A theorem in representation theory for DG algebras, with an application to a question of Vasconcelos

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Much of Algebra Reduces to Linear Algebra

Assumption

R is a d-dimensional commutative algebra over a field $F = \overline{F}$.

Facts

1. Every R-module has a canonical F-vector space structure by restriction of scalars.
2. Every non-zero F-vector space has many distinct R-module structures.

Example

Let $R = F[x]/(x^2)$ and $M = (R/xR)^2$ and $N = R$.
Then M and N are isomorphic over F, but not over R.

Slogan

To study R-modules, fix $V = F^n$ and study all the ways to make V into an R-module.

Algebraic Information Can Be Encoded Geometrically

Facts

1. An R-module structure on V is a bilinear map $R \times V \to V$ satisfying certain axioms (associative and unital).
2. The bilinear maps $R \times V \to V$ are in bijection with the linear maps $R \otimes_F V \to V$, i.e., the $n \times dn$ matrices over F.
3. An R-module structure on V is a matrix in $\mathrm{M}_{n\times dn}(F)$ satisfying certain axioms (associative and unital).

Notation

$\mathrm{Mod}_R(V) \subseteq \mathrm{M}_{n\times dn}(F)$ is the set of R-module structures on V.

Fact

Given variables x_{ij} to represent the entries of a matrix in $\mathrm{M}_{n\times dn}(F)$, the R-module axioms are characterized by polynomial equations in the x_{ij}, so $\mathrm{Mod}_R(V) \subseteq \mathrm{M}_{n\times dn}(F)$ is Zariski closed.

When are two modules in $\mathrm{Mod}_R(V)$ isomorphic?

Facts

1. $\mathrm{GL}_n(F)$ acts on $\mathrm{Mod}_R(V)$ by conjugation: Given $\phi \in \mathrm{GL}_n(F)$ and $\mu \in \mathrm{Mod}_R(V)$, set
$$\phi \cdot \mu = \phi \circ \mu \circ (R \otimes_F \phi^{-1}).$$
2. Two module structures $\mu, \lambda \in \mathrm{Mod}_R(V)$ are isomorphic over R if and only if $\lambda = \phi \cdot \mu$ for some $\phi \in \mathrm{GL}_n(F)$.
3. The isomorphism classes in $\mathrm{Mod}_R(V)$ are precisely the orbits under the action of $\mathrm{GL}_n(F)$.
4. Each orbit in $\mathrm{Mod}_R(V)$ is locally closed.
5. For $M \in \mathrm{Mod}_R(V)$, there is an inclusion of tangent spaces
$$\mathrm{T}_M^{\mathrm{GL}_n(F)\cdot M} \subseteq \mathrm{T}_M^{\mathrm{Mod}_R(V)}.$$

Geometric Information Can Be Encoded Algebraically

Theorem

Given $M \in \mathrm{Mod}_R(V)$*, there is an isomorphism*

$$\mathrm{T}_M^{\mathrm{Mod}_R(V)} \,/\, \mathrm{T}_M^{\mathrm{GL}_n(F)\cdot M} \cong \mathrm{Ext}^1_R(M, M).$$

Corollary

Given $M \in \mathrm{Mod}_R(V)$*, the orbit* $\mathrm{GL}_n(F) \cdot M$ *is open in* $\mathrm{Mod}_R(V)$ *if and only if* $\mathrm{Ext}^1_R(M, M) = 0$*.*

Corollary

The set of isomorphism classes of R*-modules* M *such that* $\mathrm{Hom}_R(M, M) \cong R$ *and* $\mathrm{Ext}^1_R(M, M) = 0$ *is finite.*

Question

How to prove the second corollary for rings that are not finite dimensional algebras over a field?

An Extension

Answer

When R is local, replace R with an appropriate finite dimensional differential graded (DG) F-algebra U:

1. U is a a graded commutative F-algebra $U = \oplus_{i=0}^{e} U_i$,
2. U has a differential, i.e., a sequence of R-linear maps $\partial_i^U \colon U_i \to U_{i-1}$ such that $\partial_i^U \partial_{i+1}^U = 0$ for all i, and
3. ∂^U satisfies the Leibniz Rule: for all $a_i \in U_i$ and $a_j \in U_j$

$$\partial_{i+j}^U(a_i a_j) = \partial_i^U(a_i)a_j + (-1)^i a_i \partial_j^U(a_j).$$

Note

The starting point for this replacement is to take the Koszul complex on a minimal generating sequence for the maximal ideal $\mathfrak{m} \subset R$.

An Extension, cont.

Solution

1. One needs to work with DG U-modules: U-modules with extra data (a differential that satisfies the Leibniz Rule), and one has to encode the extra data into the geometric object $\mathrm{DGMod}_U(V)$.
2. One has to consider a product of GL's for the group action.
3. The quotient of tangent spaces is still isomorphic to an Ext-module, but it is in general the wrong Ext-module.
4. There are two distinct kinds of Ext over U!
   DG-Ext corresponds to $\mathrm{Ext}^1_R(M, M)$ under passage to U.
   Yoneda-Ext parametrizes extensions. They are not generally the same.
5. By using truncations of semiprojective DG U-modules we can reduce to the case where DG Ext and Yoneda Ext are the same, and the rest of the proof goes through.

Conclusion

Remarks

1. Commutative algebra does not exist in an algebraic vacuum.
2. Much of algebra reduces to linear algebra.
3. Geometry can encode algebraic information.
4. Group actions are not only useful for the Algebra prelim.
5. Algebra can encode geometric information.
6. Sometime to prove a theorem about rings, you have to be flexible about your definition of “ring”.