Homology of artinian and Matlis reflexive modules, I

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Abstract

Let \( R \) be a commutative local noetherian ring, and let \( L \) and \( L' \) be \( R \)-modules. We investigate the properties of the functors \( \text{Tor}_i^R(L, -) \) and \( \text{Ext}_i^R(L, -) \). For instance, we show the following:

(a) if \( L \) and \( L' \) are artinian, then \( \text{Tor}_i^R(L, L') \) is artinian, and \( \text{Ext}_i^R(L, L') \) is noetherian over the completion \( \hat{R} \);

(b) if \( L \) is artinian and \( L' \) is Matlis reflexive, then \( \text{Ext}_i^R(L, L'), \text{Ext}_i^R(L', L) \), and \( \text{Tor}_i^R(L, L') \) are Matlis reflexive.

Also, we study the vanishing behavior of these functors, and we include computations demonstrating the sharpness of our results.

0. Introduction

Throughout this paper, let \( R \) be a commutative noetherian local ring with maximal ideal \( m \) and residue field \( k = R/m \).

The \( m \)-adic completion of \( R \) is denoted by \( \hat{R} \), the injective hull of \( k \) is \( E = E_R(k) \), and the Matlis duality functor is \((\cdot)^\vee = \text{Hom}_E(\cdot, E) \).

This paper is concerned, in part, with the properties of the functors \( \text{Hom}_R(A, -) \) and \( A \otimes_R - \), where \( A \) is an artinian \( R \)-module. To motivate this, recall that [8, Proposition 6.1] shows that if \( A \) and \( A' \) are artinian \( R \)-modules, then \( A \otimes_R A' \) has finite length. It follows that if \( N \) is a noetherian \( R \)-module, then \( \text{Hom}_R(A, N) \) also has finite length (see also Corollaries 2.12 and 3.9). In light of this, it is natural to investigate the properties of \( \text{Ext}_i^R(A, -) \) and \( \text{Tor}_i^R(A, -) \). In general, the modules \( \text{Ext}_i^R(A, N) \) and \( \text{Tor}_i^R(A, A') \) will not have finite length. However, we have the following (see Theorems 2.2 and 3.1).

Theorem 1. Let \( A \) be an artinian \( R \)-module, and let \( i \geq 0 \). Let \( L \) and \( L' \) be \( R \)-modules such that \( \mu_i^L(L) \) and \( \beta_i^L(L') \) are finite. Then \( \text{Ext}_i^R(A, L) \) is a noetherian \( \hat{R} \)-module, and \( \text{Tor}_i^R(A, L') \) is artinian.

In this result, we use the \( i \)th Bass number \( \mu_i^L(k) \) and the \( i \)th Betti number \( \beta_i^L(k) \).

For instance, these are both finite for all \( i \) when \( L \) and \( L' \) are either artinian or noetherian. In particular, when \( A \) and \( A' \) are artinian, Theorem 1 implies that \( \text{Ext}_i^R(A, A') \) is a noetherian \( \hat{R} \)-module. The next result, contained in Theorem 4.3, gives another explanation for this fact.

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EPS-0814442. Micah Leamer was supported by a GAANN grant from the Department of Education. Sean Sather-Wagstaff was supported by a grant from the NSA.

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Theorem 2. Let $A$ and $A'$ be artinian $R$-modules, and let $i \geq 0$. Then there is an isomorphism $\Ext^i_R(A, A') \cong \Ext^i_R(A', A)$. Hence, there are noetherian $R$-modules $N$ and $N'$ such that $\Ext^i_R(A, A') \cong \Ext^i_R(N, N')$.

This result proves useful for studying the vanishing of $\Ext^i_R(A, N)$, since the vanishing of $\Ext^i_R(N, N')$ is somewhat well understood.

Our next result shows how extra conditions on the modules in Theorem 1 imply that $\Ext^i_R(A, L)$ and $\Tor^R_i(A, L')$ are Matlis reflexive; see Corollaries 2.4 and 3.3.

Theorem 3. Let $A$, $L$, and $L'$ be $R$-modules such that $A$ is artinian. Assume that $R/(\Ann_R(A) + \Ann_R(L))$ and $R/(\Ann_R(A) + \Ann_R(L'))$ are complete. Given an index $i \geq 0$ such that $\mu^i_E(L)$ and $\beta^i_{L'}(L')$ are finite, the modules $\Ext^i_R(A, L)$ and $\Tor^R_i(A, L')$ are Matlis reflexive.

A key point in the proof of this theorem is a result of Belshoff et al. [4]: An $R$-module $M$ is Matlis reflexive if and only if it is mini-max and $R/\Ann_R(M)$ is complete. Here $M$ is mini-max when $M$ has a noetherian submodule $N$ such that $M/N$ is artinian. In particular, noetherian modules are mini-max, as are artinian modules.

The last result singled out for this introduction describes the Matlis dual of $\Ext^i_R(M, M')$ in some special cases. It is contained in Corollary 4.11.

Theorem 4. Let $M$ and $M'$ be mini-max $R$-modules, and fix an index $i \geq 0$. If either $M$ or $M'$ is Matlis reflexive, then $\Ext^i_R(M, M') \cong \Tor^R_i(M, M')$.

We do not include a description of the Matlis dual of $\Tor^R_i(M, M')$, as a standard application of Hom–tensor adjointness shows that $\Tor^R_i(M, M') \cong \Ext^i_R(M, M')$.

Many of our results generalize to the non-local setting. As this generalization requires additional tools, we treat it separately in [11].

1. Background material and preliminary results

Torsion modules

Definition 1.1. Let $a$ be a proper ideal of $R$. We denote the $a$-adic completion of $R$ by $\widehat{R}^a$. Given an $R$-module $L$, set $\Gamma_a(L) = \{x \in L \mid a^n x = 0 \text{ for } n > 0\}$. We say that $L$ is a torsion if $L = \Gamma_a(L)$. We set $\Supp_a(L) = \{p \in \Spec(R) \mid L_p \neq 0\}$.

Fact 1.2. Let $a$ be a proper ideal of $R$, and let $L$ be an $a$-torsion $R$-module.

(a) Every artinian $R$-module is $m$-torsion. In particular, the module $E$ is $m$-torsion.

(b) We have $\Supp_a(L) \subseteq V(a)$. Hence, if $L$ is $m$-torsion, then $\Supp_a(L) \subseteq (m)$.

(c) The module $L$ has an $R$-module structure that is compatible with its $R$-module structure, as follows. For each $x \in L$, fix an exponent $n$ such that $a^n x = 0$. For each $r \in \widehat{R}^a$, the isomorphism $\widehat{R}^a/a^n \widehat{R}^a \cong R/a^n$ provides an element $r_0 \in R$ such that $r - r_0 \in a^n \widehat{R}^a$, and we set $rx := r_0x$.

(d) If $R/a$ is complete, then $\widehat{R}^a$ is naturally isomorphic to $\widehat{R}$. To see this, assume that $R/a$ is complete. By induction on $n$, it follows that $R/a^n$ is complete for all $n$, and this explains the second step in the next display:

$$\widehat{R}^a \cong \lim_{\leftarrow} R/a^n \cong \lim_{\leftarrow} \widehat{R}/a^n \widehat{R} \cong (\widehat{R})^{\hat{\cdot}} \cong \widehat{R}.$$ 

For the last step in this display, see, e.g., [1, Exercise 10.5].

Lemma 1.3. Let $a$ be a proper ideal of $R$, and let $L$ be an $a$-torsion $R$-module.

(a) A subset $Z \subseteq L$ is an $R$-submodule if and only if it is an $\widehat{R}^a$-submodule.

(b) The module $L$ is noetherian over $R$ if and only if it is noetherian over $\widehat{R}^a$.

Proof. (a) Every $\widehat{R}^a$-submodule of $L$ is an $R$-submodule by restriction of scalars. Conversely, fix an $R$-submodule $Z \subseteq L$. Since $L$ is $a$-torsion, so is $Z$, and Fact 1.2(c) implies that $Z$ is an $\widehat{R}^a$-submodule.

(b) The set of $R$-submodules of $L$ equals the set of $\widehat{R}^a$-submodules of $L$, so they satisfy the ascending chain condition simultaneously.

Lemma 1.4. Let $a$ be a proper ideal of $R$, and let $L$ be an $a$-torsion $R$-module.

(a) The natural map $L \to \widehat{R}^a \otimes_R L$ is an isomorphism.

(b) The left and right $\widehat{R}^a$-module structures on $\widehat{R}^a \otimes_R L$ are the same.

Proof. The natural map $L \to \widehat{R}^a \otimes_R L$ is injective, as $\widehat{R}^a$ is faithfully flat over $R$. To show surjectivity, it suffices to show that each generator $r \otimes x \in \widehat{R}^a \otimes_R L$ is of the form $1 \otimes x'$ for some $x' \in L$. Let $n \geq 1$ such that $a^n x = 0$, and let $r_0 \in R$ such that $r - r_0 \in a^n \widehat{R}^a$. It follows that $r \otimes x = r_0 \otimes x = 1 \otimes (r_0 x)$, and this yields the conclusion of part (a). This also proves (b) because $1 \otimes (r_0 x) = 1 \otimes (rx)$. □
Lemma 1.5. Let $a$ be a proper ideal of $R$, and let $L$ and $L'$ be $R$-modules such that $L$ is $a$-torsion.

(a) If $L'$ is $a$-torsion, then $\text{Hom}_R(L, L') = \text{Hom}_{R_a}(L, L')$; thus $L' = \text{Hom}_{R_a}(L, E)$.

(b) One has $\text{Hom}_{R_a}(L, L') \cong \text{Hom}_{R_a}(L, \Gamma_a(L')) = \text{Hom}_{R_a}(L, \Gamma_a(L'))$.

Proof. (a) It suffices to verify the inclusion $\text{Hom}_{R_a}(L, L') \subseteq \text{Hom}_{R_a}(L, L')$. Let $x \in L$ and $r \in R$, and fix $\psi \in \text{Hom}_{R_a}(L, L')$. Let $n \geq 1$ such that $a^n x = 0$ and $a^n \psi(x) = 0$. Choose an element $r_0 \in R$ such that $r - r_0 \in a^n R$. It follows that $\psi(x) = \psi(r_0 x) = r_0 \psi(x) = r \psi(x)$; hence $\psi \in \text{Hom}_{R_a}(L, L')$.

(b) For each $f \in \text{Hom}_{R_a}(L, L')$, one has $\text{Im}(f) \subseteq \Gamma_a(L')$. This yields the desired isomorphism, and the equality is from part (a). $\square$

A Natural Map from $\text{Tor}^R_a(L, L'^\vee)$ to $\text{Ext}^i_R(L, L'^\vee)$

Definition 1.6. Let $L$ be an $R$-module, and let $J$ be an $R$-complex. The $\text{Hom}$-evaluation morphism

$$\theta_{LJ} : L \otimes_R \text{Hom}_R(J, E) \to \text{Hom}_R(\text{Hom}_R(L, J), E)$$

is given by $\theta_{LJ}(l \otimes \psi)(\phi) = \psi(l(\phi))$.

Remark 1.7. Let $L$ and $L'$ be $R$-modules, and let $J$ be an injective resolution of $L'$. Using the notation $(-)^\vee$, we have

$$\theta_{LJ} : L \otimes_R J^\vee \to \text{Hom}_R(L, J^\vee).$$

The complex $J^\vee$ is a flat resolution of $L^\vee$; see, e.g., [7, Theorem 3.2.16]. This explains the first isomorphism in the following sequence:

$$\text{Tor}^i_R(L, L'^\vee) \cong H_i(L \otimes_R J^\vee) \xrightarrow{H_i(\theta_{LJ})} H_i(\text{Hom}_R(L, J)^\vee) \cong \text{Ext}^i_R(L, L'^\vee).$$

For the second isomorphism, the exactness of $(-)^\vee$ implies that $H_i(\text{Hom}_R(L, J)^\vee) \cong H^i(\text{Hom}_R(L, J))^\vee \cong \text{Ext}^i_R(L, L'^\vee)$.

Definition 1.8. Let $L$ and $L'$ be $R$-modules, and let $J$ be an injective resolution of $L'$. The $R$-module homomorphism

$$\theta^i_{LL'} : \text{Tor}^i_R(L, L'^\vee) \to \text{Ext}^i_R(L, L'^\vee)$$

is defined to be the composition of the maps displayed in Remark 1.7.

Remark 1.9. Let $L$, $L'$, and $N$ be $R$-modules such that $N$ is noetherian. It is straightforward to show that the map $\theta^i_{LL'}$ is natural in $L$ and in $L'$.

The fact that $E$ is injective implies that $\theta^i_{LL'}$ is an isomorphism; see [17, Lemma 3.60]. This explains the first of the following isomorphisms:

$$\text{Ext}^i_R(N, L'^\vee) \cong \text{Tor}^i_R(N, L'^\vee) \quad \text{Tor}^i_R(L, L'^\vee) \cong \text{Ext}^i_R(L, L'^\vee).$$

The second isomorphism is a consequence of Hom–tensor adjointness, $\text{Hom}$. $\text{Tor}$. $\text{Ext}$.

Numerical invariants

Definition 1.10. Let $L$ be an $R$-module. For each integer $i$, the $i$th Bass number of $L$ and the $i$th Betti number of $L$ are respectively

$$\mu^i_R(L) = \text{len}_R(\text{Ext}^i_R(k, L)) \quad \beta^i_R(L) = \text{len}_R(\text{Tor}^i_R(k, L))$$

where $\text{len}_R(L)$ denotes the length of an $R$-module $L'$.

Remark 1.11. Let $L$ be an $R$-module.

(a) If $I$ is a minimal injective resolution of $L$, then for each index $i \geq 0$ such that $\mu^i_R(L) < \infty$, we have $I^i \cong \text{Ext}^{i+1}(i) \oplus J^i$ where $J^i$ does not have $E$ as a summand, that is, $\Gamma^i(J^i) = 0$; see, e.g., [14, Theorem 18.7]. Similarly, the Betti numbers of a noetherian module are the ranks of the free modules in a minimal free resolution. The situation for Betti numbers of non-noetherian modules is more subtle; see, e.g., Lemma 1.19.

(b) Then $\mu^i_R(L) < \infty$ for all $i \geq 0$ if and only if $\beta^i_R(L) < \infty$ for all $i \geq 0$; see [12, Proposition 1.1].

When $a = m$, the next invariants can be interpreted in terms of (non)vanishing Bass and Betti numbers.

Definition 1.12. Let $a$ be an ideal of $R$. For each $R$-module $L$, set

$$\text{depth}_R(a; L) = \inf \{ i \geq 0 \mid \text{Ext}^i_R(R/a, L) \neq 0 \}$$
$$\text{width}_R(a; L) = \inf \{ i \geq 0 \mid \text{Tor}^i_R(R/a, L) \neq 0 \}.$$
Lemma 1.13. Let $L$ be an $R$-module, and let $a$ be an ideal of $R$.

(a) Then $\text{width}_R(a; L) = \text{depth}_R(a; L^\vee)$ and $\text{width}_R(a; L^\vee) = \text{depth}_R(a; L)$.
(b) For each index $i \geq 0$ we have $\beta_i^L(L) = \mu_i^L(L^\vee)$ and $\beta_i^L(L^\vee) = \mu_i^L(L)$.
(c) $L = \oplus a_i$ if and only if $\text{depth}_R(a; L^\vee) > 0$.
(d) $L^\vee = a(L^\vee)$ if and only if $\text{depth}_R(a; L) > 0$.
(e) $\text{depth}_R(a; L) > 0$ if and only if $a$ contains a non-zero-divisor for $L$.

Proof. Part (a) is from [9, Proposition 4.4], and part (b) follows directly from this.

(c)--(d) These follow from part (a) since $L = \oplus a_i$ if and only if $\text{depth}_R(a; L) > 0$.

(e) By definition, we need to show that $\text{Hom}_R(R/a, L) = 0$ if and only if $a$ contains a non-zero-divisor for $L$. One implication is explicitly stated in [6, Proposition 1.2.3(a)]. One can prove the converse like [6, Proposition 1.2.3(b)], using the fact that $R/a$ is finitely generated. □

The next result characterizes artinian modules in terms of Bass numbers.

Lemma 1.14. Let $L$ be an $R$-module. The following conditions are equivalent:

(i) $L$ is an artinian $R$-module;
(ii) $L$ is an artinian $\hat{R}$-module;
(iii) $\hat{R} \otimes_R L$ is an artinian $\hat{R}$-module; and
(iv) $L$ is $m$-torsion and $\mu_m^L(L) < \infty$.

Proof. (i) $\iff$ (iv) If $L$ is artinian over $R$, then it is $m$-torsion by Fact 1.2(a), and we have $\mu_m^L(L) < \infty$ by [7, Theorem 3.4.3].

For the converse, assume that $L$ is $m$-torsion and $\mu_m^L(L) < \infty$. Since $L$ is $m$-torsion, so is $E_R(L)$. Thus, we have $E_R(L) \cong E_{\hat{R}}^0$, which is artinian since $\mu_m^0 < \infty$. Since $L$ is a submodule of the artinian module $E_R(L)$, it is also artinian.

To show the equivalence of the conditions (i)--(iii), first note that each of these conditions implies that $L$ is $m$-torsion. (For condition (iii), use the monomorphism $L \rightarrow \hat{R} \otimes_R L$.) Thus, for the rest of the proof, we assume that $L$ is $m$-torsion.

Because of the equivalence (i) $\iff$ (iv), it suffices to show that

$$\mu_m^L(L) = \mu_m^0(L) = \mu_m^0(\hat{R} \otimes_R L).$$

These equalities follow from the next isomorphisms

$$\text{Hom}_R(k, L) \cong \text{Hom}_{\hat{R}}(k, L) \cong \text{Hom}_R(k, \hat{R} \otimes_R L)$$

which are from Lemmas 1.5(a) and 1.4, respectively. □

Lemma 1.15. Let $L$ be an $R$-module.

(a) The module $L$ is noetherian over $R$ if and only if $L^\vee$ is artinian over $R$.
(b) If $L^\vee$ is noetherian over $R$, then $L$ is artinian over $R$.
(c) Let $a$ be a proper ideal of $R$ such that $R/a$ is complete. If $L$ is $a$-torsion, then $L$ is artinian over $R$ if and only if $L^\vee$ is noetherian over $R$.

Proof. (a) This is [7, Corollary 3.4.4].
(b) If $L^\vee$ is noetherian over $R$, then we conclude from [7, Corollary 3.4.5] that $L$ is artinian over $R$. To complete the proof of (b), we assume that $L^\vee$ is noetherian over $R$ and show that $L$ is artinian. Fix a descending chain $L_1 \supseteq L_2 \supseteq \cdots$ of submodules of $L$. Dualize the surjections $L \rightarrow \cdots \rightarrow L/L_2 \rightarrow L/L_1$ to obtain a sequence of $R$-module monomorphisms $(L/L_1)^\vee \hookrightarrow (L/L_2)^\vee \hookrightarrow \cdots \hookrightarrow L^\vee$. The corresponding ascending chain of submodules must stabilize since $L^\vee$ is noetherian over $R$, and it follows that the original chain $L_1 \supseteq L_2 \supseteq \cdots$ of submodules of $L$ also stabilizes. Thus $L$ is artinian.
(c) Assume that $L$ is $a$-torsion. One implication is from part (b). For the converse, assume that $L$ is artinian over $R$. From [14, Theorem 18.6(v)] we know that $\text{Hom}_{\hat{R}}(L, E)$ is noetherian over $R$, and Lemma 1.5(a) implies that $L^\vee = \text{Hom}_{\hat{R}}(L, E)$. Thus, Lemma 1.5(b) implies that $L^\vee$ is noetherian over $R$. □

Mini-max and Matlis reflexive modules

Definition 1.16. An $R$-module $M$ is mini-max if there is a noetherian submodule $N \subseteq M$ such that $M/N$ is artinian.

Definition 1.17. An $R$-module $M$ is Matlis reflexive provided that the natural biduality map $\delta_M: M \rightarrow M^{**}$, given by $\delta_M(x)(\psi) = \psi(x)$, is an isomorphism.

Fact 1.18. An $R$-module $M$ is Matlis reflexive if and only if it is mini-max and $R/\text{Ann}_R(M)$ is complete; see [4, Theorem 12].

Thus, if $M$ is mini-max over $R$, then $R \otimes_R M$ is Matlis reflexive over $R$.

Lemma 1.19. If $M$ is mini-max over $R$, then $\beta_i^R(M)$, $\mu_i^R(M) < \infty$ for all $i \geq 0$. 

Proof. We show that \( \mu_k^i(M) < \infty \) for all \( i \geq 0 \); then Remark 1.11(b) implies that \( \beta_k^i(M) < \infty \) for all \( i \geq 0 \). The noetherian case is standard. If \( M \) is artinian, then we have \( \mu_0^0 = \mu_0^0(M) < \infty \) by Lemma 1.14; since \( E^0 \) is artinian, an induction argument shows that \( \mu_k^i(M) < \infty \) for all \( i \geq 0 \). One deduces the mini-max case from the artinian and noetherian cases, using a long exact sequence.  \( \square \)

**Lemma 1.20.** Let \( L \) be an \( R \)-module such that \( R/\Ann_R(L) \) is complete. The following conditions are equivalent:

(i) \( L \) is Matlis reflexive over \( R \);
(ii) \( L \) is mini-max over \( \hat{R} \);
(iii) \( L \) is mini-max over \( R \); and
(iv) \( L \) is Matlis reflexive over \( R \).

Proof. The equivalences (i) \( \iff \) (ii) and (iii) \( \iff \) (iv) are from Fact 1.18. Note that conditions (iii) and (iv) make sense since \( L \) is an \( \hat{R} \)-module; see Fact 1.2.

(ii) \( \implies \) (iii) Assume that \( L \) is mini-max over \( R \), and fix a noetherian \( R \)-submodule \( N \subseteq L \) such that \( L/N \) is artinian over \( R \). As \( R/\Ann_R(L) \) is complete and surjects onto \( R/\Ann_R(N) \), we conclude that \( R/\Ann_R(N) \) is complete. Fact 1.2(d) and Lemma 1.3(a) imply that \( N \) is an \( \hat{R} \)-submodule. Similarly, Lemmas 1.3(b) and 1.14 imply that \( N \) is noetherian over \( \hat{R} \), and \( L/N \) is an artinian over \( \hat{R} \). Thus \( L \) is mini-max over \( \hat{R} \).

(iii) \( \implies \) (ii) Assume that \( L \) is mini-max over \( \hat{R} \), and fix a noetherian \( \hat{R} \)-submodule \( L' \subseteq L \) such that \( L/L' \) is artinian over \( \hat{R} \). Lemmas 1.3(b) and 1.14 imply that \( L' \) is noetherian over \( \hat{R} \), and \( L/L' \) is artinian over \( \hat{R} \), so \( L \) is mini-max over \( R \).  \( \square \)

**Lemma 1.21.** Let \( L \) be an \( R \)-module such that \( m^tL = 0 \) for some integer \( t \geq 1 \). Then the following conditions are equivalent:

(i) \( L \) is mini-max over \( R \) (equivalently, over \( \hat{R} \));
(ii) \( L \) is artinian over \( R \) (equivalently, over \( \hat{R} \));
(iii) \( L \) is noetherian over \( R \) (equivalently, over \( \hat{R} \)); and
(iv) \( L \) has finite length over \( R \) (equivalently, over \( \hat{R} \)).

Proof. Lemma 1.20 shows that \( L \) is mini-max over \( R \) if and only if it is mini-max over \( \hat{R} \). Also, \( L \) is artinian (resp., noetherian or finite length) over \( R \) if and only if it is artinian (resp., noetherian or finite length) over \( \hat{R} \) by Lemmas 1.14 and 1.3(b).

The equivalence of conditions (ii)–(iv) follows from an application of [7, Proposition 2.3.20] over the artinian ring \( R/m^t \). The implication (ii) \( \implies \) (i) is evident. For the implication (i) \( \implies \) (ii), assume that \( L \) is mini-max over \( R \). Given a noetherian submodule \( N \subseteq L \) such that \( L/N \) is artinian, the implication (iii) \( \implies \) (ii) shows that \( N \) is artinian; hence so is \( L \).  \( \square \)

**Definition 1.22.** A full subcategory of the category of \( R \)-modules is a Serre subcategory if it is closed under submodules, quotients, and extensions.

**Lemma 1.23.** The category of mini-max (resp., noetherian, artinian, finite length, or Matlis reflexive) \( R \)-modules is a Serre subcategory.

Proof. The noetherian, artinian, and finite length cases are standard, as is the Matlis reflexive case; see [7, p. 92, Exercise 2]. For the mini-max case, fix an exact sequence \( 0 \to L' \xrightarrow{f} L \xrightarrow{g} L'' \to 0 \). Identify \( L' \) with \( \Im(f) \). Assume first that \( L \) is mini-max, and fix a noetherian submodule \( N \) such that \( L/N \) is artinian. Then \( L' \cap N \) is noetherian, and the quotient \( L'/L' \cap N \equiv (L' + N)/N \) is artinian, since it is a submodule of \( L/N \). Thus \( L' \) is mini-max. Also, \((N + L')/L' \) is noetherian and \([L'/L' \cap N]/(L' + N)/L' \equiv L/N + L' \) is artinian, so \( L''/L' \) is mini-max.

Next, assume that \( L' \) and \( L'' \) are mini-max, and fix noetherian submodules \( N' \subseteq L' \) and \( N'' \subseteq L'' \) such that \( L'/N' \) and \( L''/N'' \) are artinian. Let \( x_1, \ldots, x_l \) be coset representatives in \( L \) of a generating set for \( N'' \). Let \( N = N' + Rx_1 + \ldots + Rx_l \). Then \( N \) is noetherian and the following commutative diagram has exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & N & \to & N' & \to & 0 \\
\downarrow & & & & & & \\
0 & \to & L' \cap N & \to & L' & \to & 0 \\
\downarrow & & & & & & \\
0 & \to & L' & \to & L'' & \to & 0.
\end{array}
\]

The sequence \( 0 \to L'/L' \cap N \to L/N \to L''/N'' \to 0 \) is exact by the Snake Lemma. The module \( L'/L' \cap N \) is artinian, being a quotient of \( L'/N' \). Since the class of artinian modules is closed under extensions, the module \( L/N \) is artinian. It follows that \( L \) is mini-max.  \( \square \)

The next two lemmas apply to the classes of modules from Lemma 1.23.

**Lemma 1.24.** Let \( \mathcal{C} \) be a Serre subcategory of the category of \( R \)-modules.

(a) Given an exact sequence \( L' \xrightarrow{f} L \xrightarrow{g} L'' \), if \( L' \subseteq \mathcal{C} \), then \( L \subseteq \mathcal{C} \).
(b) Given an \( R \)-complex \( X \) and an integer \( i \), if \( X_i \subseteq \mathcal{C} \), then \( H_i(X) \subseteq \mathcal{C} \).
(c) Given a noetherian \( R \)-module \( N \), if \( L \subseteq \mathcal{C} \), then \( \Ext^i_R(N, L), \Tor^i_R(N, L) \subseteq \mathcal{C} \).
**Proof.** (a) Assume that $L', L'' \in \mathcal{C}$. By assumption, $\text{Im}(f), \text{Im}(g) \in \mathcal{C}$. Using the exact sequence $0 \to \text{Im}(f) \to L \to \text{Im}(g) \to 0$, we conclude that $L$ is in $\mathcal{C}$.

(b) The module $H_i(X)$ is a subquotient of $X_i$, so it is in $\mathcal{C}$ by assumption.

(c) If $F$ is a minimal free resolution of $N$, then the modules in the complexes $\text{Hom}_R(F, L)$ and $F \otimes_R L$ are in $\mathcal{C}$, so their homologies are in $\mathcal{C}$ by part (b). □

**Lemma 1.25.** Let $R \to S$ be a local ring homomorphism, and let $\mathcal{C}$ be a Serre subcategory of the category of $S$-modules. Fix an $S$-module $L$, an $R$-module $L'$, and an $R$-submodule $L'' \subseteq L'$, and an index $i \geq 0$.

(a) If $\text{Ext}_R^i(L, L''), \text{Ext}_R^i(L', L') \in \mathcal{C}$, then $\text{Ext}_R^i(L', L'') \in \mathcal{C}$.

(b) If $\text{Ext}_R^i(L'', L), \text{Ext}_R^i(L'/L'', L) \in \mathcal{C}$, then $\text{Ext}_R^i(L', L) \in \mathcal{C}$.

(c) If $\text{Tor}_i^R(L, L'), \text{Tor}_i^R(L', L'') \in \mathcal{C}$, then $\text{Tor}_i^R(L, L') \in \mathcal{C}$.

**Proof.** We prove part (a); the other parts are proved similarly. Apply $\text{Ext}_R^i(L, -)$ to the exact sequence $0 \to L'' \to L' \to L'/L'' \to 0$ to obtain the next exact sequence:

$$\text{Ext}_R^i(L, L'') \to \text{Ext}_R^i(L, L') \to \text{Ext}_R^i(L', L'').$$

Since $L$ is an $S$-module, the maps in this sequence are $S$-module homomorphisms. Now, apply Lemma 1.24(a). □

2. **Properties of $\text{Ext}_R^i(M, -)$**

This section documents properties of the functors $\text{Ext}_R^i(M, -)$ where $M$ is a mini-max $R$-module.

**Noetherianness of $\text{Ext}_R^i(A, L)$**

**Lemma 2.1.** Let $A$ and $L$ be $R$-modules such that $A$ is artinian and $L$ is $m$-torsion.

(a) Then $\text{Hom}_R(L, A) = \text{Hom}_R(L, A) \cong \text{Hom}_R(A, L')$.

(b) If $L$ is artinian, then $\text{Hom}_R(L, A)$ is a noetherian $\hat{R}$-module.

**Proof.** (a) The first equality is from Lemma 1.5(a). For the second equality, the fact that $A$ is Matlis reflexive over $\hat{R}$ explains the first step below:

$$\text{Hom}_R(L, A) \cong \text{Hom}_R(L, A^{\omega}) \cong \text{Hom}_R(A^{\omega}, L') \cong \text{Hom}_R(A', L')$$

where $(\cdot)^\omega = \text{Hom}_R(\cdot, E)$. The second step follows from Hom–tensor adjointness, and the third step is from Lemma 1.5(a).

(b) If $L$ is artinian, then $L'$ and $A'$ are noetherian over $\hat{R}$, so $\text{Hom}_R(A', L')$ is also noetherian over $\hat{R}$. □

The next result contains part of Theorem 1 from the introduction. When $R$ is not complete, the example $\text{Hom}_R(E, E) \cong \hat{R}$ shows that $\text{Ext}_R^i(A, L)$ is not necessarily noetherian or artinian over $R$.

**Theorem 2.2.** Let $A$ and $L$ be $R$-modules such that $A$ is artinian. For each index $i \geq 0$ such that $\mu_i^R(L) < \infty$, the module $\text{Ext}_R^i(A, L)$ is a noetherian $R$-module.

**Proof.** Let $J$ be a minimal $R$-injective resolution of $L$. Remark 1.11(a) implies that $\Gamma_m^R(J) \cong E^{\mu_i^R(L)}$. Lemma 1.5(b) explains the first isomorphism below:

$$\text{Hom}_R(A, J)^\omega \cong \text{Hom}_R(A, \Gamma_m^R(J)^\omega) \cong \text{Hom}_R(A, E)^{\mu_i^R(L)}.$$  

**Lemma 2.1** implies that these are noetherian $\hat{R}$-modules. The differentials in the complex $\text{Hom}_R(A, \Gamma_m^R(J))$ are $\hat{R}$-linear because $A$ is an $R$-module. Thus, the subquotient $\text{Ext}_R^i(A, L)$ is a noetherian $\hat{R}$-module. □

**Corollary 2.3.** Let $A$ and $M$ be $R$-modules such that $A$ is artinian and $M$ is mini-max. For each index $i \geq 0$, the module $\text{Ext}_R^i(A, M)$ is a noetherian $\hat{R}$-module.

**Proof.** Apply Theorem 2.2 and Lemma 1.19. □

The next result contains part of Theorem 3 from the introduction.

**Corollary 2.4.** Let $A$ and $L$ be $R$-modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is complete and $A$ is artinian. For each index $i \geq 0$ such that $\mu_i^R(L) < \infty$, the module $\text{Ext}_R^i(A, L)$ is noetherian and Matlis reflexive over $R$ and $\hat{R}$.

**Proof.** Theorem 2.2 shows that $\text{Ext}_R^i(A, L)$ is noetherian over $\hat{R}$; so, it is Matlis reflexive over $\hat{R}$. As $\text{Ann}_R(A) + \text{Ann}_R(L) \subseteq \text{Ann}_R(\text{Ext}_R^i(A, L))$, Lemmas 1.3(b) and 1.20 imply that $\text{Ext}_R^i(A, L)$ is noetherian and Matlis reflexive over $R$. □

**Corollary 2.5.** Let $A$ and $L$ be $R$-modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is artinian and $A$ is artinian. Given an index $i \geq 0$ such that $\mu_i^R(L) < \infty$, one has $\text{len}_R(\text{Ext}_R^i(A, L)) < \infty$.

**Proof.** Apply Theorem 2.2 and Lemma 1.21. □
Matlis reflexivity of \( \text{Ext}^s_R(M, M') \)

**Theorem 2.6.** Let \( A \) and \( M \) be \( R \)-modules such that \( A \) is artinian and \( M \) is mini-max. For each \( s \geq 0 \), the module \( \text{Ext}^s_R(M, A) \) is Matlis reflexive over \( \hat{R} \).

**Proof.** Fix a noetherian submodule \( N \subseteq M \) such that \( M/N \) is artinian. Since \( A \) is artinian, it is an \( \hat{R} \)-module. Corollary 2.3 implies that \( \text{Ext}^s_R(M/N, A) \) is a noetherian \( \hat{R} \)-module. As \( \text{Ext}^s_R(N, A) \) is artinian, Lemma 1.25(b) says that \( \text{Ext}^s_R(M, A) \) is a mini-max \( \hat{R} \)-module and hence is Matlis reflexive over \( \hat{R} \) by Fact 1.18. \( \Box \)

**Theorem 2.7.** Let \( M \) and \( N' \) be \( R \)-modules such that \( M \) is mini-max and \( N' \) is noetherian. Fix an index \( s \geq 0 \). If \( R/(\text{Ann}_R(M) + \text{Ann}_R(N')) \) is complete, then \( \text{Ext}^s_R(M, N') \) is noetherian and Matlis reflexive over \( R \) and \( \hat{R} \).

**Proof.** Fix a noetherian submodule \( N \subseteq M \) such that \( M/N \) is artinian. If the ring \( R/(\text{Ann}_R(M) + \text{Ann}_R(N')) \) is complete, then so is \( R/(\text{Ann}_R(M/N) + \text{Ann}_R(N')) \). Corollary 2.4 implies that \( \text{Ext}^s_R(M/N, N') \) is noetherian over \( R \). Since \( \text{Ext}^s_R(N, N') \) is noetherian over \( R \), Lemma 1.25(b) implies that \( \text{Ext}^s_R(M, N') \) is noetherian over \( R \). As \( R/(\text{Ann}_R(\text{Ext}^s_R(M, N'))) \) is complete, Fact 1.18 implies that \( \text{Ext}^s_R(M, N') \) is also Matlis reflexive over \( R \). Thus \( \text{Ext}^s_R(M, N') \) is noetherian and Matlis reflexive over \( \hat{R} \) by Lemmas 1.3(b) and 1.20. \( \Box \)

**Theorem 2.8.** Let \( M \) and \( M' \) be mini-max \( R \)-modules, and fix an index \( s \geq 0 \).

(a) If \( R/(\text{Ann}_R(M) + \text{Ann}_R(M')) \) is complete, then \( \text{Ext}^s_R(M, M') \) is Matlis reflexive over \( R \) and \( \hat{R} \).

(b) If \( R/(\text{Ann}_R(M) + \text{Ann}_R(M')) \) is artinian, then \( \text{Ext}^s_R(M, M') \) has finite length.

**Proof.** Fix a noetherian submodule \( N \subseteq M \) such that \( M/N \) is artinian. If the ring \( R/(\text{Ann}_R(M) + \text{Ann}_R(M')) \) is complete, then \( R/(\text{Ann}_R(M/N) + \text{Ann}_R(N')) \) is complete. Theorem 2.7 implies that \( \text{Ext}^s_R(M/N, N') \) is noetherian and Matlis reflexive over \( R \). Therefore, \( \text{Ext}^s_R(M, N') \) is Matlis reflexive over \( \hat{R} \); hence, it is Matlis reflexive over \( R \) by Lemma 1.20. Thus, Lemmas 1.25(a) and 1.20 imply that \( \text{Ext}^s_R(M, M') \) is Matlis reflexive over \( R \) and \( \hat{R} \).

(b) This follows from part (a), because of Fact 1.18 and Lemma 1.21. \( \Box \)

A special case of the next result can be found in [3, Theorem 3].

**Corollary 2.9.** Let \( M \) and \( M' \) be \( R \)-modules such that \( M \) is mini-max and \( M' \) is Matlis reflexive. For each index \( s \geq 0 \), the modules \( \text{Ext}^s_R(M, M') \) and \( \text{Ext}^s_R(M', M) \) are Matlis reflexive over \( R \) and \( \hat{R} \).

**Proof.** Apply Theorem 2.8(a) and Fact 1.18. \( \Box \)

**Length Bounds for** \( \text{Hom}_R(A, L) \)

**Lemma 2.10.** Let \( A \) and \( L \) be \( R \)-modules such that \( A \) is artinian and \( m^n \Gamma_m(L) = 0 \) for some \( n \geq 1 \). Fix an index \( t \geq 0 \) such that \( m^t A = m^{t+1} A \), and let \( s \) be an integer such that \( s \geq \min(n, t) \). Then

\[
\text{Hom}_R(A, L) \cong \text{Hom}_R(A/m^t A, L) \cong \text{Hom}_R(A/m^s A, 0 :\text{m}^t).
\]

**Proof.** Given any map \( \psi \in \text{Hom}_R(A/m^t A, L) \), the image of \( \psi \) is annihilated by \( m^s \). That is, \( \text{Im}(\psi) \subseteq (0 :\text{m}^s) \); hence \( \text{Hom}_R(A/m^t A, L) \cong \text{Hom}_R(A/m^s A, 0 :\text{m}^t) \). In the next sequence, the first and third isomorphisms are from Lemma 1.5(b):

\[
\text{Hom}_R(A, L) \cong \text{Hom}_R(A, \Gamma_m(L)) \cong \text{Hom}_R(A/m^t A, \Gamma_m(L)) \cong \text{Hom}_R(A/m^s A, \Gamma_m(L)).
\]

For the second isomorphism, we argue by cases. If \( s \geq n \), then we have \( m^n \Gamma_m(L) = 0 \) because \( m^n \Gamma_m(L) = 0 \), and the isomorphism is evident. If \( s < n \), then we have \( n > s \geq t \), so \( m^t A = m^s A \) since \( m^t A = m^{t+1} A \); it follows that \( \text{Hom}_R(A, \Gamma_m(L)) \cong \text{Hom}_R(A/m^t A, \Gamma_m(L)) \cong \text{Hom}_R(A/m^s A, \Gamma_m(L)) \). \( \Box \)

For the next result, the example \( \text{Hom}_R(E, E) \cong \hat{R} \) shows that the condition \( m^n \Gamma_m(L) = 0 \) is necessary.

**Theorem 2.11.** Let \( A \) and \( L \) be \( R \)-modules such that \( A \) is artinian and \( m^n \Gamma_m(L) = 0 \) for some \( n \geq 1 \). Fix an index \( t \geq 0 \) such that \( m^t A = m^{t+1} A \), and let \( s \) be an integer such that \( s \geq \min(n, t) \). Then there is an inequality

\[
\text{len}_R(\text{Hom}_R(A, L)) \leq \beta_0^R(A) \text{len}_R(0 :\text{m}^s).
\]

Here, we use the convention \( 0 :\infty = 0 \).

**Proof.** We deal with the degenerate case first. If \( \beta_0^R(A) = 0 \), then \( A/m^t A = 0 \), so

\[
\text{Hom}_R(A, L) \cong \text{Hom}_R(A/m^t A, L) = \text{Hom}_R(0, L) = 0
\]

by Lemma 2.10. So, we assume for the rest of the proof that \( \beta_0^R(A) \neq 0 \). We also assume without loss of generality that \( \text{len}_R(0 :\text{m}^s) < \infty \).
Lemma 2.10 explains the first step in the following sequence:
\[
\text{len}_\beta(\text{Hom}_R(A, L)) = \text{len}_\beta(\text{Hom}_R(A/m^iA, (0 :_LM^i))) \\
\leq \beta^R_0(A/m^iA) \text{len}_\beta(0 :_LM^i) \\
= \beta^R_0(A) \text{len}(0 :_LM^i).
\]
The second step can be proved by induction on $\beta^R_0(A/m^iA)$ and $\text{len}_\beta(0 :_LM^i)$. □

The next result can also be obtained as a corollary to [8, Proposition 6.1]. Example 6.3 shows that $\text{len}_\beta(\text{Ext}_R^i(A, N))$ can be infinite when $i \geq 1$.

**Corollary 3.2.** If $A$ and $N$ are $R$-modules such that $A$ is artinian and $N$ is noetherian, then $\text{len}_\beta(\text{Hom}_R(A, N)) < \infty$.

**Proof.** Apply Theorem 2.11 and Lemma 1.19. □

3. Properties of $\text{Tor}^R_i(M, -)$

This section focuses on properties of the functors $\text{Tor}^R_i(M, -)$ where $M$ is a mini-max $R$-module.

**Artinniness of $\text{Tor}^R_i(A, L)$**

The next result contains part of Theorem 1 from the introduction. Recall that a module is artinian over $R$ if and only if it is artinian over $\widehat{R}$; see Lemma 1.14.

**Theorem 3.1.** Let $A$ and $L$ be $R$-modules such that $A$ is artinian. For each index $i \geq 0$ such that $\beta^R_i(L) < \infty$, the module $\text{Tor}^R_i(A, L)$ is artinian.

**Proof.** Lemma 1.13(b) implies that $\mu_{\infty}(L) = \beta^R_0(L) < \infty$. By Remark 1.9, we have $\text{Ext}^R_0(A, L^\vee) \cong \text{Tor}^R_0(A, L)^\vee$. Thus, $\text{Tor}^R_0(A, L)^\vee$ is a noetherian $\widehat{R}$-module by Theorem 2.2, and we conclude that $\text{Tor}^R_0(A, L)$ is artinian by Lemma 1.15(b). □

For the next result, the example $E \otimes_R R \cong E$ shows that $\text{Tor}^R_0(A, L)$ is not necessarily noetherian over $R$ or $\widehat{R}$.

**Corollary 3.2.** Let $A$ and $M$ be $R$-modules such that $A$ is artinian and $M$ mini-max. For each index $i \geq 0$, the module $\text{Tor}^R_i(A, M)$ is artinian.

**Proof.** Apply Theorem 3.1 and Lemma 1.19. □

The proofs of the next two results are similar to those of Corollaries 2.4 and 2.5. The first result contains part of Theorem 3 from the introduction.

**Corollary 3.3.** Let $A$ and $L$ be $R$-modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is complete and $A$ is artinian. For each index $i \geq 0$ such that $\beta^R_i(L) < \infty$, the module $\text{Tor}^R_i(A, L)$ is artinian and Matlis reflexive over $R$ and $\widehat{R}$.

**Corollary 3.4.** Let $A$ and $L$ be $R$-modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is artinian and $A$ is artinian. Given an index $i \geq 0$ such that $\beta^R_i(L) < \infty$, one has $\text{len}_\beta(\text{Tor}^R_i(A, L)) < \infty$.

**Theorem 3.5.** Let $M$ and $M'$ be mini-max $R$-modules, and fix an index $i \geq 0$.

(a) The $R$-module $\text{Tor}^R_i(M, M')$ is mini-max over $R$.

(b) If $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is complete, then $\text{Tor}^R_i(M, M')$ is Matlis reflexive over $R$ and $\widehat{R}$.

(c) If $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is artinian, then $\text{Tor}^R_i(M, M')$ has finite length.

**Proof.** (a) Choose a noetherian submodule $N \subseteq M$ such that $M/N$ is artinian. Lemmas 1.23 and 1.24(c) say that $\text{Tor}^R_i(N, M')$ is mini-max. Corollary 3.2 implies that $\text{Tor}^R_i(M/N, M')$ mini-max, so $\text{Tor}^R_i(M, M')$ is mini-max by Lemma 1.25(c).

Parts (b) and (c) now follow from Lemmas 1.20 and 1.21. □

A special case of the next result is contained in [3, Theorem 3].

**Corollary 3.6.** Let $M$ and $M'$ be $R$-modules such that $M$ is mini-max and $M'$ is Matlis reflexive. For each index $i \geq 0$, the module $\text{Tor}^R_i(M, M')$ is Matlis reflexive over $R$ and $\widehat{R}$.

**Proof.** Apply Theorem 3.5(b) and Fact 1.18. □
Length Bounds for $A \otimes_R L$

**Lemma 3.7.** Let $A$ be an artinian module, and let $a$ be a proper ideal of $R$. Fix an integer $t \geq 0$ such that $a^iA = a^{i+1}A$. Given an $a$-torsion $R$-module $L$, one has

$$A \otimes_R L \cong (A/a^iA) \otimes_R L \cong (A/a^iA) \otimes_R (L/a^iL).$$

**Proof.** The isomorphism $(A/a^iA) \otimes_R L \cong (A/a^iA) \otimes_R (L/a^iL)$ is from the following:

$$(A/a^iA) \otimes_R L \cong [(A/a^iA) \otimes_R (R/a^i) \otimes_R L]
\cong (A/a^iA) \otimes_R (L/a^iL).$$

For the isomorphism $A \otimes_R L \cong (A/a^iA) \otimes_R L$, consider the exact sequence:

$$0 \to a^iA \to A \to A/a^iA \to 0.$$

The exact sequence induced by $- \otimes_R L$ has the form

$$(a^iA) \otimes_R L \to A \otimes_R L \to (A/a^iA) \otimes_R L \to 0. \tag{3.7.1}$$

The fact that $L$ is $a$-torsion and $a^iA = a^{i+1}A$ for all $i \geq 1$ implies that $(a^iA) \otimes_R L = 0$, so the sequence (3.7.1) yields the desired isomorphism. \hfill \□

The example $E \otimes_R R \cong R$ shows that the $m$-torsion assumption on $L$ is necessary in the next result.

**Theorem 3.8.** Let $A$ be an artinian $R$-module, and let $L$ be an $m$-torsion $R$-module. Fix an integer $t \geq 0$ such that $m^iA = m^{i+1}A$. Then there are inequalities

$$\text{len}_R(A \otimes_R L) \leq \text{len}_R(A/m^tA) \beta_0^R(L) \tag{3.8.1}$$

$$\text{len}_R(A \otimes_R L) \leq \beta_0^R(A \otimes_R L/m^tL). \tag{3.8.2}$$

Here we use the convention $0 \cdot \infty = 0$.

**Proof.** From Lemma 3.7 we have

$$A \otimes_R L \cong (A/m^tA) \otimes_R (L/m^tL). \tag{3.8.3}$$

**Lemmas 3.19 and 1.21** imply that $\text{len}_R(A/m^tA) < \infty$ and $\beta_0^R(A) < \infty$.

For the degenerate cases, first note that $\text{len}_R(A/m^tA) = 0$ if and only if $\beta_0^R(A) = 0$. When $\text{len}_R(A/m^tA) = 0$, the isomorphism (3.8.3) implies that $A \otimes_R L = 0$; hence the desired inequalities. Thus, we assume without loss of generality that $1 \leq \beta_0^R(A) \leq \text{len}_R(A/m^tA)$. Further, we assume that $\beta_0^R(L) < \infty$.

The isomorphism (3.8.3) provides the first step in the next sequence:

$$\text{len}_R(A \otimes_R L) = \text{len}_R((A/m^tA) \otimes_R (L/m^tL)) \leq \text{len}_R(A/m^tA) \beta_0^R(L).$$

The second step in this sequence can be verified by induction on $\text{len}_R(A/m^tA)$ and $\beta_0^R(L)$. This explains the inequality (3.8.1), and (3.8.2) is verified similarly. \hfill \□

The next corollary recovers [8, Proposition 6.1]. Note that Example 6.4 shows that $\text{len}_R(\text{Tor}_1^R(A, A'))$ can be infinite when $i > 1$.

**Corollary 3.9.** If $A$ and $A'$ are artinian $R$-modules, then $\text{len}_R(A \otimes_R A') < \infty$.

**Proof.** Apply Theorem 3.8 and Lemmas 1.19 and 1.21. (Alternatively, apply Corollary 2.12 and Matlis duality.) \hfill \□

4. The Matlis dual of $\text{Ext}_R^t(L, L')$

This section contains the proof of Theorem 4 from the introduction; see Corollary 4.11. Most of the section is devoted to technical results for use in the proof.

**Lemma 4.1.** Let $L$ be an $R$-module. If $I$ is an $R$-injective resolution of $L$, and $J$ is an $R$-injective resolution of $\widehat{R} \otimes_R L$, then there is a homotopy equivalence $I_m^t(I) \to \Gamma_m^t(I) = \Gamma_m^t(J).

**Proof.** Each injective $\widehat{R}$-module $J$ is injective over $R$; this follows from the isomorphism $\text{Hom}_R(\cdot, J') \cong \text{Hom}_R(\cdot, \text{Hom}_R(\widehat{R}, J')) \cong \text{Hom}_R(\widehat{R} \otimes_R \cdot, J')$ since $\widehat{R}$ is flat over $R$. Hence, there is a lift $f : I \to J$ of the natural map $\xi : L \to \widehat{R} \otimes_R L$. This lift is a chain map of $R$-complexes.
We show that the induced map $\Gamma_m(f) : \Gamma_m(l) \to \Gamma_m(j) = \Gamma_m(R)$ is a homotopy equivalence. As $\Gamma_m(l)$ and $\Gamma_m(j)$ are bounded above complexes of injective $R$-modules, it suffices to show that $\Gamma_m(f)$ induces an isomorphism on homology in each degree. The induced map on homology is compatible with the following sequence:

$$H^i(\Gamma_m(I)) \cong H^i_m(l) \xrightarrow{\delta} H^i_m(R \otimes_R L) \cong H^i(\Gamma_m(j)).$$

The map $H^i_m(\xi) : H^i_m(l) \to H^i_m(R \otimes_R L)$ is an isomorphism (see the proof of [6, Proposition 3.5.4(d)]) so we have the desired homotopy equivalence. □

**Lemma 4.2.** Let $L$ and $L'$ be $R$-modules such that $L$ is $m$-torsion. Then for each index $i \geq 0$, there are $R$-module isomorphisms

$$\text{Ext}^i_R(L, L') \cong \text{Ext}^i_R(R, L') \cong \text{Ext}^i_R(R, L \otimes_R L').$$

**Proof.** Let $I$ be an $R$-injective resolution of $L$, and let $J$ be an $R$-injective resolution of $R \otimes_R L'$. Because $L$ is $m$-torsion, Lemma 1.5(b) explains the first, third and sixth steps in the next display:

$$\text{Hom}_R(l, I) \cong \text{Hom}_R(l, \Gamma_m(I)) \sim \text{Hom}_R(l, \Gamma_m(j)) \cong \text{Hom}_R(l, J).$$

$$\text{Hom}_R(R, I \otimes_R J) \cong \text{Hom}_R(R, I \otimes_R J) \cong \text{Hom}_R(R, I \otimes_R J).$$

The homotopy equivalence in the second step is from Lemma 4.1. The fifth step is from Lemma 1.5(a). Since $L$ is $m$-torsion, it is an $R$-module, so the isomorphisms and the homotopy equivalence in this sequence are $R$-linear. In particular, the complexes $\text{Hom}_R(l, I)$ and $\text{Hom}_R(l, J)$ and $\text{Hom}_R(l, I \otimes_R J)$ have isomorphic cohomology over $R$; so one has the desired isomorphisms. □

The next result contains Theorem 2 from the introduction. It shows, for instance, that given artinian $R$-modules $A$ and $A'$, there are noetherian $\tilde{R}$-modules $N$ and $N'$ such that $\text{Ext}^i_R(A, A') \cong \text{Ext}^i_R(N, N')$; thus, it provides an alternate proof of Corollary 2.3.

**Theorem 4.3.** Let $A$ and $M$ be $R$-modules such that $A$ is artinian and $M$ is mini-max. Then for each index $i \geq 0$, we have $\text{Ext}^i_R(A, M) \cong \text{Ext}^i_R(M', A')$.

**Proof.** Case 1: $R$ is complete. Let $F$ be a free resolution of $A$. It follows that each $F_i$ is flat, so the complex $F^\vee$ is an injective resolution of $A^\vee$; see [7, Theorem 3.2.9]. We obtain the isomorphism $\text{Ext}^i_R(R, M) \cong \text{Ext}^i_R(M^\vee, A^\vee)$ by taking cohomology in the next sequence:

$$\text{Hom}_R(F, M) \cong \text{Hom}_R(F, M^\vee) \cong \text{Hom}_R(M^\vee, F^\vee).$$

The first step follows from the fact that $M$ is Matlis reflexive; see Fact 1.18. The second step is from Hom–tensor adjointness Case 2: the general case. The first step below is from Lemma 4.2:

$$\text{Ext}^i_R(A, M) \cong \text{Ext}^i_R(A, R \otimes_R M) \cong \text{Ext}^i_R(R \otimes_R M^\vee, A^\vee) \cong \text{Ext}^i_R(M^\vee, A^\vee).$$

Here $(\cdot)^\vee \cong \text{Hom}_R(\cdot, E)$. Since $M$ is mini-max, it follows that $R \otimes_R M$ is mini-max over $R$. Thus, the second step is from Case 1. For the third step use Hom–tensor adjointness and Lemma 1.5(a) to see that $(R \otimes_R M)^\vee \cong M^\vee$ and $A^\vee \cong A^\vee$. □

**Fact 4.4.** Let $L$ and $L'$ be $R$-modules, and fix an index $i \geq 0$. Then the following diagram commutes, where the unlabeled isomorphism is from Remark 1.9:

$$\begin{array}{c}
\text{Ext}^i_R(L', L) \xrightarrow{\delta} \text{Ext}^i_R(L', L)^\vee \\
\downarrow \text{Ext}^i_R(L', A) \quad \downarrow (\Theta^i_{L,k})^\vee \\
\text{Ext}^i_R(L', L'^\vee) \xrightarrow{\cong} \text{Tor}^i_{L'}(L', L'^\vee).
\end{array}$$

**Lemma 4.5.** Let $L$ be an $R$-module, and fix an index $i \geq 0$. If $\mu^i_L(L) < \infty$, then the map $\text{Ext}^i_R(k, d_i) : \text{Ext}^i_R(k, L) \to \text{Ext}^i_R(k, L'^\vee)$ is an isomorphism.

**Proof.** The assumption $\mu^i_L(L) < \infty$ says that $\text{Ext}^i_R(k, L)$ is a finite dimensional $k$-vector space, so it is Matlis reflexive over $R$; that is, the map

$$\delta : \text{Ext}^i_R(k, L) \to \text{Ext}^i_R(k, L'^\vee)$$

is an isomorphism. Since $k$ is finitely generated, Remark 1.9 implies that

$$\Theta^i_{kd} : \text{Tor}^i_{k}(L, L'^\vee) \to \text{Ext}^i_R(k, L'^\vee)$$

is an isomorphism. Hence $(\Theta^i_{kd})^\vee$ is also an isomorphism. Using Fact 4.4 with $L' = k$, we conclude that $\text{Ext}^i_R(k, d_i)$ is an isomorphism, as desired. □
Lemma 4.6. Let \( A \) and \( L \) be \( R \)-modules such that \( A \) is artinian. Fix an index \( i \geq 0 \) such that \( \mu^-_R(i, L) \) and \( \mu^+_R(i, L) \) are finite. Then the map

\[
\text{Ext}^i_R(A, \delta_i) : \text{Ext}^i_R(A, L) \to \text{Ext}^i_R(A, L^{\vee})
\]

is an isomorphism.

**Proof.** Lemma 4.5 implies that for \( t = i - 1, i, i + 1 \) the maps

\[
\text{Ext}^i_R(k, \delta_i) : \text{Ext}^i_R(k, L) \to \text{Ext}^i_R(k, L^{\vee})
\]

are isomorphisms. As the biduality map \( \delta_i \) is injective, we have an exact sequence

\[
0 \to L \to L^{\vee} \to \text{Coker} \delta_i \to 0. \tag{4.6.1}
\]

Using the long exact sequence associated to \( \text{Ext}^i_R(k, -) \), we conclude that for \( t = i - 1, i \) we have \( \text{Ext}^i_R(k, \text{Coker} \delta_i) = 0 \). In other words, we have \( \mu^+_R(\text{Coker} \delta_i) = 0 \).

Let \( J \) be a minimal injective resolution of \( \text{Coker} \delta_i \). The previous paragraph shows that for \( t = i - 1, i \) the module \( J^t \) does not have \( E \) as a summand by Remark 1.11(a). That is, we have \( \Gamma^t_\infty(J^t) = 0 \), so Lemma 1.5(b) implies that

\[
\text{Hom}_R(A, J^t) \cong \text{Hom}_R(A, \Gamma^t_\infty(J^t)) = 0.
\]

It follows that \( \text{Ext}^i_R(A, \text{Coker}(\delta_i)) = 0 \) for \( t = i - 1, i \). From the long exact sequence associated to \( \text{Ext}^i_R(A, -) \) with respect to (4.6.1), it follows that \( \text{Ext}^i_R(A, \delta_i) \) is an isomorphism, as desired. \( \square \)

We are now ready to tackle the main results of this section.

Theorem 4.7. Let \( A \) and \( L \) be \( R \)-modules such that \( A \) is artinian. Fix an index \( i \geq 0 \) such that \( \mu^-_R(i, L) \) and \( \mu^+_R(i, L) \) are finite.

(a) There is an \( R \)-module isomorphism \( \text{Ext}^i_R(A, L^{\vee}) \cong \text{Tor}^i_R(A, L^{\vee}) \) where \((-)^{\vee} = \text{Hom}_R(-, E)\).

(b) If \( R/(\text{Ann}_R(A) + \text{Ann}_R(L)) \) is complete, then \( \theta^i_{M} \) provides an isomorphism \( \text{Tor}^i_R(A, L^{\vee}) \cong \text{Ext}^i_R(A, L^{\vee}) \).

**Proof.** (b) Corollary 2.4 and Lemma 4.6 show that the maps

\[
\delta_{\text{Ext}^i_R(A, L)} : \text{Ext}^i_R(A, L) \to \text{Ext}^i_R(A, L^{\vee})
\]

are isomorphisms. Fact 4.4 implies that \((\delta_{\text{Ext}^i_R(A, L)})^{\vee} \) is an isomorphism, so we conclude that \( \theta^i_{M} \) is also an isomorphism.

(a) Lemma 4.2 explains the first step in the next sequence:

\[
\text{Ext}^i_R(A, L^{\vee}) \cong \text{Ext}^i_R(A, \widehat{R} \otimes_R L^{\vee})
\]

\[
\cong \text{Tor}^i_R(A, (\widehat{R} \otimes_R L^{\vee}))
\]

\[
\cong \text{Tor}^i_R(A, (\widehat{R} \otimes_R L)^{\vee})
\]

\[
\cong \text{Tor}^i_R(A, L^{\vee}).
\]

The second step is from part (b), as \( \widehat{R} \) is complete and \( \mu^+_R(\widehat{R} \otimes_R L) = \mu^+_R(L) < \infty \) for \( t = i - 1, i, i + 1 \). The fourth step is from Hom–tensor adjointness. For the third step, let \( P \) be a projective resolution of \( A \) over \( R \). Since \( \widehat{R} \) is flat over \( R \), the complex \( \widehat{R} \otimes_R P \) is a projective resolution of \( \widehat{R} \otimes_R A \cong A \) over \( \widehat{R} \); see Lemma 1.4(a). Thus, the third step follows from the isomorphism \( (\widehat{R} \otimes_R P) \otimes_R \widehat{R} \otimes_R L^{\vee} \cong P \otimes_R (\widehat{R} \otimes_R L)^{\vee}. \)

□

Question 4.8. Do the conclusions of Lemma 4.6 and Theorem 4.7 hold when one only assumes that \( \mu^+_R(L) \) is finite?

Corollary 4.9. Let \( A \) and \( M \) be \( R \)-modules such that \( A \) is artinian and \( M \) is mini-max. For each index \( i \geq 0 \), one has \( \text{Ext}^i_R(A, M^{\vee}) \cong \text{Tor}^i_R(A, M^{\vee}) \), where \((-)^{\vee} = \text{Hom}_R(-, E)\).

**Proof.** Apply Theorem 4.7(a) and Lemma 1.19. \( \square \)

Theorem 4.10. Let \( M \) and \( M' \) be mini-max \( R \)-modules, and fix an index \( i \geq 0 \). If \( R/(\text{Ann}_R(M) + \text{Ann}_R(M')) \) is complete, then \( \theta^i_{M^{\vee}M'} \) is an isomorphism, so

\[
\text{Ext}^i_R(M, M^{\vee}) = \text{Ext}^i_R(M, M')^{\vee} \cong \text{Tor}^i_R(M, M')^{\vee}
\]

where \((-)^{\vee} = \text{Hom}_R(-, E)\).
Lemma 5.2. Let $A$ be an artinian $R$-module such that $R$ is flat over $R$. Then, it remains to show that $\Theta_{MM'}^i$ is an isomorphism. Case 1: $M$ is noetherian. In the next sequence, the first and last steps are from $\text{Hom}^{-}$-tensor adjointness. The second step is standard since $M$ is noetherian:

$$\text{Ext}_R^i(M, M')^\vee \cong (\widehat{R} \otimes_R \text{Ext}_R^i(M, M'))^\vee$$

$$\cong \text{Ext}_R^i(\widehat{R} \otimes_R M, \widehat{R} \otimes_R M')^\vee$$

$$\cong \text{Tor}_R^i(\widehat{R} \otimes_R M, (\widehat{R} \otimes_R M')^\vee)$$

$$\cong \text{Tor}_R^i(M, (\widehat{R} \otimes_R M')^\vee)$$

$$\cong \text{Tor}_R^i(M, M^\vee).$$

Since $M$ and $M'$ are mini-max over $R$, the modules $\widehat{R} \otimes_R M$ and $\widehat{R} \otimes_R M'$ are Matlis reflexive over $\widehat{R}$; see Fact 1.18. Thus [2, Theorem 4(c)] explains the third step. The fourth step is from the fact that $\widehat{R}$ is flat over $R$. Since these isomorphisms are compatible with $\Theta_{MM'}^i$, it follows that $\Theta_{MM'}^i$ is an isomorphism.

Case 2: the general case. Since $M$ is mini-max over $R$, there is an exact sequence of $R$-modules homomorphisms

$$0 \to N \to M \to A \to 0$$

such that $N$ is noetherian and $A$ is artinian. The long exact sequences associated to $\text{Tor}_R^i(\_, M^\vee)$ and $\text{Ext}_R^i(\_, M')$ fit into the following commutative diagram:

$$\cdots \to \text{Tor}_R^i(N, M^\vee) \to \text{Tor}_R^i(M, M^\vee) \to \text{Tor}_R^i(A, M^\vee) \to \cdots$$

$$\cdots \to \text{Ext}_R^i(N, M')^\vee \to \text{Ext}_R^i(M, M')^\vee \to \text{Ext}_R^i(A, M')^\vee \to \cdots$$

Case 1 shows that $\Theta_{N,M'}^i$ and $\Theta_{M,M'}^{i+1}$ are isomorphisms. Theorem 4.7(b) implies that $\Theta_{AM'}^i$ and $\Theta_{A'M'}^{i+1}$ are isomorphisms. Hence, the Five Lemma shows that $\Theta_{AM'}^i$ is an isomorphism. □

The next result contains Theorem 4 from the introduction. A special case of it can be found in [3, Theorem 3].

Corollary 4.11. Let $M$ and $M'$ be mini-max $R$-modules, and fix an index $i \geq 0$. If either $M$ or $M'$ is Matlis reflexive, then $\Theta_{MM'}^i$ is an isomorphism, so one has $\text{Ext}_R^i(M, M')^\vee = \text{Ext}_R^i(M, M')^\vee \cong \text{Tor}_R^i(M, M'^\vee)$, where $(-)^\vee = \text{Hom}_R(\_, E)$. Proof. Apply Theorem 4.10 and Fact 1.18. □

The next example shows that the modules $\text{Ext}_R^i(L, L'^\vee)$ and $\text{Tor}_R^i(L, L'^\vee)$ are not isomorphic in general.

Example 4.12. Assume that $R$ is not complete. We have $\text{Ann}_R(E) = 0$, so the ring $R/\text{Ann}_R(E) \cong R$ is not complete, by assumption. Thus, Fact 1.18 implies that $E$ is not Matlis reflexive, that is, the biduality map $\delta_E : E \leftrightarrow E'^{\vee}$ is not an isomorphism. Since $E'^{\vee}$ is injective, we have $E'^{\vee} \cong E \oplus J$ for some non-zero injective $R$-module $J$. The uniqueness of direct sum decompositions of injective $R$-modules implies that $E'^{\vee} \not\cong E$. This provides the second step below:

$$\text{Hom}_R(E, E'^{\vee}) \cong E'^{\vee} \not\cong E \cong E \otimes_R \widehat{R} \cong E \otimes_R E'^{\vee}.$$  

The third step is from Lemma 1.4(a), and the remaining steps are standard.

5. Vanishing of Ext and Tor

In this section we describe the sets of associated primes of $\text{Hom}_R(A, M)$ and attached primes of $A \otimes_R M$ over $\widehat{R}$. The section concludes with some results on the related topic of vanishing for $\text{Ext}_R^i(A, M)$ and $\text{Tor}_R^i(A, M)$.

Associated and attached primes

The following is dual to the notion of associated primes of noetherian modules; see, e.g., [13] or [14, Appendix to §6] or [16].

Definition 5.1. Let $A$ be an artinian $R$-module. A prime ideal $p \in \text{Spec}(R)$ is attached to $A$ if there is a submodule $A' \subseteq A$ such that $p = \text{Ann}_R(A/A')$. We let $\text{Att}_R(A)$ denote the set of prime ideals attached to $A$.

Lemma 5.2. Let $A$ be an artinian $R$-module such that $R/\text{Ann}_R(A)$ is complete, and let $N$ be a noetherian $R$-module. There are equalities

$$\text{Supp}_R(A^\vee) = \bigcup_{p \in \text{Att}_R(A^\vee)} V(p) = \bigcup_{p \in \text{Att}_R(A)} V(p)$$

$$\text{Att}_R(N^\vee) = \text{Ass}_R(N)$$

$$\text{Att}_R(A) = \text{Ass}_R(A^\vee).$$
Proof. The R-module $A^\vee$ is noetherian by Lemma 1.15(c), so the first equality is standard, and the second equality follows from the fourth one. The third equality is from [18, (2.3) Theorem]. This also explains the second step in the next sequence

$$\text{Att}_R(A) = \text{Att}_R(A^\vee) = \text{Ass}_R(A^\vee)$$

since $A^\vee$ is noetherian. The first step in this sequence follows from the fact that $A$ is Matlis reflexive; see Fact 1.18. □

The next proposition can also be deduced from a result of Melkersson and Schenzel [15, Proposition 5.2].

**Proposition 5.3.** Let $A$ and $L$ be $R$-modules such that $\mu_R^0(L) < \infty$ and $A$ is artinian. Then

$$\text{Ass}_R(\text{Hom}_R(A, L)) = \text{Ass}_R(A^\vee) \cap \text{Supp}_R(\Gamma_m(L)^\vee) = \text{Att}_R(A) \cap \text{Supp}_R(\Gamma_m(L)^\vee).$$

**Proof.** The assumption $\mu_R^0(L) < \infty$ implies that $\Gamma_m(L)$ is artinian. This implies that $\Gamma_m(L)^\vee$ is a noetherian $\widetilde{R}$-module, so a result of Bourbaki [5, IV 1.4 Proposition 10] provides the third equality in the next sequence; see also [6, Exercise 1.2.27]:

$$\text{Ass}_R(\text{Hom}_R(A, L)) = \text{Ass}_R(\text{Hom}_R(A, \Gamma_m(L)))
= \text{Ass}_R(\text{Hom}_R(\Gamma_m(L)^\vee, A^\vee))
= \text{Ass}_R(A^\vee) \cap \text{Supp}_R(\Gamma_m(L)^\vee)
= \text{Att}_R(A) \cap \text{Supp}_R(\Gamma_m(L)^\vee).$$

The remaining equalities are from Lemmas 1.5(b), 2.1(a) and 5.2, respectively. □

**Corollary 5.4.** Let $M$ and $M'$ be mini-max $R$-modules such that the quotient $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is complete.

(a) For each index $i \geq 0$, one has $\text{Ext}_R^i(M, M') \cong \text{Ext}_R^i(M^\vee, M'^\vee)$.

(b) If $M'$ is noetherian, then

$$\text{Ass}_R(\text{Hom}_R(M, M')) = \text{Att}_R(M'^\vee) \cap \text{Supp}_R(\Gamma_m(M'^\vee)^\vee).$$

**Proof.** (a) The first step in the next sequence comes from Theorem 2.8(a):

$$\text{Ext}_R^i(M, M') \cong \text{Ext}_R^i(M, M'^\vee)^\vee \cong (\text{Tor}_R^i(M, M'^\vee))^\vee \cong \text{Ext}_R^i(M'^\vee, M').$$

The remaining steps are from Theorem 4.10 and Remark 1.9, respectively.

(b) This follows from the case $i = 0$ in part (a) because of Proposition 5.3. □

**Proposition 5.5.** Let $A$ and $L$ be $R$-modules such that $A$ is artinian and $\mu_R^0(L) < \infty$. Then

$$\text{Att}_R(A \otimes_R L) = \text{Ass}_R(A^\vee) \cap \text{Supp}_R(\Gamma_m(L)^\vee) = \text{Att}_R(A) \cap \text{Supp}_R(\Gamma_m(L)^\vee).$$

**Proof.** Theorem 3.1 implies that $A \otimes_R L$ is artinian. Hence, we have

$$\text{Hom}_R(A \otimes_R L, E) \cong \text{Hom}_R(A \otimes_R L, E) \cong \text{Hom}_R(A, L^\vee)$$

by Lemma 1.5(a), and this explains the second step in the next sequence:

$$\text{Att}_R(A \otimes_R L) = \text{Ass}_R(\text{Hom}_R(A \otimes_R L, E)) = \text{Ass}_R(\text{Hom}_R(A, L^\vee)).$$

The first step is from Lemma 5.2. Since $\mu_R^0(L^\vee) < \infty$ by Lemma 1.13(b), we obtain the desired equalities from Proposition 5.3. □

Next, we give an alternate description of the module $\Gamma_m(L)^\vee$ from the previous results. See Lemma 5.2 for a description of its support.

**Remark 5.6.** Let $L$ be an $R$-module. There is an isomorphism $\Gamma_m(L)^\vee \cong \hat{L}^\vee$. In particular, given a noetherian $R$-module $N$, one has $\Gamma_m(N)^\vee \cong \hat{R} \otimes_R N$. When $R$ is Cohen–Macaulay with a dualizing module $D$, Grothendieck’s local duality theorem implies that $\Gamma_m(N)^\vee \cong \hat{R} \otimes_R \text{Ext}^{\text{dim}(R)}_R(N, D)$; see, e.g., [6, Theorem 3.5.8]. A similar description is available when $R$ is not Cohen–Macaulay, provided that it has a dualizing complex; see [10, Chapter V, §6].

**Vanishing of Hom and Tensor product**

For the next result note that if $L$ is noetherian, then the conditions on $\mu_R^0(L)$ and $R/(\text{Ann}_R(A) + \text{Ann}_R(\Gamma_m(L)))$ are automatically satisfied. Also, the example $\text{Hom}_R(E, E) \cong R$ when $R$ is complete shows the necessity of the condition on $R/(\text{Ann}_R(A) + \text{Ann}_R(\Gamma_m(L)))$. 
**Proposition 5.7.** Let $A$ be an artinian $R$-module. Let $L$ be an $R$-module such that $R/(\text{Ann}_R(A) + \text{Ann}_R(I_m(L)))$ is artinian and $\mu^0_R(L) < \infty$. Then $\text{Hom}_R(A, L) = 0$ if and only if $A = mA$ or $I_m(L) = 0$.

**Proof.** If $I_m(L) = 0$, then we are done by Lemma 1.5(b), so assume that $I_m(L) \neq 0$. Theorem 2.2 and Lemma 1.21 show that $\text{Hom}_R(A, L)$ has finite length. Thus Proposition 5.3 implies that $\text{Hom}_R(A, L) \neq 0$ if and only if $mR \in \text{Ass}_R(A')$, that is, if and only if depth$_R(A') = 0$. Lemma 1.13(c) shows that depth$_R(A') = 0$ if and only if $mA \neq A$, that is, if and only if $mA \neq A$. □

For the next result note that the conditions on $L$ are satisfied when $L$ is artinian.

**Proposition 5.8.** Let $A$ be an artinian $R$-module, and let $L$ be an $m$-torsion $R$-module. The following conditions are equivalent:

(i) $A \otimes_R L = 0$;
(ii) either $A = mA$ or $L = mL$; and
(iii) either $\text{depth}_R(A') > 0$ or $\text{depth}_R(L') > 0$.

**Proof.** (i) $\iff$ (ii) if $A \otimes_R L = 0$, then we have

$$0 = \text{len}_R(A \otimes_R L) \geq \beta^0_R(A) \beta^0_R(L)$$

so either $\beta^0_R(A) = 0$ or $\beta^0_R(L) = 0$, that is, $A/mA = 0$ or $L/mlL = 0$. Conversely, if $A/mA = 0$ or $L/mlL = 0$, then we have either $\beta^0_R(A) = 0$ or $\beta^0_R(L) = 0$, so Theorem 3.8 implies that $\text{len}_R(A \otimes_R L) = 0$.

The implication (ii) $\iff$ (iii) is from Lemma 1.13(c). □

The next result becomes simpler when $L$ is artinian, as $I_m(L) = L$ in this case.

**Theorem 5.9.** Let $A$ and $L$ be $R$-modules such that $A$ is artinian and $\mu^0_R(L) < \infty$. The following conditions are equivalent:

(i) $\text{Hom}_R(A, L) = 0$;
(ii) $\text{Hom}_R(A, I_m(L)) = 0$;
(iii) $\text{Hom}_R(I_m(L), A') = 0$;
(iv) there is an element $x \in \text{Ann}_R(I_m(L))$ such that $A = xA$;
(v) $\text{Ann}_R(I_m(L))A = A$;
(vi) $\text{Ann}_R(I_m(L))$ contains a non-zero-divisor for $A'$; and
(vii) $\text{Att}_R(A) \cap \text{Supp}_R(I_m(L)) = \emptyset$.

**Proof.** The equivalence (i) $\iff$ (ii) is from Lemma 1.5(b). The equivalence (ii) $\iff$ (vii) follows from Proposition 5.3, and the equivalence (ii) $\iff$ (iii) follows from Lemma 2.1(a). The equivalence (iv) $\iff$ (vi) follows from the fact that the map $A \rightarrow A$ is surjective if and only if the map $A' \rightarrow A'$ is injective. The equivalence (v) $\iff$ (vi) follows from Lemma 1.13, parts (c) and (e).

The module $I_m(L)$ is artinian as $\mu^0_R(L) < \infty$. Since $A'$ and $I_m(L)^\vee$ are noetherian over $\hat{R}$, the equivalence (iii) $\iff$ (vi) is standard; see [6, Proposition 1.2.3]. □

As with Theorem 5.9, the next result simplifies when $L$ is noetherian. Also, see Remark 5.6 for some perspective on the module $I_m(L)^\vee$.

**Corollary 5.10.** Let $A$ be a non-zero artinian $R$-module, and let $L$ be an $R$-module such that $\beta^0_R(L) < \infty$. The following conditions are equivalent:

(i) $A \otimes_R L = 0$;
(ii) $\text{Ann}_R(I_m(L))A = A$;
(iii) there is an element $x \in \text{Ann}_R(I_m(L))$ such that $xA = A$;
(iv) $\text{Ann}_R(I_m(L))$ contains a non-zero-divisor for $A'$; and
(v) $\text{Att}_R(A) \cap \text{Supp}_R(I_m(L)) = \emptyset$.

**Proof.** For an artinian $R$-module $A'$, one has $\text{Att}_R(A') = \emptyset$ if and only if $A' = 0$ by Lemma 5.2. Thus, Proposition 5.5 explains the equivalence (i) $\iff$ (v); see [16, Corollary 2.3]. Since one has $A \otimes_R L = 0$ if and only if $(A \otimes_R L)^\vee = 0$, the isomorphism $(A \otimes_R L)^\vee \cong \text{Hom}_R(A, L')$ from Remark 1.9 in conjunction with Theorem 5.9 shows that the conditions (i)–(iv) are equivalent. □

**Depth and vanishing**

**Proposition 5.11.** Let $A$ and $L$ be $R$-modules such that $A$ is artinian. Then $\text{Ext}^i_R(A, L) = 0$ for all $i < \text{depth}_R(L)$.  

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*Note: The above text is a transcription of a portion of a document, likely pertaining to the study of artinian modules and related algebraic structures.*
Proof. Let $J$ be a minimal $R$-injective resolution of $L$, and let $i < \text{depth}_R(L)$. It follows that $\text{Ext}_R^i(k, L) = 0$, that is $\mu_R^i(L) = 0$, so the module $E$ does not appear as a summand of $f$. As in the proof of Theorem 2.2, this implies that $\text{Hom}_R(A, J)^i = 0$, so $\text{Ext}_R^i(A, L) = 0$. □

The next example shows that, in Proposition 5.11 one may have $\text{Ext}_R^i(A, L) = 0$ when $i = \text{depth}_R(L)$. See also Eq. (5.14.1).

**Example 5.12.** Assume that depth$(R) \geq 1$. Then $mE = E$ by Lemma 1.13(c), so Lemma 2.10 implies that

$$\text{Ext}_R^0(E, k) \cong \text{Hom}_R(k, E) \cong \text{Hom}_R(E/mE, k) = 0$$

even though depth$^R(k) = 0$.

**Proposition 5.13.** Let $A$ and $L$ be R-modules such that $A$ is artinian. Then for all $i < \text{depth}_R(L^\vee)$ one has $\text{Tor}_i^R(A, L) = 0$.

**Proof.** When $i < \text{depth}_R(L^\vee)$, one has $\text{Tor}_i^R(A, L^\vee) \cong \text{Ext}_R^i(A, L^\vee) = 0$ by Remark 1.9 and Proposition 5.11, so $\text{Tor}_i^R(A, L) = 0$. □

**Theorem 5.14.** Let $A$ and $A'$ be artinian $R$-modules, and let $N$ and $N'$ be noetherian $R$-modules. Then one has

$$\text{depth}_R(\text{Ann}_R(A'); A') = \inf\{i \geq 0 \mid \text{Ext}_R^i(A', A') \neq 0\} \quad (5.14.1)$$

$$\text{depth}_R(\text{Ann}_R(N'); A') = \inf\{i \geq 0 \mid \text{Ext}_R^i(A, N') \neq 0\} \quad (5.14.2)$$

$$\text{depth}_R(\text{Ann}_R(N'); N) = \inf\{i \geq 0 \mid \text{Ext}_R^i(N', N') \neq 0\}. \quad (5.14.3)$$

**Proof.** We verify Eq. (5.14.1) first. For each index $i$, Theorem 4.3 implies that

$$\text{Ext}_R^i(A, A') \cong \text{Ext}_R^i(A', A').$$

Since $A'$ and $A'$ are noetherian over $\hat{R}$, this explains the first equality below:

$$\inf\{i \geq 0 \mid \text{Ext}_R^i(A', A') \neq 0\} = \text{depth}_R(\text{Ann}_R(A'); A') = \text{depth}_R(\text{Ann}_R(A'); A').$$

The second equality is standard since $A'' = \text{Hom}_R(A', E)$ by Lemma 1.5(a).

Next, we verify Eq. (5.14.2). Since $N''$ is artinian, Eq. (5.14.1) shows that we need only verify that

$$\text{depth}_R(\text{Ann}_R(N'')); A') = \text{depth}_R(\text{Ann}_R(N')); A'). \quad (5.14.4)$$

For this, we compute as follows:

$$\hat{R} \otimes_R N' \overset{(1)}{\cong} \text{Hom}_R(\text{Hom}_R(\hat{R} \otimes_R N', E), E) \overset{(2)}{=} \text{Hom}_R(N'''); E).$$

Step (1) follows from the fact that $\hat{R} \otimes_R N'$ is noetherian (hence, Matlis reflexive) over $\hat{R}$, and step (2) is from Hom–tensor adjointness. This explains step (4) below:

$$\text{Ann}_R(N'') = \text{Ann}_R(\text{Hom}_R(N'''); E)) \overset{(4)}{=} \text{Ann}_R(\hat{R} \otimes_R N') \overset{(5)}{=} \text{Ann}_R(N') \hat{R}.$$

Steps (3) and (5) are standard. This explains step (6) in the next sequence:

$$\text{depth}_R(\text{Ann}_R(N'')); A') \overset{(6)}{=} \text{depth}_R(\text{Ann}_R(N') \hat{R}; A') \overset{(7)}{=} \text{depth}_R(\text{Ann}_R(N'); A').$$

Step (7) is explained by the following, where step (8) is standard, and step (9) is a consequence of Hom–tensor adjointness:

$$\text{Ext}_R^i(\hat{R}/ \text{Ann}_R(N'), A') \overset{(8)}{=} \text{Ext}_R^i(\hat{R} \otimes_R (R/ \text{Ann}_R(N'))), A') \overset{(9)}{=} \text{Ext}_R^i(R/ \text{Ann}_R(N'), A').$$

This establishes Eq. (5.14.4) and thus Eq. (5.14.2).

**Corollary 5.15.** Let $A$ and $A'$ be artinian $R$-modules, and let $N$ and $N'$ be noetherian $R$-modules. Then

$$\text{depth}_R(\text{Ann}_R(A'); A') = \inf\{i \geq 0 \mid \text{Tor}_i^R(A, A') \neq 0\} \quad (5.15.1)$$

$$\text{depth}_R(\text{Ann}_R(N'); A') = \inf\{i \geq 0 \mid \text{Tor}_i^R(A, N') \neq 0\} \quad (5.15.2)$$

$$\text{depth}_R(\text{Ann}_R(N'); N) = \inf\{i \geq 0 \mid \text{Tor}_i^R(N', N') \neq 0\}. \quad (5.15.3)$$
Proof. We verify Eq. (5.15.1); the others are verified similarly.

Since $\text{Ext}^i_R(A, A') \neq 0$ if and only if $\text{Hom}_R(\text{Ext}^i_R(A, A'), E) \neq 0$, the isomorphism $\text{Hom}_R(\text{Ext}^i_R(A, A'), E) \cong \text{Tor}^R_i(A, A')$ from Corollary 4.9 shows that

$$\text{inf}\{i \geq 0 \mid \text{Ext}^i_R(A, A') \neq 0\} = \text{inf}\{i \geq 0 \mid \text{Tor}^R_i(A, A') \neq 0\}.$$ 

Thus Eq. (5.15.1) follows from (5.14.1). □

6. Examples

This section contains some explicit computations of Ext and Tor for the classes of modules discussed in this paper. Our first example shows that $\text{Ext}^i_R(A, A')$ need not be mini-max over $R$.

Example 6.1. Let $k$ be a field, and set $R = k[X_1, \ldots, X_d]_{(x_1, \ldots, x_d)}$. We show that $\text{Hom}_R(E, E) \cong \widehat{R}$ is not mini-max over $R$. Note that $R$ is countably generated over $k$, and $\widehat{R} \cong k[X_1, \ldots, X_d]$ is not countably generated over $k$. So, $\widehat{R}$ is not countably generated over $R$. Also, every artinian $R$-module $A$ is a countable union of the finite length submodules $(0 \colon_A m^n)$, so $A$ is countably generated. It follows that every mini-max $R$-module is also countably generated. Since $R$ is not countably generated, it is not mini-max over $R$.

Our next example describes $\text{Ext}^i_R(A, A')$ for some special cases.

Example 6.2. Assume that $\text{depth}(R) \geq 1$, and let $A$ be an artinian $R$-module. Let $x \in m$ be an $R$-regular element. The map $E \xrightarrow{d} E$ is surjective since $E$ is divisible, and the kernel $(0 \colon_E x)$ is artinian, being a submodule of $E$. Using the injective resolution $0 \to E \xrightarrow{d} E \to 0$ for $(0 \colon_E x)$, one can check that

$$\text{Ext}^i_R(A, (0 \colon_E x)) \cong \begin{cases} (0 \colon A^\vee) & \text{if } i = 0 \\ A^\vee/xA^\vee & \text{if } i = 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

For instance, in the case $A = (0 \colon_E x)$, the isomorphism $\text{Hom}_R((0 \colon_E x)^\vee) \cong \widehat{R}/x\widehat{R}$ implies

$$\text{Ext}^i_R((0 \colon_E x), (0 \colon_E x)) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = 0, 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

On the other hand, if $x, y$ is an $R$-regular sequence, then $(0 \colon_E y)^\vee \cong \widehat{R}/y\widehat{R}$; it follows that $x$ is $(0 \colon_E y)^\vee$-regular, so one has

$$\text{Ext}^i_R((0 \colon_E y), (0 \colon_E x)) \cong \begin{cases} \widehat{R}/(x, y)\widehat{R} & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

The next example shows that $\text{Ext}^i_R(A, N)$ need not be mini-max over $R$.

Example 6.3. Assume that $R$ is Cohen–Macaulay with $d = \text{dim}(R)$, and let $A$ be an artinian $R$-module. Assume that $R$ admits a dualizing (i.e., canonical) module $D$. (For instance, this is so when $R$ is Gorenstein, in which case $D = R$.) A minimal injective resolution of $D$ has the form

$$F = 0 \to \bigoplus_{ht(p) = 0} E_R(R/p) \to \cdots \to \bigoplus_{ht(p) = d - 1} E_R(R/p) \to E \to 0.$$ 

In particular, we have $F_m(j) = (0 \to 0 \to 0 \to \cdots \to 0 \to E \to 0)$ where the copy of $E$ occurs in degree $d$. Since $\text{Hom}_R(A, F) \cong \text{Hom}_R(A, F_m(j))$, it follows that

$$\text{Ext}^i_R(A, D) \cong \begin{cases} A^\vee & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Assume that $d \geq 1$, and let $x \in m$ be an $R$-regular element. It follows that the map $D \xrightarrow{x} D$ is injective, and the cokernel $D/xD$ is noetherian. Consider the exact sequence $0 \to D \xrightarrow{x} D \to D/xD \to 0$. The long exact sequence associated to $\text{Ext}^i_R(A, -)$ shows that

$$\text{Ext}^i_R(A, D/xD) \cong \begin{cases} (0 \colon_A x) & \text{if } i = d - 1 \\ A^\vee/xA^\vee & \text{if } i = d \\ 0 & \text{if } i \neq d - 1, d. \end{cases}$$

As in Example 6.2, we have $(0 \colon_E x)^\vee \cong \widehat{R}/x\widehat{R}$ and

$$\text{Ext}^i_R((0 \colon_E x), D/xD) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = d - 1, d \\ 0 & \text{if } i \neq d - 1, d. \end{cases}$$
Also, if $x, y$ is an $R$-regular sequence, then $(0 :_E y)^{\vee} \cong \hat{R}/y\hat{R}$ and
\[
\Ext^i_R((0 :_E y), D/xD) \cong \begin{cases} \hat{R}/(x, y)\hat{R} & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}
\]

Next, we show that $\Tor^R_i(A, A')$ need not be noetherian over $R$ or $\hat{R}$.

**Example 6.4.** Assume that $R$ is Gorenstein and complete with $d = \dim(R)$. (Hence $D = R$ is a dualizing $R$-module.) Given two artinian $R$-modules $A$ and $A'$, Theorem 3.1 implies that $\Tor^R_i(A, A')$ is artinian, hence Matlis reflexive for each index $i$, since $R$ is complete. This explains the first isomorphism below, and Remark 1.9 provides the second isomorphism:
\[
\Tor^R_i(A, E) \cong \Tor^R_i(A, E)^{\vee} \cong \Ext^i_R(A, E^{\vee}) \cong \Ext^i_R(A, R)^{\vee} \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}
\]

Example 6.3 explains the fourth isomorphism. Assume that $d \geq 1$, and let $x \in m$ be an $R$-regular element. Then $(0 :_E x)^{\vee} \cong R/xR$, so Example 6.3 implies that
\[
\Tor^R_i((0 :_E x), (0 :_E x)) \cong \Ext^i_R((0 :_E x), (0 :_E x)^{\vee}) \cong \begin{cases} (0 :_A x) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}
\]

On the other hand, if $x, y$ is an $R$-regular sequence, then
\[
\Tor^R_i((0 :_E y), (0 :_E x)) \cong \begin{cases} (R/(x, y)R)^{\vee} \cong \Ext^i_R(k/(x, y)R) \cong k & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}
\]

Lastly, we provide an explicit computation of $E \otimes_R E$.

**Example 6.5.** Let $k$ be a field and set $R = k[[X, Y]]/(XY, Y^2)$. This is the completion of the multi-graded ring $R' = k[X, Y]/(XY, Y^2)$ with homogeneous maximal ideal $m' = (X, Y)R'$. The multi-graded structure on $R'$ is represented in the following diagram:

\[
\begin{array}{ccccccccc}
R' \\
\bullet & \cdots & \bullet \\
\end{array}
\]

where each bullet represents the corresponding monomial in $R'$. It follows that $E \cong E_R(k) \cong k[X^{-1}] \oplus kY^{-1}$ with graded module structure given by the formulas
\[
\begin{align*}
X \cdot 1 &= 0 \\
Y \cdot 1 &= 0 \\
Y \cdot X^{-n} &= X^{1-n} \\
Y \cdot Y^{-1} &= 1 \\
Y \cdot X^n &= 0 \\
X \cdot Y^{-1} &= 0 \\
X \cdot X^n &= 0
\end{align*}
\]

for $n \geq 1$. Using this grading, one can show that $mE = m'E \cong k[X^{-1}]$ and $m^2E = mE$. These modules are represented in the next diagrams:

\[
\begin{array}{ccccccccc}
E \\
\bullet & \cdots & \bullet & \cdots & \bullet \\
\end{array} \quad mE \\
\end{array}
\]

It follows that $E/mE \cong k$, so Lemma 3.7 implies that
\[
E \otimes_R E \cong (E/mE) \otimes_R (E/mE) \cong k \otimes_R k \cong k.
\]

A similar computation shows the following: Fix positive integers $a, b, c$ such that $c > b$, and consider the ring $S = k[[X, Y]]/(X^a, Y^b, Y^c)$ with maximal ideal $n$ and $E_S = E_S(k)$. Then $n^{-c-b}E_S = n^{-c-b+1}E_S$ and we get the following:
\[
\begin{align*}
E_S/n^{-c-b}E_S &\cong S/(X^a, Y^{c-b})S \cong k[X, Y]/(X^a, Y^{c-b}) \\
E_S \otimes_SE_S &\cong (E_S/n^{-c-b}E_S) \otimes_S (E_S/n^{-c-b}E_S) \cong S/(X^a, Y^{c-b})S.
\end{align*}
\]


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Acknowledgements

We are grateful to Luchezar Avramov, Cătălin Ciupercă, Edgar Enochs, Srikanth Iyengar, Amir Mafi, and Roger Wiegand for useful feedback about this research. We also thank the anonymous referee for valuable comments.

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