4)}{=} \inf(\mathbf{L}\Lambda^{\mathfrak{a}}(M)) \overset{(5)}{=} \inf(\mathbf{L}\Lambda^{\mathfrak{a}}(\Sigma^s \operatorname{Ext}^{-s}_R(\widehat{R}^{\mathfrak{a}}, M))) \overset{(6)}{\geqslant} s \overset{(7)}{=} \sup(M) \overset{(8)}{\geqslant} \inf(M)$$

while (4) is by assumption, (6) is by 2.1(g), (7) is proved above, and (8) is trivial. It follows that $\inf(M) = \sup(M) = s$ and so M is homologically concentrated in degree s. Finally, one has $M \simeq \Sigma^s \mathrm{H}_s(M)$ and so

$$\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \simeq \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, \Sigma^s \mathrm{H}_s(M)) \simeq \Sigma^s \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, \mathrm{H}_s(M)).$$

Since this is homologically concentrated in degree s, one has $\mathrm{Ext}_i^R(\widehat{R}^{\mathfrak{a}}, \mathrm{H}_s(M)) = 0$ for each $i \neq 0$. Theorem A implies that $\mathrm{H}_s(M)$ is $\mathfrak{a}$-adically complete. □

The next result contains Theorem C from the Introduction.

Corollary 3.9. *Let M, N be homologically finite R-complexes and $s \in \mathbb{Z}$ such that $s \geqslant \sup(\mathbf{R}\mathrm{Hom}_R(N, M))$. If $\mathbf{R}\mathrm{Hom}_R(\widehat{N}^{\mathfrak{a}}, M)$ is homologically concentrated in degree s, then so is $\mathbf{R}\mathrm{Hom}_R(N, M)$, and $\mathrm{Ext}_R^{-s}(N, M)$ is $\mathfrak{a}$-adically complete.*

Proof. 2.1(e) and adjunction provide the following sequence:

$$\mathbf{R}\mathrm{Hom}_R(\widehat{N}^{\mathfrak{a}}, M) \simeq \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}} \otimes_R^{\mathbf{L}} N, M) \simeq \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, \mathbf{R}\mathrm{Hom}_R(N, M)).$$

Lemma 2.3 shows that Proposition 3.8 applies to the complex $\mathbf{R}\mathrm{Hom}_R(N, M)$, yielding the desired conclusion. □

Corollary 3.10. *Assume that R is local and M, N are nonzero finitely generated R-modules with $\mathrm{pd}_R(N) < \infty$. If $\mathrm{Ext}_R^i(\widehat{N}^{\mathfrak{a}}, M) = 0$ for each $i \neq 0$, then N is free and M is $\mathfrak{a}$-adically complete.*

Proof. Using $s = 0$ in Corollary 3.9, one concludes that $\mathrm{Ext}_R^i(N, M) = 0$ for each $i \neq 0$ and that $\mathrm{Hom}_R(N, M)$ is $\mathfrak{a}$-adically complete. Since $\mathrm{pd}_R(N)$ is finite, one has

$$\mathbf{R}\mathrm{Hom}_R(N, M) \simeq \mathbf{R}\mathrm{Hom}_R(N, R) \otimes_R^{\mathbf{L}} M$$

by tensor-evaluation [2, (4.4)]. The next equalities are from [7, (2.1)] and [5, (2.13)]:

$$0 = \inf(\mathbf{R}\mathrm{Hom}_R(N, M)) = \inf(\mathbf{R}\mathrm{Hom}_R(N, R)) + \inf(M) = -\,\mathrm{pd}_R(N).$$

Since R is local, the module $N \neq 0$ is free and $\mathrm{Hom}_R(N, M) \cong M^n$ for some $n > 0$. Because M^n is $\mathfrak{a}$-adically complete, the same is true of M. □

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