A new characterization of Gorenstein rings

Anders Frankild

Sean Sather-Wagstaff

15 May 2004

math.AC/0404361

Throughout, $(R, \mathfrak{m}, k)$ is a local ring.

Much research in commutative algebra is devoted to duality.

(Grothendieck-Hartshorne) Investigate $\mathrm{Hom}_R(-, D)$ when R admits a dualizing complex D.

(Auslander-Bridger) Investigate $\mathrm{Hom}_R(-, R)$.

(Foxby-Golod) Investigate $\mathrm{Hom}_R(-, K)$ when K is semidualizing.

Definition. A homologically finite complex of R-modules K is *semidualizing* if the natural homothety momorphism $\chi^R_K \colon R \to \mathbf{R}\mathrm{Hom}_R(K, K)$ is an isomorphism in $\mathcal{D}(R)$.

Definition. A finite (i.e., finitely generated) R-module K is *semidualizing* if the natural homothety homomorphism $\chi^R_K \colon R \to \mathrm{Hom}_R(K, K)$ is an isomorphism and $\mathrm{Ext}^i_R(K, K) = 0$ for each $i \neq 0$.

Example. D and R are semidualizing.

Example. (Avramov-Foxby) The dualizing complex D_φ of a local homomorphism $\varphi\colon Q \to R$ of finite G-dimension is semidualizing.

Notation. $\mathfrak{S}(R)$ is the set of shift-isomorphism classes of semidualizing R-complexes.

Example. If R is Gorenstein, then $\mathfrak{S}(R) = \{R\}$.

Example. If $R \sim D$, then R is Gorenstein.

Example. If R is CM, then $\mathfrak{S}(R) \leftrightarrow \{\text{semidualizing } R\text{-modules}\}$.

Example. If D is shift-isomorphic to a module, then so is every semidualizing complex and R is CM.

Example. (Foxby) $\mathfrak{S}(R)$ may have elements other than D and R. Let $\varphi\colon Q \to R$ be a finite flat local homomorphism and C a dualizing complex for Q. The following R-complexes are in $\mathfrak{S}(R)$:

$$D = \mathbf{R}\mathrm{Hom}_Q(R, C), \quad \mathbf{R}\mathrm{Hom}_Q(R, Q), \quad C \otimes^{\mathbf{L}}_Q R, \quad R = Q \otimes^{\mathbf{L}}_Q R.$$

Question. Is $\mathfrak{S}(R)$ finite?

Let K, M be homologically finite complexes with K semidualizing.

Definition. M is *K-reflexive* or has *finite G_K-dim* if

(a) The complex $\mathbf{R}\mathrm{Hom}_R(M,K)$ is homologically bounded, and

(b) The natural biduality momorphism

$$\delta_M^K \colon M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,K),K)$$

is an isomorphism in $\mathcal{D}(R)$.

Definition. When M, K are modules, M is *totally K-reflexive* if

(a) The natural biduality homomorphism

$$\delta_M^K \colon M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,K),K)$$

is an isomorphism, and

(b) $\mathrm{Ext}^i_R(M,K) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M,K),K)$ for each $i \neq 0$.

M has *finite G_K-dim* if it admits a finite resolution by totally K-reflexive modules.

Example. R and K are K-reflexive.

Example. Each homologically finite complex is D-reflexive.

Example. The nonzero totally R-reflexive modules are the modules of G-dimension 0. The R-reflexive modules are the modules of finite G-dimension.

Example. For semidualizing complexes K, L the complex $\mathbf{R}\mathrm{Hom}_R(K, L)$ is semidualizing if and only if K is L-reflexive. In particular, $\mathbf{R}\mathrm{Hom}_R(K, D)$ is semidualizing.

Theorem 1. $\mathfrak{S}(R)$ *admits a (nontrivial) metric.*

Theorem 2. *The assignment* $K \mapsto \mathbf{R}\mathrm{Hom}_R(K, D)$ *yields an isometric involution of* $\mathfrak{S}(R)$.

Triviality. When R is Gorenstein, this isometry has fixed point.

Question. Does the converse hold?

Theorem 3. *The following conditions are equivalent:*

(i) *R is Gorenstein;*

(ii) *R admits a dualizing complex D and a semidualizing complex K such that $K \sim \mathbf{R}\mathrm{Hom}_R(K, D)$.*

Note. K must be semidualizing for the implication (ii) $\Longrightarrow$ (i) to hold since $k \sim \mathbf{R}\mathrm{Hom}_R(k, D)$.

The theorem is a corollary of a more general result.

Theorem 4. *For semidualizing complexes K, L the following conditions are equivalent:*

(i) *$K \sim R \sim L$;*

(ii) *$K \sim \mathbf{R}\mathrm{Hom}_R(K, L)$.*

Proof of Theorem 3. (ii) $\Longrightarrow$ (i). If $K \sim \mathbf{R}\mathrm{Hom}_R(K, D)$, then Theorem 4 yields $K \sim R \sim D$, so R is Gorenstein. $\square$

Sketch of proof of Theorem 4. (ii) $\Longrightarrow$ (i). Let $P \xrightarrow{\simeq} K$ be a minimal free resolution. By a result of Gerko, the hypothesis $K \sim \mathbf{R}\mathrm{Hom}_R(K, L)$ yields isomorphisms

$$L \simeq K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(K, L) \sim K \otimes^{\mathbf{L}}_R K \simeq P \otimes_R P.$$

In particular, $P \otimes_R P$ is semidualizing.

For two complexes X, Y let

$$\theta_{XY} \colon X \otimes_R Y \to Y \otimes_R X$$

denote the natural "commutativity" isomorphism.

Since $P \otimes_R P$ is semidualizing, the isomorphism

$$\chi^R_{P\otimes_R P} \colon R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(P \otimes_R P, P \otimes_R P) \simeq \underbrace{\mathrm{Hom}_R(P \otimes_R P, P \otimes_R P)}_{\theta_{PP} \text{is a cycle}}$$

yields an element $v \in R$ such that θ_{PP} is homotopic to the homothety $\mu_v \colon P \otimes_R P \to P \otimes_R P$.

Set $\overline{P} = P \otimes_R k$ to obtain a homotopy between the morphisms

$$\theta_{\overline{P}\overline{P}} \text{ and } \mu_{\overline{v}} \colon \overline{P} \otimes_k \overline{P} \to \overline{P} \otimes_k \overline{P}.$$

Since the differential on $\overline{P}$ is zero, it follows that $\theta_{\overline{P}\overline{P}} = \mu_{\overline{v}}$.

It is now straightforward to check that

$$\overline{P} = \Sigma^i k$$

for some i, so

$$K \simeq P \sim R.$$

Finally,

$$L \sim K \otimes^{\mathbf{L}}_R K \sim R \otimes_R R \simeq R$$

completing the proof. $\square$