On the structure of S_2-ifications of complete local rings

Sean Sather-Wagstaff[1] Sandra Spiroff[2]

[1]North Dakota State University

[2]University of Mississippi

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S_2-ifications

Assumption

$(R, \mathfrak{m}, k)$ is a commutative noetherian local ring that is complete, equidimensional, and unmixed with canonical module ω and total ring of fractions $Q(R)$.

Definition (Hochster and Huneke, '94)

An R-subalgebra $T \subseteq Q(R)$ is an S_2-ification of R if:

(1) T is module finite and (S_2) over R; and

(2) the inclusion $R \to T$ is an isomorphism in codimension 2.

Fact (HH)

(a) *R has a unique S_2-ification T.*

(b) *If R is (R_1), then T is the integral closure of R in $Q(R)$.*

(c) *In general, one has $T \cong \mathrm{Hom}_R(\omega, \omega)$.*

Local S_2-ifications

Definition (HH)

Γ_R is the graph with vertex set $\mathrm{Min}(R)$ such that distinct vertices $\mathfrak{p}$ and $\mathfrak{q}$ are adjacent in Γ_R if and only if $\mathrm{ht}_R(\mathfrak{p}+\mathfrak{q}) = 1$.

Fact (HH)

The following conditions are equivalent:

(i) *T is local;*

(ii) *ω is indecomposable;*

(iii) $\mathrm{H}_{\mathfrak{m}}^{\dim(R)}(R)$ *is indecomposable;*

(iv) *For every ideal J of height at least two,* $\mathrm{Spec}(R) - V(J)$ *is connected; and*

(v) Γ_R *is connected.*

Question

Can one similarly obtain more information about $\text{m-Spec}(T)$?

Maximal Ideals of S_2-ifications

Theorem (SW-Spiroff)

The following quantities are equal:

(i) $|m\text{-}\operatorname{Spec}(T)|$;

(ii) *the number of summands in an indecomposable decomposition of* ω;

(iii) *the number of summands in an indecomposable decomposition of* $\mathrm{H}_{\mathfrak{m}}^{\dim(R)}(R)$;

(iv) *the maximum number of components of* $\operatorname{Spec}(R) - V(J)$ *where* J *ranges through the ideals of height at least 2; and*

(v) *the number of connected components of* Γ_R.

Remark (Lyubeznik '06, Zhang '07)

If R is the completion of the strict hensilization of an equicharacteristic local ring A, then the above quantity is also the top "Lyubeznik number" $\lambda_{d,d}(A)$.

Examples of Γ_R

Definition

Γ_R is the graph with vertex set $\mathrm{Min}(R)$ such that distinct vertices $\mathfrak{p}$ and $\mathfrak{q}$ are adjacent in Γ_R if and only if $\mathrm{ht}_R(\mathfrak{p}+\mathfrak{q}) = 1$.

Example (complete graphs)

If R is a hypersurface or $\dim(R) \leq 1$, then $\Gamma_R = K_{|\,\mathrm{Min}(R)|}$.

Example (2-vertex graphs)

$$k[\![X_1, X_2]\!]/(X_1X_2) \qquad\qquad (X_1) \text{ ——— } (X_2)$$

$$k[\![X_1, X_2, X_3, X_4]\!]/(X_1X_2, X_2X_3, X_3X_4, X_1X_4)$$

$$(X_1, X_3) \qquad\qquad (X_2, X_4)$$

More Examples of Γ_R

Example (paths)

Let $n \geq 1$ and $R = \mathsf{k}[\![X_1, \ldots, X_n]\!]/J$ where

$$J = (X_1, X_2) \bigcap (X_2, X_3) \bigcap \cdots \bigcap (X_{n-1}, X_n).$$

Then $\Gamma_R \cong P_n$.

$$(X_1, X_2) \text{ ---- } (X_2, X_3) \text{ ---- } \cdots \text{ ---- } (X_{n-1}, X_n)$$

Example (cycles)

Let $n \geq 3$ and $R = \mathsf{k}[\![X_1, \ldots, X_n]\!]/J$ where

$$J = (X_1, X_2) \bigcap (X_2, X_3) \bigcap \cdots \bigcap (X_{n-1}, X_n) \bigcap (X_n, X_1).$$

Then $\Gamma_R \cong C_n$.

Graph Labelings

Question

How to decide whether a graph G is of the form Γ_R?

Definition (address labeling of G, intuitive version)

Each vertex of G is assigned a distinct "address" of s distinct numbers from $[n] = \{1, \ldots, n\}$, so that two vertices are adjacent if and only if their addresses differ by exactly one number.

Example (paths)

$$\{1,2\} \text{ —— } \{2,3\} \text{ —— } \cdots \text{ —— } \{n-1, n\}$$

Graph Labelings, cont.

Theorem (SW-Spiroff)

If G admits an address labeling, then there is a complete local equidimensional unmixed ring R such that $\Gamma_R \cong G$. Moreover, the ring R is of the form $\mathsf{k}[\![X_1,\ldots,X_n]\!]/I$ where I is a square-free monomial ideal.

Example (paths)

$$\{1,2\} \text{——} \{2,3\} \text{——} \cdots \text{——} \{n-1,n\}$$

$$(X_1,X_2) \text{——} (X_2,X_3) \text{——} \cdots \text{——} (X_{n-1},X_n)$$

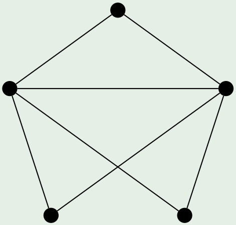
$R = \mathsf{k}[\![X_1,\ldots,X_n]\!]/J$ where

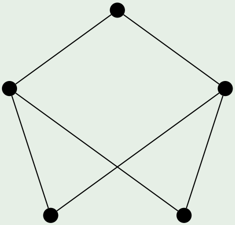
$$J = (X_1,X_2)\bigcap(X_2,X_3)\bigcap\cdots\bigcap(X_{n-1},X_n).$$

Graph Labelings, cont.

Example

The following graphs do not have address labelings.





Thus, one cannot realize these graphs as Γ_R for any unmixed equidimensional monomial ideal. Note that the first graph is chordal and the second one is complete bipartite.

Question

Can these graphs be realized as Γ_R?

Graph Labelings, cont.

Notation

Let G be a graph with vertex set V. Fix positive integers n and s, and let $\binom{[n]}{s}$ denote the set of subsets of $[n]$ with cardinality s.

Definition

An address labeling of G is an injective function $\phi\colon V \hookrightarrow \binom{[n]}{s}$, for some choice of n and s such that for all $v, w \in V$, we have v adjacent to w in G if and only if $|\phi(v) \bigcap \phi(w)| = s - 1$, that is, if and only if $|\phi(v) \bigcup \phi(w)| = s + 1$.