A Conjecture of Kurano and Roberts Related to Positivity of the Intersection Multiplicity of Serre (preliminary report)

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Abstract. Kurano and Roberts have conjectured the following: Assume that \((R, m)\) is a regular local ring with prime ideals \(p\) and \(q\) such that \(\dim(R/p) + \dim(R/q) = \dim(R)\) and \(\sqrt{p + q} = m\), then for all \(n > 0\), \(p^n \cap q \subseteq m^{n+1}\). We shall discuss progress on this and a related conjecture.
I. Background and Introduction

Assume that \((R, \mathfrak{m})\) is a regular local ring with ideals \(p\) and \(q\) such that \(\sqrt{p + q} = \mathfrak{m}\). Under these assumptions, Serre defined the intersection multiplicity of \(R/p\) and \(R/q\) as

\[
\chi(R/p, R/q) = \sum_{i=0}^{\text{dim}(R)} (-1)^i \text{len}(\text{Tor}_i^R(R/p, R/q))
\]

and proved that

\[
\text{dim}(R/p) + \text{dim}(R/q) \leq \text{dim}(R).
\]

Furthermore, he conjectured:

(Vanishing) If \(\text{dim}(R/p) + \text{dim}(R/q) < \text{dim}(R)\), then \(\chi(R/p, R/q) = 0\).

(Nonnegativity) \(\chi(R/p, R/q) \geq 0\).

(Positivity) If \(\text{dim}(R/p) + \text{dim}(R/q) = \text{dim}(R)\), then \(\chi(R/p, R/q) > 0\).
Recently, K. Kurano and P. Roberts proved the following theorem.

**Theorem 1.** Assume that \((R, \mathfrak{m})\) is a regular local ring which either contains a field or is ramified, and that \(\mathfrak{p}\) and \(\mathfrak{q}\) are prime ideals in \(R\) such that \(\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R\) and \(\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}\). If \(\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0\) then

\[
\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} \text{ for all } n \geq 0. \tag{1}
\]

As a result, they conjectured that (1) should hold in all regular local rings.

**Conjecture 2.** (Kurano-Roberts) Assume that \((R, \mathfrak{m})\) is a regular local ring and that \(\mathfrak{p}\) and \(\mathfrak{q}\) are prime ideals in \(R\) such that \(\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R\) and \(\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}\). Then \(\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}, \text{ for all } n \geq 0.\)
**Definition 3.** Assume that \((R, m)\) is a Noetherian local ring. The *Hilbert polynomial* of \(R\), denoted \(H[R](n)\), is the polynomial in \(n\) of degree \(d = \dim(R)\) with rational coefficients such that for \(n \gg 0\)

\[ H[R](n) = \text{len}(R/m^{n+1}). \]

If \(e_d\) is the leading coefficient of \(H[R](n)\), then the *multiplicity* of \(R\) is \(e(R) = d!e_d\).

If \(R\) is regular with prime ideal \(p\) and \(0 \neq x \in p\), then \(e = e(R_p/xR_p)\) if and only if \(x \in p(e) \setminus p(e+1)\). Thus, the conjecture of Kurano and Roberts may be rephrased.

**Conjecture 2′** Assume that \((R, m)\) is a regular local ring with prime ideals \(p\) and \(q\) such that \(\sqrt{p + q} = m\). Assume that \(0 \neq x \in p \cap q\) such that \(e(R_p/xR_p) = e(R/xR)\). Then

\[ \dim(R/p) + \dim(R/q) \leq \dim(R/xR). \]
With this restatement in mind, Conjecture 2 can be generalized.

**Conjecture 4.** *(Roberts, S-W)* Assume that \((R, \mathfrak{m})\) is a Cohen-Macaulay local ring with prime ideals \(\mathfrak{p}\) and \(\mathfrak{q}\) such that \(\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}\) and \(e(R_\mathfrak{p}) = e(R)\). Then

\[
\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).
\]

**Example 5.** If we do not require that \(e(R_\mathfrak{p}) = e(R)\), then Conjecture 4 does not hold. Let \(k\) be a field, \(R = k[[X,Y,Z,W]]/(XY-ZW) = k[[x,y,z,w]]\) with \(\mathfrak{p} = (x,z)R\) and \(\mathfrak{q} = (y,w)R\). Then \(e(R) = 2 > 1 = e(R_\mathfrak{p})\) and \(\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 4 > 3 = \dim(R)\).

**Example 6.** If we do not require that \(\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}\), then Conjecture 4 does not hold. Let \(R = k[[X]]\) and \(\mathfrak{p} = \mathfrak{q} = (0)\). Then \(e(R_\mathfrak{p}) = e(R)\) and \(\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 > \dim(R)\).
Example 7. If we do not require that $R$ is at least equidimensional, then Conjecture 4 does not hold. Let $R = k[[X]] \times_k k[[Y, Z]]$ with $p = ((X, 0))R$ and $q = ((0, Y), (0, Z))R$. Then $R$ is a local ring with maximal ideal $m = p + q$ and $e(R_p) = e(R_q) = e(R) = 1$. However, $\dim(R/p) + \dim(R/q) = 3 > 2 = \dim(R)$.

If $R$ is excellent then we have verified Conjecture 4 in the following cases:

(i) either $p$ or $q$ is minimal.
(ii) either $\dim(R/p) = 1$ or $\dim(R/q) = 1$.
(iii) $R/p$ is regular. (Here the condition “Cohen-Macaulay and excellent” may be loosened to “quasi-unmixed”.)
(iv) $R$ contains a field.

The rest of this talk consists of a sketch of the proof of (iv).
II. The Equicharacteristic Case

**Theorem 8.** Assume that \((R, \mathfrak{m})\) is an excellent local Cohen-Macaulay ring which contains a field. Also, assume that \(\mathfrak{p}\) and \(\mathfrak{q}\) are prime ideals of \(R\) such that \(\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}\) and \(e(R_\mathfrak{p}) = e(R)\). Then
\[
\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).
\]

**Proof.** Step 1: Reduce to the case where the residue field of \(R\) is infinite by passing to \(R[X]_{\mathfrak{m}}[X]\) if necessary. This preserves dimensions and multiplicities.

Step 2: Reduce to the case where \(R/\mathfrak{p}\) is a normal domain by passing to a polynomial ring \(R[X_1, \ldots, X_n]\) which surjects onto the integral closure \(\widetilde{R/\mathfrak{p}}\) of \(R/\mathfrak{p}\) and localizing at a maximal ideal. The fact that \(R\) is excellent implies that \(\widetilde{R/\mathfrak{p}}\) is module finite over \(R/\mathfrak{p}\). This preserves dimensions and multiplicities.
Step 3: Reduce to the case where $R$ is complete by passing to the completion $\hat{R}$. Because $R$ is excellent and $R/p$ is normal, the ideal $p\hat{R}$ is prime and it follows that multiplicities are preserved, as well as dimensions.

Step 4: By completeness, $R$ contains a coefficient field $K$. Let $y_1, \ldots, y_n$ be a system of parameters of $R$ which form part of a minimal generating set of $m$ and such that $e(R) = \text{len}(R/(y_1, \ldots, y_d))$. Extend this to a minimal generating set $y_1, \ldots, y_d, z_1, \ldots, z_q$ of $m$. The fact that $R$ is complete and Cohen-Macaulay implies that the natural map $K[Y] = K[Y_1, \ldots, Y_n] \rightarrow R$ is injective and $R$ is a finite free module over $K[Y]$. The natural map $K[Y, Z] \rightarrow R$ gives a commutative diagram.
If we can show that $\sqrt{p'+q'}$ is primary to the maximal ideal of $K[[Y,Z]]$, then Serre's Intersection Theorem for regular local rings implies that

$$\dim(R/p) + \dim(R/q) = \dim(K[[Y,Z]]/p') - q + \dim(K[[Y,Z]]/q') \leq \dim(K[[Y,Z]]) - q = \dim(R).$$

as desired.

Chasing the diagram and using the fact that $R$ is a free $K[[Y]]$-module we see that it suffices to show that $p$ is the unique prime of $R$ which contracts to $p \cap K[[Y]]$ in $K[[Y]]$. 
We shall use the following result and the fact
that $e(R_p) = e(R)$ to give the desired
uniqueness.

**Theorem 9.** Assume that $A$ is a regular local
ring with overring $B$ such that $B$ is a finite
free $A$-module. Assume that $p$ is a prime ideal
of $B$ such that $e(B_p) = \text{rank}_A(B)$. Then $p$ is
the unique prime ideal of $B$ contracting to
$p \cap A$ in $A$.

It follows that we need only verify that
$e(R_p) = r$ where $r = \text{rank}_K[\![Y]\!]\!(R)$:

$$
e(R_p) = e(R) = \text{len}(R/(y))
= \dim_K(R/(y))
= \dim_K(R \otimes_{K[\![Y]\!]\!} K[\![Y]\!]/(Y))
= \dim_K(K[\![Y]\!]^r \otimes_{K[\![Y]\!]\!} K[\![Y]\!]/(Y))
= \dim_K(K^r) = r.$$

This gives the desired conclusion.