Weakly Spherical Modules

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Definition (Seidel and Thomas, 2000)

Let $X$ be a smooth complex projective variety, and let $\mathcal{D}^b(X)$ denote the bounded derived category of coherent sheaves on $X$. An object $\mathcal{E} \in \mathcal{D}^b(X)$ is spherical if $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ and

$$\text{Hom}^{r}_{\mathcal{D}^b(X)}(\mathcal{E}, \mathcal{E}) \cong \begin{cases} \mathbb{C} & \text{if } r = 0, \dim(X) \\ 0 & \text{if } r \neq 0, \dim(X). \end{cases}$$

Motivation

Given certain collections of spherical objects, Seidel and Thomas construct braid group actions on $\mathcal{D}^b(X)$ related to homological mirror symmetry, the McKay correspondence, etc.

Motivation

I love Ext.
Assumption

\((R, \mathfrak{m}, k)\) is a commutative local noetherian ring and \(n\) is a positive integer.

Definition

A finitely generated \(R\)-module \(M\) is weakly \(n\)-spherical if

\[
\operatorname{Ext}_R^r(M, M) \cong \begin{cases} 
  k & \text{if } r = 0, n \\
  0 & \text{if } r \neq 0, n.
\end{cases}
\]

Example

If \(R\) is a DVR, then \(k\) is weakly 1-spherical over \(R\).

Question

Are there any other examples?
Theorem

Assume that $R$ admits a weakly $n$-spherical module $M$. Then $R$ is a DVR and $M \cong k$.

Proof.

Assumption: $\text{Ext}_R^r(M, M) \cong \begin{cases} k & \text{if } r = 0, n \\ 0 & \text{if } r \neq 0, n. \end{cases}$

We have $0 \neq R/\text{Ann}_R(M) \hookrightarrow \text{Hom}_R(M, M) \cong k$.

So $M \cong k^\beta$ for some $\beta \neq 0$.

It follows that $k \cong \text{Hom}_R(k^\beta, k^\beta) \cong k^{\beta^2}$, so $\beta = 1$ and $M = k$.

Hence $0 = \text{Ext}_R^r(M, M) = \text{Ext}_R^r(k, k)$ for all $r \gg 0$.

We conclude that $R$ is regular. Let $d = \text{dim}(R)$.

Then $\text{Ext}_R^r(k, k) = k^{\binom{d}{r}}$ for all $r$, so $\text{Ext}_R^r(k, k) \neq 0$ for $0 \leq r \leq d$.

The Assumption implies that $d = 1$, so $R$ is a DVR.
Weakly Spherical Complexes

Definition

A homologically finite $R$-complex $X$ is weakly $n$-spherical if

$$\text{Ext}_R^r(X, X) \cong \begin{cases} k & \text{if } r = 0, n \\ 0 & \text{if } r \neq 0, n. \end{cases}$$

Here $\text{Ext}_R^i(X, X) = \text{H}_{-i}(\text{RHom}_R(X, X))$.

Example

If $R$ is a DVR, then $\Sigma^i k$ is weakly 1-spherical over $R$ for all $i$.

Question

Are there any other examples?
Theorem

Assume that \( R \) admits a weakly \( n \)-spherical complex \( X \). Then \( R \) is a DVR and \( M \cong \Sigma^j k \) for some \( j \in \mathbb{Z} \).

Sketch of proof.

The fact that \( X \) is homologically finite such that \( \text{Ext}_R^r(X, X) \) has finite length for all \( r \) implies that \( H_i(X) \) has finite length for all \( i \). From this, one deduces that there is an integer \( j \) such that \( H_i(X) = 0 \) for all \( i \neq j \). So there is a finitely generated \( R \)-module \( M \) such that \( X \cong \Sigma^j M \). The previous theorem implies that \( M \cong k \).

Assumption

Let \( \mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m} \) and set \( K = K^R(\mathbf{x}) \), the Koszul complex. We view \( K \) as a DG \( R \)-algebra via the wedge product.
Definition

A homologically finite DG $K$-module $X$ is weakly $n$-spherical if

$$\text{Ext}_K^r(X, X) \cong \begin{cases} k & \text{if } r = 0, n \\ 0 & \text{if } r \neq 0, n. \end{cases}$$

Example

Let $R$ be a regular local ring with regular system of parameters $x_1, \ldots, x_n, t$. Then the DG $K$-module $K^R(x) \otimes_R K^R(t) = K^R(x_1, \ldots, x_n, t)$ is a weakly 1-spherical.

Question

Are there any other examples?
Theorem

Assume that $K$ admits a weakly $n$-spherical DG module $X$.

(a) Then $n = 1$ or $n = 2$.

(b) If $n = 1$, then $R$ is a regular local ring with rsop $x_1, \ldots, x_n, t$ and $X \cong \Sigma^i K^R(x_1, \ldots, x_n, t)$ for some $i \in \mathbb{Z}$.

Hint of proof.

Assume that $R$ is complete and that $n \neq 2$. It follows that $\text{Ext}^2_R(X, X) = 0$ so a lifting result of Nasseh and Sather-Wagstaff (à la Auslander, Ding, and Solberg) provides a homologically finite $R$-complex $M$ such that $X \cong K \otimes^L_R M$. Prove that $n = 1$ and $M \cong \Sigma^i K^R(t)$, as described above.

Note

The case $n = 2$ is still open.