A Conjecture of Kurano and Roberts Related to Positivity of the Intersection Multiplicity of Serre (preliminary report)

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Abstract. Kurano and Roberts have conjectured the following: Assume that $(R, \mathfrak{m})$ is a regular local ring with prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$ and $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$, then for all $n > 0$, $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$. We shall discuss progress on this and a related conjecture.

I. Background and Introduction

Assume that $(R, \mathfrak{m})$ is a regular local ring with ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. Under these assumptions, Serre defined the *intersection multiplicity* of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ as

$$\chi(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{\dim(R)} (-1)^i \operatorname{len}(\operatorname{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q}))$$

and proved that

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).$$

Furthermore, he conjectured:

(Vanishing) If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) < \dim(R)$, then $\chi(R/\mathfrak{p}, R/\mathfrak{q}) = 0$.

(Nonnegativity) $\chi(R/\mathfrak{p}, R/\mathfrak{q}) \geq 0$.

(Positivity) If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$, then $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

Recently, K. Kurano and P. Roberts proved the following theorem.

Theorem 1. *Assume that* $(R, \mathfrak{m})$ *is a regular local ring which either contains a field or is ramified, and that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$ *and* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. *If* $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$ *then*

$$\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} \text{ for all } n \geq 0. \qquad (1)$$

As a result, they conjectured that (1) should hold in all regular local rings.

Conjecture 2. *(Kurano-Roberts) Assume that* $(R, \mathfrak{m})$ *is a regular local ring and that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$ *and* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. *Then* $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$, *for all* $n \geq 0$.

Definition 3. Assume that $(R, \mathfrak{m})$ is a Noetherian local ring. The *Hilbert polynomial* of R, denoted $H[R](n)$, is the polynomial in n of degree $d = \dim(R)$ with rational coefficients such that for $n \gg 0$

$$H[R](n) = \text{len}(R/\mathfrak{m}^{n+1}).$$

If e_d is the leading coefficient of $H[R](n)$, then the *multiplicity* of R is $e(R) = d!e_d$.

If R is regular with prime ideal $\mathfrak{p}$ and $0 \neq x \in \mathfrak{p}$, then $e = e(R_\mathfrak{p}/xR_\mathfrak{p})$ if and only if $x \in \mathfrak{p}^{(e)} \smallsetminus \mathfrak{p}^{(e+1)}$. Thus, the conjecture of Kurano and Roberts may be rephrased.

Conjecture 2′ *Assume that* $(R, \mathfrak{m})$ *is a regular local ring with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$. *Assume that* $0 \neq x \in \mathfrak{p} \cap \mathfrak{q}$ *such that* $e(R_\mathfrak{p}/xR_\mathfrak{p}) = e(R/xR)$. *Then*

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R/xR).$$

With this restatement in mind, Conjecture 2 can be generalized.

Conjecture 4. *(Roberts, S-W) Assume that* $(R, \mathfrak{m})$ *is a Cohen-Macaulay local ring with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $e(R_\mathfrak{p}) = e(R)$*. Then*

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).$$

Example 5. If we do not require that $e(R_\mathfrak{p}) = e(R)$, then Conjecture 4 does not hold. Let k be a field, $R = k[\![X, Y, Z, W]\!]/(XY - ZW) = k[\![x, y, z, w]\!]$ with $\mathfrak{p} = (x, z)R$ and $\mathfrak{q} = (y, w)R$. Then $e(R) = 2 > 1 = e(R_\mathfrak{p})$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 4 > 3 = \dim(R)$.

Example 6. If we do not require that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$, then Conjecture 4 does not hold. Let $R = k[\![X]\!]$ and $\mathfrak{p} = \mathfrak{q} = (0)$. Then $e(R_\mathfrak{p}) = e(R)$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 > \dim(R)$.

Example 7. If we do not require that R is at least equidimensional, then Conjecture 4 does not hold. Let $R = k[\![X]\!] \times_k k[\![Y, Z]\!]$ with $\mathfrak{p} = ((X, 0))R$ and $\mathfrak{q} = ((0, Y), (0, Z))R$. Then R is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$ and $e(R_{\mathfrak{p}}) = e(R_{\mathfrak{q}}) = e(R) = 1$. However, $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 3 > 2 = \dim(R)$.

If R is excellent then we have verified Conjecture 4 in the following cases:
(i) either $\mathfrak{p}$ or $\mathfrak{q}$ is minimal.
(ii) either $\dim(R/\mathfrak{p}) = 1$ or $\dim(R/\mathfrak{q}) = 1$.
(iii) $R/\mathfrak{p}$ is regular. (Here the condition "Cohen-Macaulay and excellent" may be loosened to "quasi-unmixed".)
(iv) R contains a field.

The rest of this talk consists of a sketch of the proof of (iv).

II. The Equicharacteristic Case

Theorem 8. *Assume that* $(R, \mathfrak{m})$ *is an excellent local Cohen-Macaulay ring which contains a field. Also, assume that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals of* R *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $e(R_{\mathfrak{p}}) = e(R)$*. Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$*.*

Proof. Step 1: Reduce to the case where the residue field of R is infinite by passing to $R[X]_{\mathfrak{m}[X]}$ if necessary. This preserves dimensions and multiplicities.

Step 2: Reduce to the case where $R/\mathfrak{p}$ is a normal domain by passing to a polynomial ring $R[X_1, \ldots, X_n]$ which surjects onto the integral closure $\widetilde{R/\mathfrak{p}}$ of $R/\mathfrak{p}$ and localizing at a maximal ideal. The fact that R is excellent implies that $\widetilde{R/\mathfrak{p}}$ is module finite over $R/\mathfrak{p}$. This preserves dimensions and multiplicities.

Step 3: Reduce to the case where R is complete by passing to the completion $\widehat{R}$. Because R is excellent and $R/\mathfrak{p}$ is normal, the ideal $\mathfrak{p}\widehat{R}$ is prime and it follows that multiplicities are preserved, as well as dimensions.

Step 4: By completeness, R contains a coefficient field K. Let $y_1, \ldots, y_n$ be a system of parameters of R which form part of a minimal generating set of $\mathfrak{m}$ and such that $e(R) = \text{len}(R/(y_1, \ldots, y_d))$. Extend this to a minimal generating set $y_1, \ldots, y_d, z_1, \ldots, z_q$ of $\mathfrak{m}$. The fact that R is complete and Cohen-Macaulay implies that the natural map $K[\![Y]\!] = K[\![Y_1, \ldots, Y_n]\!] \to R$ is injective and R is a finite free module over $K[\![Y]\!]$. The natural map $K[\![Y, Z]\!] \to R$ gives a commutative diagram.

$$\begin{array}{ccccc} \mathfrak{p}\cap K[\![Y]\!] \subset & K[\![Y]\!] & \longrightarrow & K[\![Y,Z]\!] & \supset \mathfrak{p}',\mathfrak{q}' \\ & & \searrow & \downarrow & \\ & & & R & \supset \mathfrak{p},\mathfrak{q} \end{array}$$

If we can show that $\sqrt{\mathfrak{p}'+\mathfrak{q}'}$ is primary to the maximal ideal of $K[\![Y,Z]\!]$, then Serre's Intersection Theorem for regular local rings implies that

$$\begin{aligned} &\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \\ &\qquad = \dim(K[\![Y,Z]\!]/\mathfrak{p}') - q + \dim(K[\![Y,Z]\!]/\mathfrak{q}') \\ &\qquad \leq \dim(K[\![Y,Z]\!]) - q = \dim(R). \end{aligned}$$

as desired.

Chasing the diagram and using the fact that R is a free $K[\![Y]\!]$-module we see that it suffices to show that $\mathfrak{p}$ is the unique prime of R which contracts to $\mathfrak{p}\cap K[\![Y]\!]$ in $K[\![Y]\!]$.

We shall use the following result and the fact that $e(R_{\mathfrak{p}}) = e(R)$ to give the desired uniqueness.

Theorem 9. *Assume that A is a regular local ring with overring B such that B is a finite free A-module. Assume that $\mathfrak{p}$ is a prime ideal of B such that $e(B_{\mathfrak{p}}) = \mathrm{rank}_A(B)$. Then $\mathfrak{p}$ is the unique prime ideal of B contracting to $\mathfrak{p} \cap A$ in A.*

It follows that we need only verify that $e(R_{\mathfrak{p}}) = r$ where $r = \mathrm{rank}_{K[\![Y]\!]}(R)$:

$$\begin{aligned} e(R_{\mathfrak{p}}) &= e(R) = \mathrm{len}(R/(y)) \\ &= \dim_K(R/(y)) \\ &= \dim_K(R \otimes_{K[\![Y]\!]} K[\![Y]\!]/(Y)) \\ &= \dim_K(K[\![Y]\!]^r \otimes_{K[\![Y]\!]} K[\![Y]\!]/(Y)) \\ &= \dim_K(K^r) = r. \end{aligned}$$

This gives the desired conclusion. □