Descent of semidualizing modules

Sean Sather-Wagstaff

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Joint with Lars W. Christensen

Assumption. $(R, \mathfrak{m})$ is a Cohen-Macaulay (CM) local ring.

Definition. A finitely generated R-module C is *semidualizing* if

(a) The natural homothety homomorphism

$$\chi_C^R \colon R \to \operatorname{Hom}_R(C, C)$$

is an isomorphism, and

(b) $\operatorname{Ext}_R^i(C, C) = 0$ for each $i \neq 0$.

Notation. $\mathfrak{S}_0(R)$ is the set of isomorphism classes of semidualizing R-modules.

Example. R is semidualizing.

Fact. ω_R is a canonical (dualizing) module if and only if it is semidualizing and $\operatorname{id}_R(\omega_R)$ is finite.

Assumption. R admits a canonical module ω_R.

Fact. R is Gorenstein if and only if $\mathfrak{S}_0(R) = \{[R]\}$.

Fact. If R is a normal domain, then $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$.

Example. Let $\varphi\colon Q \to R$ be a finite local homomorphism of finite projective dimension and ω_Q a canonical module for Q. If $d = \operatorname{depth} Q - \operatorname{depth} R$, then $\mathfrak{S}_0(R)$ contains the following:

$$\omega_R \cong \mathrm{Ext}^d_Q(R, \omega_Q) \qquad \omega_Q \otimes_Q R$$

$$\omega_\varphi := \mathrm{Ext}^d_Q(R, Q) \qquad R \cong Q \otimes_Q R$$

These modules can be pairwise nonisomorphic.

Functoriality. Let $\varphi\colon Q \to R$ be a local homomorphism of finite flat dimension. The assignment $[C] \mapsto [C \otimes_Q R]$ describes an injective map $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(Q) \to \mathfrak{S}_0(R)$.

Question. When is $\mathfrak{S}_0(\varphi)$ surjective?

Fact. If Q is complete, then $\mathfrak{S}_0(\pi)$ is surjective.

Fact. $[\omega_R]$ is in the image of $\mathfrak{S}_0(\varphi)$ if and only if Q admits a canonical module ω_Q and $\omega_R \cong \omega_Q \otimes_Q R$.

First obstruction. Even if Q admits a canonical module, the map $\mathfrak{S}_0(\varphi)$ will usually not be surjective.

Example. Let Q be a field and R a non-Gorenstein local Q-algebra. The structure map $\varphi\colon Q \to R$ is flat and $\omega_Q \cong Q$, but $\omega_Q \otimes_Q R \cong R \ncong \omega_R$.

Moral. Restrict focus to sufficiently nice homomorphisms.

Refined Question. Let Q be a local CM ring and fix a Q-sequence $\mathbf{x} \in \mathfrak{m}_Q$. If $\varphi\colon Q \to \widehat{Q}$ and $\pi\colon Q \to Q/(\mathbf{x})$ are the natural maps, then when are $\mathfrak{S}_0(\varphi)$ and $\mathfrak{S}_0(\pi)$ surjective?

Second obstruction. Q may not admit a canonical module.

Side Question. With φ and π as in the Refined Question, if Q does not admit a canonical module, how badly can surjectivity of $\mathfrak{S}_0(\varphi)$ and $\mathfrak{S}_0(\pi)$ fail?

Example. Fix an integer $n \geq 1$. There exists a complete CM normal domain R containing $\mathbb{Q}$ such that $\dim(R) \geq 2$ and

$$\operatorname{card}(\mathfrak{S}_0(R)) = 2^n.$$

There exists a local UFD Q such that $\widehat{Q} \cong R$. In particular,

$$\mathfrak{S}_0(Q) \subseteq \operatorname{Cl}(Q) = \{[Q]\} \quad \Longrightarrow \quad \operatorname{card}(\mathfrak{S}_0(Q)) = 1.$$

A maximal Q-sequence $\mathbf{x} \in \mathfrak{m}_Q$ provides commutative diagrams

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & R \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ Q/(\mathbf{x}) & \xrightarrow{\cong} & R/(\mathbf{x}) \end{array} \qquad \begin{array}{ccc} \mathfrak{S}_0(Q) & \xrightarrow{\mathfrak{S}_0(\varphi)} & \mathfrak{S}_0(R) \\ \downarrow{\scriptstyle \mathfrak{S}_0(\pi)} & & {\scriptstyle \cong}\downarrow \\ \mathfrak{S}_0(Q/(\mathbf{x})) & \xrightarrow{\cong} & \mathfrak{S}_0(R/(\mathbf{x})) \end{array}$$

so the images of $\mathfrak{S}_0(\varphi)$ and $\mathfrak{S}_0(\pi)$ are trivial.

Fact. If Q is a local CM ring with the approximation property, then Q admits a canonical module.

Theorem. *Let Q be local CM with the approximation property.*

1. *For an ideal $I \subseteq \mathfrak{m}_Q$, let $\widehat{Q}^I$ denote the I-adic completion of Q with completion map $\varphi\colon Q \to \widehat{Q}^I$. Then $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(Q) \to \mathfrak{S}_0(\widehat{Q}^I)$ is surjective.*

2. *For a Q-sequence $\mathbf{x} \in \mathfrak{m}_Q$, let $\pi\colon Q \to Q/(\mathbf{x})$ be the natural map. Then $\mathfrak{S}_0(\pi)\colon \mathfrak{S}_0(Q) \to \mathfrak{S}_0(Q/(\mathbf{x}))$ is surjective.*