On the structure of $S_2$-ifications of complete local rings

Sean Sather-Wagstaff\textsuperscript{1}  Sandra Spiroff\textsuperscript{2}

\textsuperscript{1} North Dakota State University
\textsuperscript{2} University of Mississippi

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Assumption

\((R, m, k)\) is a commutative noetherian local ring that is complete, equidimensional, and unmixed with canonical module \(\omega\) and total ring of fractions \(Q(R)\).

Definition (Hochster and Huneke, ’94)

An \(R\)-subalgebra \(T \subseteq Q(R)\) is an \(S_2\)-ification of \(R\) if:

1. \(T\) is module finite and \((S_2)\) over \(R\); and
2. the inclusion \(R \rightarrow T\) is an isomorphism in codimension 2.

Fact (HH)

(a) \(R\) has a unique \(S_2\)-ification \(T\).
(b) If \(R\) is \((R_1)\), then \(T\) is the integral closure of \(R\) in \(Q(R)\).
(c) In general, one has \(T \cong \text{Hom}_R(\omega, \omega)\).
### Local $S_2$-ifications

**Definition (HH)**

$\Gamma_R$ is the graph with vertex set $\text{Min}(R)$ such that distinct vertices $p$ and $q$ are adjacent in $\Gamma_R$ if and only if $\text{ht}_R(p + q) = 1$.

**Fact (HH)**

*The following conditions are equivalent:*

1. $T$ is local;
2. $\omega$ is indecomposable;
3. $H^{\dim(R)}_m(R)$ is indecomposable;
4. For every ideal $J$ of height at least two, $\text{Spec}(R) - V(J)$ is connected; and
5. $\Gamma_R$ is connected.

**Question**

Can one similarly obtain more information about $m$-$\text{Spec}(T)$?
Theorem (SW-Spiroff)

The following quantities are equal:

(i) \( |m\text{-Spec}(T)| \);

(ii) the number of summands in an indecomposable decomposition of \( \omega \);

(iii) the number of summands in an indecomposable decomposition of \( H_{m}^{\dim(R)}(R) \);

(iv) the maximum number of components of \( \text{Spec}(R) \setminus V(J) \) where \( J \) ranges through the ideals of height at least 2; and

(v) the number of connected components of \( \Gamma_{R} \).

Remark (Lyubeznik ’06, Zhang ’07)

If \( R \) is the completion of the strict hensilization of an equicharacteristic local ring \( A \), then the above quantity is also the top “Lyubeznik number” \( \lambda_{d,d}(A) \).
Examples of $\Gamma_R$

**Definition**

$\Gamma_R$ is the graph with vertex set $\text{Min}(R)$ such that distinct vertices $p$ and $q$ are adjacent in $\Gamma_R$ if and only if $\text{ht}_R(p + q) = 1$.

**Example (complete graphs)**

If $R$ is a hypersurface or $\dim(R) \leq 1$, then $\Gamma_R = K_{|\text{Min}(R)|}$.

**Example (2-vertex graphs)**

$$k[[X_1, X_2]]/(X_1 X_2) \quad (X_1) \xrightarrow{} (X_2)$$

$$k[[X_1, X_2, X_3, X_4]]/(X_1 X_2, X_2 X_3, X_3 X_4, X_1 X_4) \quad (X_1, X_3) \xrightarrow{} (X_2, X_4)$$
More Examples of $\Gamma_R$

### Example (paths)

Let $n \geq 1$ and $R = k[[X_1, \ldots, X_n]]/J$ where

$$J = (X_1, X_2) \cap (X_2, X_3) \cap \cdots \cap (X_{n-1}, X_n).$$

Then $\Gamma_R \cong P_n$.

$$\begin{align*}
(X_1, X_2) & \quad (X_2, X_3) \quad \cdots \quad (X_{n-1}, X_n)
\end{align*}$$

### Example (cycles)

Let $n \geq 3$ and $R = k[[X_1, \ldots, X_n]]/J$ where

$$J = (X_1, X_2) \cap (X_2, X_3) \cap \cdots \cap (X_{n-1}, X_n) \cap (X_n, X_1).$$

Then $\Gamma_R \cong C_n$. 
**Question**
How to decide whether a graph $G$ is of the form $\Gamma_R$?

**Definition (address labeling of $G$, intuitive version)**
Each vertex of $G$ is assigned a distinct “address” of $s$ distinct numbers from $[n] = \{1, \ldots, n\}$, so that two vertices are adjacent if and only if their addresses differ by exactly one number.

**Example (paths)**

$\{1, 2\} \rightarrow \{2, 3\} \rightarrow \cdots \rightarrow \{n-1, n\}$
Theorem (SW-Spiroff)

If $G$ admits an address labeling, then there is a complete local equidimensional unmixed ring $R$ such that $\Gamma_R \cong G$. Moreover, the ring $R$ is of the form $k[\![X_1, \ldots, X_n]\!] / I$ where $I$ is a square-free monomial ideal.

Example (paths)

\[
\{1, 2\} \longrightarrow \{2, 3\} \longrightarrow \cdots \longrightarrow \{n-1, n\}
\]

\[
(X_1, X_2) \longrightarrow (X_2, X_3) \longrightarrow \cdots \longrightarrow (X_{n-1}, X_n)
\]

$R = k[\![X_1, \ldots, X_n]\!] / J$ where

\[
J = (X_1, X_2) \cap (X_2, X_3) \cap \cdots \cap (X_{n-1}, X_n).
\]
Example

The following graphs do not have address labelings.

![Graph 1](image1)
![Graph 2](image2)

Thus, one cannot realize these graphs as $\Gamma_R$ for any unmixed equidimensional monomial ideal. Note that the first graph is chordal and the second one is complete bipartite.

Question

Can these graphs be realized as $\Gamma_R$?
Graph Labelings, cont.

Notation

Let $G$ be a graph with vertex set $V$. Fix positive integers $n$ and $s$, and let $\binom{n}{s}$ denote the set of subsets of $[n]$ with cardinality $s$.

Definition

An address labeling of $G$ is an injective function $\phi: V \rightarrow \binom{n}{s}$, for some choice of $n$ and $s$ such that for all $v, w \in V$, we have $v$ adjacent to $w$ in $G$ if and only if $|\phi(v) \cap \phi(w)| = s - 1$, that is, if and only if $|\phi(v) \cup \phi(w)| = s + 1$. 