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Homology of artinian and Matlis reflexive modules, I ☆

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ABSTRACT

Let R be a commutative local noetherian ring, and let L and L' be R-modules. We investigate the properties of the functors $\operatorname{Tor}_i^R(L,-)$ and $\operatorname{Ext}_R^i(L,-)$. For instance, we show the following:

(a) if L and L' are artinian, then $\operatorname{Tor}_i^R(L,L')$ is artinian, and $\operatorname{Ext}_R^i(L,L')$ is noetherian over the completion $\widehat{R}$;
(b) if L is artinian and L' is Matlis reflexive, then $\operatorname{Ext}_R^i(L,L')$, $\operatorname{Ext}_R^i(L',L)$, and $\operatorname{Tor}_i^R(L,L')$ are Matlis reflexive.

Also, we study the vanishing behavior of these functors, and we include computations demonstrating the sharpness of our results.

0. Introduction

Throughout this paper, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The $\mathfrak{m}$-adic completion of R is denoted by $\widehat{R}$, the injective hull of k is $E = E_R(k)$, and the Matlis duality functor is $(-)^\vee = \operatorname{Hom}_R(-,E)$.

This paper is concerned, in part, with the properties of the functors $\operatorname{Hom}_R(A,-)$ and $A \otimes_R -$, where A is an artinian R-module. To motivate this, recall that [8, Proposition 6.1] shows that if A and A' are artinian R-modules, then $A \otimes_R A'$ has finite length. It follows that if N is a noetherian R-module, then $\operatorname{Hom}_R(A,N)$ also has finite length (see also Corollaries 2.12 and 3.9). In light of this, it is natural to investigate the properties of $\operatorname{Ext}_R^i(A,-)$ and $\operatorname{Tor}_i^R(A,-)$. In general, the modules $\operatorname{Ext}_R^i(A,N)$ and $\operatorname{Tor}_i^R(A,A')$ will not have finite length. However, we have the following (see Theorems 2.2 and 3.1).

Theorem 1. *Let A be an artinian R-module, and let $i \geqslant 0$. Let L and L' be R-modules such that $\mu_R^i(L)$ and $\beta_i^R(L')$ are finite. Then $\operatorname{Ext}_R^i(A,L)$ is a noetherian $\widehat{R}$-module, and $\operatorname{Tor}_i^R(A,L')$ is artinian.*

In this result, we use the ith Bass number $\mu_R^i(L) := \operatorname{len}_R(\operatorname{Ext}_R^i(k,L))$ and the ith Betti number $\beta_i^R(L') := \operatorname{len}_R(\operatorname{Tor}_i^R(k,L'))$. For instance, these are both finite for all i when L and L' are either artinian or noetherian. In particular, when A and A' are artinian, Theorem 1 implies that $\operatorname{Ext}_R^i(A,A')$ is a noetherian $\widehat{R}$-module. The next result, contained in Theorem 4.3, gives another explanation for this fact.

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Theorem 2. *Let A and A' be artinian R-modules, and let $i \geqslant 0$. Then there is an isomorphism $\mathrm{Ext}^i_R(A, A') \cong \mathrm{Ext}^i_{\widehat{R}}(A'^{\vee}, A^{\vee})$. Hence, there are noetherian $\widehat{R}$-modules N and N' such that $\mathrm{Ext}^i_R(A, A') \cong \mathrm{Ext}^i_{\widehat{R}}(N, N')$.*

This result proves useful for studying the vanishing of $\mathrm{Ext}^i_R(A, A')$, since the vanishing of $\mathrm{Ext}^i_{\widehat{R}}(N, N')$ is somewhat well understood.

Our next result shows how extra conditions on the modules in Theorem 1 imply that $\mathrm{Ext}^i_R(A, L)$ and $\mathrm{Tor}^R_i(A, L')$ are Matlis reflexive; see Corollaries 2.4 and 3.3.

Theorem 3. *Let A, L, and L' be R-modules such that A is artinian. Assume that $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(L))$ and $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(L'))$ are complete. Given an index $i \geqslant 0$ such that $\mu^i_R(L)$ and $\beta^R_i(L')$ are finite, the modules $\mathrm{Ext}^i_R(A, L)$ and $\mathrm{Tor}^R_i(A, L')$ are Matlis reflexive.*

A key point in the proof of this theorem is a result of Belshoff et al. [4]: An R-module M is Matlis reflexive if and only if it is mini-max and $R/\mathrm{Ann}_R(M)$ is complete. Here M is *mini-max* when M has a noetherian submodule N such that M/N is artinian. In particular, noetherian modules are mini-max, as are artinian modules.

The last result singled out for this introduction describes the Matlis dual of $\mathrm{Ext}^i_R(M, M')$ in some special cases. It is contained in Corollary 4.11.

Theorem 4. *Let M and M' be mini-max R-modules, and fix an index $i \geqslant 0$. If either M or M' is Matlis reflexive, then $\mathrm{Ext}^i_R(M, M')^{\vee} \cong \mathrm{Tor}^R_i(M, M'^{\vee})$.*

We do not include a description of the Matlis dual of $\mathrm{Tor}^R_i(M, M')$, as a standard application of Hom–tensor adjointness shows that $\mathrm{Tor}^R_i(M, M')^{\vee} \cong \mathrm{Ext}^i_R(M, M'^{\vee})$.

Many of our results generalize to the non-local setting. As this generalization requires additional tools, we treat it separately in [11].

1. Background material and preliminary results

Torsion modules

Definition 1.1. Let $\mathfrak{a}$ be a proper ideal of R. We denote the $\mathfrak{a}$-adic completion of R by $\widehat{R}^{\mathfrak{a}}$. Given an R-module L, set $\Gamma_{\mathfrak{a}}(L) = \{x \in L \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}$. We say that L is $\mathfrak{a}$-*torsion* if $L = \Gamma_{\mathfrak{a}}(L)$. We set $\mathrm{Supp}_R(L) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid L_{\mathfrak{p}} \neq 0\}$.

Fact 1.2. Let $\mathfrak{a}$ be a proper ideal of R, and let L be an $\mathfrak{a}$-torsion R-module.

(a) Every artinian R-module is $\mathfrak{m}$-torsion. In particular, the module E is $\mathfrak{m}$-torsion.
(b) We have $\mathrm{Supp}_R(L) \subseteq V(\mathfrak{a})$. Hence, if L is $\mathfrak{m}$-torsion, then $\mathrm{Supp}_R(L) \subseteq \{\mathfrak{m}\}$.
(c) The module L has an $\widehat{R}^{\mathfrak{a}}$-module structure that is compatible with its R-module structure, as follows. For each $x \in L$, fix an exponent n such that $\mathfrak{a}^n x = 0$. For each $r \in \widehat{R}^{\mathfrak{a}}$, the isomorphism $\widehat{R}^{\mathfrak{a}}/\mathfrak{a}^n\widehat{R}^{\mathfrak{a}} \cong R/\mathfrak{a}^n$ provides an element $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$, and we set $rx := r_0 x$.
(d) If $R/\mathfrak{a}$ is complete, then $\widehat{R}^{\mathfrak{a}}$ is naturally isomorphic to $\widehat{R}$. To see this, assume that $R/\mathfrak{a}$ is complete. By induction on n, it follows that $R/\mathfrak{a}^n$ is complete for all n, and this explains the second step in the next display:
$$\widehat{R}^{\mathfrak{a}} \cong \varprojlim R/\mathfrak{a}^n \cong \varprojlim \widehat{R}/\mathfrak{a}^n\widehat{R} \cong (\widehat{R})^{\widehat{\ }\mathfrak{a}} \cong \widehat{R}.$$
For the last step in this display, see, e.g., [1, Exercise 10.5].

Lemma 1.3. *Let $\mathfrak{a}$ be a proper ideal of R, and let L be an $\mathfrak{a}$-torsion R-module.*

(a) *A subset $Z \subseteq L$ is an R-submodule if and only if it is an $\widehat{R}^{\mathfrak{a}}$-submodule.*
(b) *The module L is noetherian over R if and only if it is noetherian over $\widehat{R}^{\mathfrak{a}}$.*

Proof. (a) Every $\widehat{R}^{\mathfrak{a}}$-submodule of L is an R-submodule by restriction of scalars. Conversely, fix an R-submodule $Z \subseteq L$. Since L is $\mathfrak{a}$-torsion, so is Z, and Fact 1.2(c) implies that Z is an $\widehat{R}^{\mathfrak{a}}$-submodule.

(b) The set of R-submodules of L equals the set of $\widehat{R}^{\mathfrak{a}}$-submodules of L, so they satisfy the ascending chain condition simultaneously. □

Lemma 1.4. *Let $\mathfrak{a}$ be a proper ideal of R, and let L be an $\mathfrak{a}$-torsion R-module.*

(a) *The natural map $L \to \widehat{R}^{\mathfrak{a}} \otimes_R L$ is an isomorphism.*
(b) *The left and right $\widehat{R}^{\mathfrak{a}}$-module structures on $\widehat{R}^{\mathfrak{a}} \otimes_R L$ are the same.*

Proof. The natural map $L \to \widehat{R}^{\mathfrak{a}} \otimes_R L$ is injective, as $\widehat{R}^{\mathfrak{a}}$ is faithfully flat over R. To show surjectivity, it suffices to show that each generator $r \otimes x \in \widehat{R}^{\mathfrak{a}} \otimes_R L$ is of the form $1 \otimes x'$ for some $x' \in L$. Let $n \geqslant 1$ such that $\mathfrak{a}^n x = 0$, and let $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$. It follows that $r \otimes x = r_0 \otimes x = 1 \otimes (r_0 x)$, and this yields the conclusion of part (a). This also proves (b) because $1 \otimes (r_0 x) = 1 \otimes (rx)$. □

Lemma 1.5. *Let $\mathfrak{a}$ be a proper ideal of R, and let L and L' be R-modules such that L is $\mathfrak{a}$-torsion.*

(a) *If L' is $\mathfrak{a}$-torsion, then $\mathrm{Hom}_R(L, L') = \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$; thus $L^\vee = \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, E)$.*
(b) *One has $\mathrm{Hom}_R(L, L') \cong \mathrm{Hom}_R(L, \Gamma_{\mathfrak{a}}(L')) = \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, \Gamma_{\mathfrak{a}}(L'))$.*

Proof. (a) It suffices to verify the inclusion $\mathrm{Hom}_R(L, L') \subseteq \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$. Let $x \in L$ and $r \in \widehat{R}^{\mathfrak{a}}$, and fix $\psi \in \mathrm{Hom}_R(L, L')$. Let $n \geqslant 1$ such that $\mathfrak{a}^n x = 0$ and $\mathfrak{a}^n \psi(x) = 0$. Choose an element $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n \widehat{R}^{\mathfrak{a}}$. It follows that $\psi(rx) = \psi(r_0 x) = r_0 \psi(x) = r\psi(x)$; hence $\psi \in \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$. (Part (a) can also be deduced from Hom–tensor adjointness, using Lemma 1.4(a).)

(b) For each $f \in \mathrm{Hom}_R(L, L')$, one has $\mathrm{Im}(f) \subseteq \Gamma_{\mathfrak{a}}(L')$. This yields the desired isomorphism, and the equality is from part (a). □

A Natural Map from $\mathrm{Tor}_i^R(L, L'^\vee)$ to $\mathrm{Ext}_R^i(L, L')^\vee$

Definition 1.6. Let L be an R-module, and let J be an R-complex. The *Hom-evaluation* morphism

$$\theta_{LJE} : L \otimes_R \mathrm{Hom}_R(J, E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(L, J), E)$$

is given by $\theta_{LJE}(l \otimes \psi)(\phi) = \psi(\phi(l))$.

Remark 1.7. Let L and L' be R-modules, and let J be an injective resolution of L'. Using the notation $(-)^\vee$, we have $\theta_{LJE} : L \otimes_R J^\vee \to \mathrm{Hom}_R(L, J)^\vee$. The complex $J^\vee$ is a flat resolution of $L'^\vee$; see, e.g., [7, Theorem 3.2.16]. This explains the first isomorphism in the following sequence:

$$\mathrm{Tor}_i^R(L, L'^\vee) \xrightarrow{\cong} \mathrm{H}_i(L \otimes_R J^\vee) \xrightarrow{\mathrm{H}_i(\theta_{LJE})} \mathrm{H}_i(\mathrm{Hom}_R(L, J)^\vee) \xrightarrow{\cong} \mathrm{Ext}_R^i(L, L')^\vee.$$

For the second isomorphism, the exactness of $(-)^\vee$ implies that $\mathrm{H}_i(\mathrm{Hom}_R(L, J)^\vee) \cong \mathrm{H}^i(\mathrm{Hom}_R(L, J))^\vee \cong \mathrm{Ext}_R^i(L, L')^\vee$.

Definition 1.8. Let L and L' be R-modules, and let J be an injective resolution of L'. The R-module homomorphism

$$\Theta_{LL'}^i : \mathrm{Tor}_i^R(L, L'^\vee) \to \mathrm{Ext}_R^i(L, L')^\vee$$

is defined to be the composition of the maps displayed in Remark 1.7.

Remark 1.9. Let L, L', and N be R-modules such that N is noetherian. It is straightforward to show that the map $\Theta_{LL'}^i$ is natural in L and in L'.

The fact that E is injective implies that $\Theta_{NL'}^i$ is an isomorphism; see [17, Lemma 3.60]. This explains the first of the following isomorphisms:

$$\mathrm{Ext}_R^i(N, L')^\vee \cong \mathrm{Tor}_i^R(N, L'^\vee) \qquad \mathrm{Tor}_i^R(L, L')^\vee \cong \mathrm{Ext}_R^i(L, L'^\vee).$$

The second isomorphism is a consequence of Hom–tensor adjointness,

Numerical invariants

Definition 1.10. Let L be an R-module. For each integer i, the ith *Bass number* of L and the ith *Betti number* of L are respectively

$$\mu_R^i(L) = \mathrm{len}_R(\mathrm{Ext}_R^i(k, L)) \qquad \beta_i^R(L) = \mathrm{len}_R(\mathrm{Tor}_i^R(k, L))$$

where $\mathrm{len}_R(L')$ denotes the length of an R-module L'.

Remark 1.11. Let L be an R-module.

(a) If I is a minimal injective resolution of L, then for each index $i \geqslant 0$ such that $\mu_R^i(L) < \infty$, we have $I^i \cong E^{\mu_R^i(L)} \oplus J^i$ where J^i does not have E as a summand, that is, $\Gamma_{\mathfrak{m}}(J^i) = 0$; see, e.g., [14, Theorem 18.7]. Similarly, the Betti numbers of a noetherian module are the ranks of the free modules in a minimal free resolution. The situation for Betti numbers of non-noetherian modules is more subtle; see, e.g., Lemma 1.19.
(b) Then $\mu_R^i(L) < \infty$ for all $i \geqslant 0$ if and only if $\beta_i^R(L) < \infty$ for all $i \geqslant 0$; see [12, Proposition 1.1].

When $\mathfrak{a} = \mathfrak{m}$, the next invariants can be interpreted in terms of (non)vanishing Bass and Betti numbers.

Definition 1.12. Let $\mathfrak{a}$ be an ideal of R. For each R-module L, set

$$\mathrm{depth}_R(\mathfrak{a}; L) = \inf\{i \geqslant 0 \mid \mathrm{Ext}_R^i(R/\mathfrak{a}, L) \neq 0\}$$
$$\mathrm{width}_R(\mathfrak{a}; L) = \inf\{i \geqslant 0 \mid \mathrm{Tor}_i^R(R/\mathfrak{a}, L) \neq 0\}.$$

We write $\mathrm{depth}_R(L) = \mathrm{depth}_R(\mathfrak{m}; L)$ and $\mathrm{width}_R(L) = \mathrm{width}_R(\mathfrak{m}; L)$.

Part (b) of the next result is known. We include it for ease of reference.

Lemma 1.13. *Let L be an R-module, and let $\mathfrak{a}$ be an ideal of R.*

(a) *Then* $\operatorname{width}_R(\mathfrak{a}; L) = \operatorname{depth}_R(\mathfrak{a}; L^\vee)$ *and* $\operatorname{width}_R(\mathfrak{a}; L^\vee) = \operatorname{depth}_R(\mathfrak{a}; L)$.
(b) *For each index $i \geqslant 0$ we have* $\beta_i^R(L) = \mu_R^i(L^\vee)$ *and* $\beta_i^R(L^\vee) = \mu_R^i(L)$.
(c) $L = \mathfrak{a}L$ *if and only if* $\operatorname{depth}_R(\mathfrak{a}; L^\vee) > 0$.
(d) $L^\vee = \mathfrak{a}(L^\vee)$ *if and only if* $\operatorname{depth}_R(\mathfrak{a}; L) > 0$.
(e) $\operatorname{depth}_R(\mathfrak{a}; L) > 0$ *if and only if $\mathfrak{a}$ contains a non-zero-divisor for L.*

Proof. Part (a) is from [9, Proposition 4.4], and part (b) follows directly from this.

(c)–(d) These follow from part (a) since $L = \mathfrak{a}L$ if and only if $\operatorname{width}_R(\mathfrak{a}; L) > 0$.

(e) By definition, we need to show that $\operatorname{Hom}_R(R/\mathfrak{a}, L) = 0$ if and only if $\mathfrak{a}$ contains a non-zero-divisor for L. One implication is explicitly stated in [6, Proposition 1.2.3(a)]. One can prove the converse like [6, Proposition 1.2.3(b)], using the fact that $R/\mathfrak{a}$ is finitely generated. □

The next result characterizes artinian modules in terms of Bass numbers.

Lemma 1.14. *Let L be an R-module. The following conditions are equivalent:*

(i) *L is an artinian R-module;*
(ii) *L is an artinian $\widehat{R}$-module;*
(iii) *$\widehat{R} \otimes_R L$ is an artinian $\widehat{R}$-module; and*
(iv) *L is $\mathfrak{m}$-torsion and $\mu_R^0(L) < \infty$.*

Proof. (i) $\Longleftrightarrow$ (iv) If L is artinian over R, then it is $\mathfrak{m}$-torsion by Fact 1.2(a), and we have $\mu_R^0(L) < \infty$ by [7, Theorem 3.4.3]. For the converse, assume that L is $\mathfrak{m}$-torsion and $\mu^0 = \mu_R^0(L) < \infty$. Since L is $\mathfrak{m}$-torsion, so is $E_R(L)$. Thus, we have $E_R(L) \cong E^{\mu^0}$, which is artinian since $\mu^0 < \infty$. Since L is a submodule of the artinian module $E_R(L)$, it is also artinian.

To show the equivalence of the conditions (i)–(iii), first note that each of these conditions implies that L is $\mathfrak{m}$-torsion. (For condition (iii), use the monomorphism $L \to \widehat{R} \otimes_R L$.) Thus, for the rest of the proof, we assume that L is $\mathfrak{m}$-torsion.

Because of the equivalence (i) $\Longleftrightarrow$ (iv), it suffices to show that

$$\mu_R^0(L) = \mu_{\widehat{R}}^0(L) = \mu_{\widehat{R}}^0(\widehat{R} \otimes_R L).$$

These equalities follow from the next isomorphisms

$$\operatorname{Hom}_R(k, L) \cong \operatorname{Hom}_{\widehat{R}}(k, L) \cong \operatorname{Hom}_{\widehat{R}}(k, \widehat{R} \otimes_R L)$$

which are from Lemmas 1.5(a) and 1.4, respectively. □

Lemma 1.15. *Let L be an R-module.*

(a) *The module L is noetherian over R if and only if $L^\vee$ is artinian over R.*
(b) *If $L^\vee$ is noetherian over R or over $\widehat{R}$, then L is artinian over R.*
(c) *Let $\mathfrak{a}$ be a proper ideal of R such that $R/\mathfrak{a}$ is complete. If L is $\mathfrak{a}$-torsion, then L is artinian over R if and only if $L^\vee$ is noetherian over R.*

Proof. (a) This is [7, Corollary 3.4.4].

(b) If $L^\vee$ is noetherian over R, then we conclude from [7, Corollary 3.4.5] that L is artinian over R. To complete the proof of (b), we assume that $L^\vee$ is noetherian over $\widehat{R}$ and show that L is artinian. Fix a descending chain $L_1 \supseteq L_2 \supseteq \cdots$ of submodules of L. Dualize the surjections $L \twoheadrightarrow \cdots \twoheadrightarrow L/L_2 \twoheadrightarrow L/L_1$ to obtain a sequence of $\widehat{R}$-module monomorphisms $(L/L_1)^\vee \hookrightarrow (L/L_2)^\vee \hookrightarrow \cdots \hookrightarrow L^\vee$. The corresponding ascending chain of submodules must stabilize since $L^\vee$ is noetherian over $\widehat{R}$, and it follows that the original chain $L_1 \supseteq L_2 \supseteq \cdots$ of submodules of L also stabilizes. Thus L is artinian.

(c) Assume that L is $\mathfrak{a}$-torsion. One implication is from part (b). For the converse, assume that L is artinian over R. From [14, Theorem 18.6(v)] we know that $\operatorname{Hom}_{\widehat{R}}(L, E)$ is noetherian over $\widehat{R}$, and Lemma 1.5(a) implies that $L^\vee = \operatorname{Hom}_{\widehat{R}}(L, E)$. Thus, Lemma 1.3(b) implies that $L^\vee$ is noetherian over R. □

Mini-max and Matlis reflexive modules

Definition 1.16. An R-module M is *mini-max* if there is a noetherian submodule $N \subseteq M$ such that M/N is artinian.

Definition 1.17. An R-module M is *Matlis reflexive* provided that the natural biduality map $\delta_M : M \to M^{\vee\vee}$, given by $\delta_M(x)(\psi) = \psi(x)$, is an isomorphism.

Fact 1.18. An R-module M is Matlis reflexive if and only if it is mini-max and $R/\operatorname{Ann}_R(M)$ is complete; see [4, Theorem 12]. Thus, if M is mini-max over R, then $\widehat{R} \otimes_R M$ is Matlis reflexive over $\widehat{R}$.

Lemma 1.19. *If M is mini-max over R, then $\beta_i^R(M), \mu_R^i(M) < \infty$ for all $i \geqslant 0$.*

Proof. We show that $\mu^i_R(M) < \infty$ for all $i \geqslant 0$; then Remark 1.11(b) implies that $\beta^R_i(M) < \infty$ for all $i \geqslant 0$. The noetherian case is standard. If M is artinian, then we have $\mu^0 = \mu^0_R(M) < \infty$ by Lemma 1.14; since E^{μ^0} is artinian, an induction argument shows that $\mu^i_R(M) < \infty$ for all $i \geqslant 0$. One deduces the mini-max case from the artinian and noetherian cases, using a long exact sequence. □

Lemma 1.20. *Let L be an R-module such that $R/\operatorname{Ann}_R(L)$ is complete. The following conditions are equivalent:*

(i) *L is Matlis reflexive over R;*
(ii) *L is mini-max over R;*
(iii) *L is mini-max over $\widehat{R}$; and*
(iv) *L is Matlis reflexive over $\widehat{R}$.*

Proof. The equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) are from Fact 1.18. Note that conditions (iii) and (iv) make sense since L is an $\widehat{R}$-module; see Fact 1.2.

(ii) $\Longrightarrow$ (iii) Assume that L is mini-max over R, and fix a noetherian R-submodule $N \subseteq L$ such that L/N is artinian over R. As $R/\operatorname{Ann}_R(L)$ is complete and surjects onto $R/\operatorname{Ann}_R(N)$, we conclude that $R/\operatorname{Ann}_R(N)$ is complete. Fact 1.2(d) and Lemma 1.3(a) imply that N is an $\widehat{R}$-submodule. Similarly, Lemmas 1.3(b) and 1.14 imply that N is noetherian over $\widehat{R}$, and L/N is an artinian over $\widehat{R}$. Thus L is mini-max over $\widehat{R}$.

(iii) $\Longrightarrow$ (ii) Assume that L is mini-max over $\widehat{R}$, and fix a noetherian $\widehat{R}$-submodule $L' \subseteq L$ such that L/L' is artinian over $\widehat{R}$. Lemmas 1.3(b), 1.14 imply that L' is noetherian over R, and L/L' is artinian over R, so L is mini-max over R. □

Lemma 1.21. *Let L be an R-module such that $\mathfrak{m}^t L = 0$ for some integer $t \geqslant 1$. Then the following conditions are equivalent:*

(i) *L is mini-max over R (equivalently, over $\widehat{R}$);*
(ii) *L is artinian over R (equivalently, over $\widehat{R}$);*
(iii) *L is noetherian over R (equivalently, over $\widehat{R}$); and*
(iv) *L has finite length over R (equivalently, over $\widehat{R}$).*

Proof. Lemma 1.20 shows that L is mini-max over R if and only if it is mini-max over $\widehat{R}$. Also, L is artinian (resp., noetherian or finite length) over R if and only if it is artinian (resp., noetherian or finite length) over $\widehat{R}$ by Lemmas 1.14 and 1.3(b).

The equivalence of conditions (ii)–(iv) follows from an application of [7, Proposition 2.3.20] over the artinian ring $R/\mathfrak{m}^t$. The implication (ii) $\Longrightarrow$ (i) is evident. For the implication (i) $\Longrightarrow$ (ii), assume that L is mini-max over R. Given a noetherian submodule $N \subseteq L$ such that L/N is artinian, the implication (iii) $\Longrightarrow$ (ii) shows that N is artinian; hence so is L. □

Definition 1.22. A full subcategory of the category of R-modules is a *Serre subcategory* if it is closed under submodules, quotients, and extensions.

Lemma 1.23. *The category of mini-max (resp., noetherian, artinian, finite length, or Matlis reflexive) R-modules is a Serre subcategory.*

Proof. The noetherian, artinian, and finite length cases are standard, as is the Matlis reflexive case; see [7, p. 92, Exercise 2]. For the mini-max case, fix an exact sequence $0 \to L' \xrightarrow{f} L \xrightarrow{g} L'' \to 0$. Identify L' with $\operatorname{Im}(f)$. Assume first that L is mini-max, and fix a noetherian submodule N such that L/N is artinian. Then $L' \cap N$ is noetherian, and the quotient $L'/(L' \cap N) \cong (L' + N)/N$ is artinian, since it is a submodule of L/N. Thus L' is mini-max. Also, $(N + L')/L'$ is noetherian and $[L/L']/[(N + L')/L'] \cong L/(N + L')$ is artinian, so $L'' \cong L/L'$ is mini-max.

Next, assume that L' and L'' are mini-max, and fix noetherian submodules $N' \subseteq L'$ and $N'' \subseteq L''$ such that L'/N' and L''/N'' are artinian. Let $x_1, \ldots, x_h$ be coset representatives in L of a generating set for N''. Let $N = N' + Rx_1 + \ldots + Rx_h$. Then N is noetherian and the following commutative diagram has exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N \cap L' \arrow[r] \arrow[d, hook] & N \arrow[r] \arrow[d, hook] & N'' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & L' \arrow[r] & L \arrow[r] & L'' \arrow[r] & 0.
\end{tikzcd}$$

The sequence $0 \to L'/(N \cap L') \to L/N \to L''/N'' \to 0$ is exact by the Snake Lemma. The module $L'/(N \cap L')$ is artinian, being a quotient of L'/N'. Since the class of artinian modules is closed under extensions, the module L/N is artinian. It follows that L is mini-max. □

The next two lemmas apply to the classes of modules from Lemma 1.23.

Lemma 1.24. *Let $\mathcal{C}$ be a Serre subcategory of the category of R-modules.*

(a) *Given an exact sequence $L' \xrightarrow{f} L \xrightarrow{g} L''$, if $L', L'' \in \mathcal{C}$, then $L \in \mathcal{C}$.*
(b) *Given an R-complex X and an integer i, if $X_i \in \mathcal{C}$, then $\mathrm{H}_i(X) \in \mathcal{C}$.*
(c) *Given a noetherian R-module N, if $L \in \mathcal{C}$, then $\operatorname{Ext}^i_R(N, L), \operatorname{Tor}^R_i(N, L) \in \mathcal{C}$.*

Proof. (a) Assume that $L', L'' \in \mathcal{C}$. By assumption, $\mathrm{Im}(f), \mathrm{Im}(g) \in \mathcal{C}$. Using the exact sequence $0 \to \mathrm{Im}(f) \to L \to \mathrm{Im}(g) \to 0$, we conclude that L is in $\mathcal{C}$.

(b) The module $\mathrm{H}_i(X)$ is a subquotient of X_i, so it is in $\mathcal{C}$ by assumption.

(c) If F is a minimal free resolution of N, then the modules in the complexes $\mathrm{Hom}_R(F, L)$ and $F \otimes_R L$ are in $\mathcal{C}$, so their homologies are in $\mathcal{C}$ by part (b). $\square$

Lemma 1.25. *Let $R \to S$ be a local ring homomorphism, and let $\mathcal{C}$ be a Serre subcategory of the category of S-modules. Fix an S-module L, an R-module L', an R-submodule $L'' \subseteq L'$, and an index $i \geqslant 0$.*

(a) *If $\mathrm{Ext}^i_R(L, L''), \mathrm{Ext}^i_R(L, L'/L'') \in \mathcal{C}$, then $\mathrm{Ext}^i_R(L, L') \in \mathcal{C}$.*
(b) *If $\mathrm{Ext}^i_R(L'', L), \mathrm{Ext}^i_R(L'/L'', L) \in \mathcal{C}$, then $\mathrm{Ext}^i_R(L', L) \in \mathcal{C}$.*
(c) *If $\mathrm{Tor}^R_i(L, L''), \mathrm{Tor}^R_i(L, L'/L'') \in \mathcal{C}$, then $\mathrm{Tor}^R_i(L, L') \in \mathcal{C}$.*

Proof. We prove part (a); the other parts are proved similarly. Apply $\mathrm{Ext}^i_R(L, -)$ to the exact sequence $0 \to L'' \to L' \to L'/L'' \to 0$ to obtain the next exact sequence:

$$\mathrm{Ext}^i_R(L, L'') \to \mathrm{Ext}^i_R(L, L') \to \mathrm{Ext}^i_R(L, L'/L'').$$

Since L is an S-module, the maps in this sequence are S-module homomorphisms. Now, apply Lemma 1.24(a). $\square$

2. Properties of $\mathrm{Ext}^i_R(M, -)$

This section documents properties of the functors $\mathrm{Ext}^i_R(M, -)$ where M is a mini-max R-module.

Noetherianness of $\mathrm{Ext}^i_R(A, L)$

Lemma 2.1. *Let A and L be R-modules such that A is artinian and L is $\mathfrak{m}$-torsion.*

(a) *Then $\mathrm{Hom}_R(L, A) = \mathrm{Hom}_{\widehat{R}}(L, A) \cong \mathrm{Hom}_{\widehat{R}}(A^\vee, L^\vee)$.*
(b) *If L is artinian, then $\mathrm{Hom}_R(L, A)$ is a noetherian $\widehat{R}$-module.*

Proof. (a) The first equality is from Lemma 1.5(a). For the second equality, the fact that A is Matlis reflexive over $\widehat{R}$ explains the first step below:

$$\mathrm{Hom}_{\widehat{R}}(L, A) \cong \mathrm{Hom}_{\widehat{R}}(L, A^{vv}) \cong \mathrm{Hom}_{\widehat{R}}(A^v, L^v) \cong \mathrm{Hom}_{\widehat{R}}(A^\vee, L^\vee)$$

where $(-)^v = \mathrm{Hom}_{\widehat{R}}(-, E)$. The second step follows from Hom–tensor adjointness, and the third step is from Lemma 1.5(a).

(b) If L is artinian, then $L^\vee$ and $A^\vee$ are noetherian over $\widehat{R}$, so $\mathrm{Hom}_{\widehat{R}}(A^\vee, L^\vee)$ is also noetherian over $\widehat{R}$. $\square$

The next result contains part of Theorem 1 from the introduction. When R is not complete, the example $\mathrm{Hom}_R(E, E) \cong \widehat{R}$ shows that $\mathrm{Ext}^i_R(A, L)$ is not necessarily noetherian or artinian over R.

Theorem 2.2. *Let A and L be R-modules such that A is artinian. For each index $i \geqslant 0$ such that $\mu^i_R(L) < \infty$, the module $\mathrm{Ext}^i_R(A, L)$ is a noetherian $\widehat{R}$-module.*

Proof. Let J be a minimal R-injective resolution of L. Remark 1.11(a) implies that $\Gamma_{\mathfrak{m}}(J)^i \cong E^{\mu^i_R(L)}$. Lemma 1.5(b) explains the first isomorphism below:

$$\mathrm{Hom}_R(A, J)^i \cong \mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(J)^i) \cong \mathrm{Hom}_R(A, E)^{\mu^i_R(L)}.$$

Lemma 2.1 implies that these are noetherian $\widehat{R}$-modules. The differentials in the complex $\mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(J))$ are $\widehat{R}$-linear because A is an $\widehat{R}$-module. Thus, the subquotient $\mathrm{Ext}^i_R(A, L)$ is a noetherian $\widehat{R}$-module. $\square$

Corollary 2.3. *Let A and M be R-modules such that A is artinian and M is mini-max. For each index $i \geqslant 0$, the module $\mathrm{Ext}^i_R(A, M)$ is a noetherian $\widehat{R}$-module.*

Proof. Apply Theorem 2.2 and Lemma 1.19. $\square$

The next result contains part of Theorem 3 from the introduction.

Corollary 2.4. *Let A and L be R-modules such that $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(L))$ is complete and A is artinian. For each index $i \geqslant 0$ such that $\mu^i_R(L) < \infty$, the module $\mathrm{Ext}^i_R(A, L)$ is noetherian and Matlis reflexive over R and $\widehat{R}$.*

Proof. Theorem 2.2 shows that $\mathrm{Ext}^i_R(A, L)$ is noetherian over $\widehat{R}$; so, it is Matlis reflexive over $\widehat{R}$. As $\mathrm{Ann}_R(A) + \mathrm{Ann}_R(L) \subseteq \mathrm{Ann}_R(\mathrm{Ext}^i_R(A, L))$, Lemmas 1.3(b) and 1.20 imply that $\mathrm{Ext}^i_R(A, L)$ is noetherian and Matlis reflexive over R. $\square$

Corollary 2.5. *Let A and L be R-modules such that $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(L))$ is artinian and A is artinian. Given an index $i \geqslant 0$ such that $\mu^i_R(L) < \infty$, one has $\mathrm{len}_R(\mathrm{Ext}^i_R(A, L)) < \infty$.*

Proof. Apply Theorem 2.2 and Lemma 1.21. $\square$

Matlis reflexivity of $\mathrm{Ext}^i_R(M, M')$

Theorem 2.6. *Let A and M be R-modules such that A is artinian and M is mini-max. For each $i \geqslant 0$, the module $\mathrm{Ext}^i_R(M, A)$ is Matlis reflexive over $\widehat{R}$.*

Proof. Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. Since A is artinian, it is an $\widehat{R}$-module. Corollary 2.3 implies that $\mathrm{Ext}^i_R(M/N, A)$ is a noetherian $\widehat{R}$-module. As $\mathrm{Ext}^i_R(N, A)$ is artinian, Lemma 1.25(b) says that $\mathrm{Ext}^i_R(M, A)$ is a mini-max $\widehat{R}$-module and hence is Matlis reflexive over $\widehat{R}$ by Fact 1.18. □

Theorem 2.7. *Let M and N' be R-modules such that M is mini-max and N' is noetherian. Fix an index $i \geqslant 0$. If $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(N'))$ is complete, then $\mathrm{Ext}^i_R(M, N')$ is noetherian and Matlis reflexive over R and over $\widehat{R}$.*

Proof. Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. If the ring $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(N'))$ is complete, then so is $R/(\mathrm{Ann}_R(M/N) + \mathrm{Ann}_R(N'))$. Corollary 2.4 implies that $\mathrm{Ext}^i_R(M/N, N')$ is noetherian over R. Since $\mathrm{Ext}^i_R(N, N')$ is noetherian over R, Lemma 1.25(b) implies that $\mathrm{Ext}^i_R(M, N')$ is noetherian over R. As $R/(\mathrm{Ann}_R(\mathrm{Ext}^i_R(M, N')))$ is complete, Fact 1.18 implies that $\mathrm{Ext}^i_R(M, N')$ is also Matlis reflexive over R. Thus $\mathrm{Ext}^i_R(M, N')$ is noetherian and Matlis reflexive over $\widehat{R}$ by Lemmas 1.3(b) and 1.20. □

Theorem 2.8. *Let M and M' be mini-max R-modules, and fix an index $i \geqslant 0$.*

(a) *If $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(M'))$ is complete, then $\mathrm{Ext}^i_R(M, M')$ is Matlis reflexive over R and $\widehat{R}$.*
(b) *If $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(M'))$ is artinian, then $\mathrm{Ext}^i_R(M, M')$ has finite length.*

Proof. Fix a noetherian submodule $N' \subseteq M'$ such that M'/N' is artinian.

(a) Assume that $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(M'))$ is complete. Theorem 2.7 implies that the module $\mathrm{Ext}^i_R(M, N')$ is Matlis reflexive over R. Theorem 2.6 shows that $\mathrm{Ext}^i_R(M, M'/N')$ is Matlis reflexive over $\widehat{R}$; hence, it is Matlis reflexive over R by Lemma 1.20. Thus, Lemmas 1.25(a) and 1.20 imply that $\mathrm{Ext}^i_R(M, M')$ is Matlis reflexive over R and $\widehat{R}$.

(b) This follows from part (a), because of Fact 1.18 and Lemma 1.21. □

A special case of the next result can be found in [3, Theorem 3].

Corollary 2.9. *Let M and M' be R-modules such that M is mini-max and M' is Matlis reflexive. For each index $i \geqslant 0$, the modules $\mathrm{Ext}^i_R(M, M')$ and $\mathrm{Ext}^i_R(M', M)$ are Matlis reflexive over R and $\widehat{R}$.*

Proof. Apply Theorem 2.8(a) and Fact 1.18. □

Length Bounds for $\mathrm{Hom}_R(A, L)$

Lemma 2.10. *Let A and L be R-modules such that A is artinian and $\mathfrak{m}^n\Gamma_{\mathfrak{m}}(L) = 0$ for some $n \geqslant 1$. Fix an index $t \geqslant 0$ such that $\mathfrak{m}^tA = \mathfrak{m}^{t+1}A$, and let s be an integer such that $s \geqslant \min(n, t)$. Then*

$$\mathrm{Hom}_R(A, L) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA, L) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA, (0 :_L \mathfrak{m}^s)).$$

Proof. Given any map $\psi \in \mathrm{Hom}_R(A/\mathfrak{m}^sA, L)$, the image of ψ is annihilated by $\mathfrak{m}^s$. That is, $\mathrm{Im}(\psi) \subseteq (0 :_L \mathfrak{m}^s)$; hence $\mathrm{Hom}_R(A/\mathfrak{m}^sA, L) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA, (0 :_L \mathfrak{m}^s))$. In the next sequence, the first and third isomorphisms are from Lemma 1.5(b):

$$\mathrm{Hom}_R(A, L) \cong \mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA, L).$$

For the second isomorphism, we argue by cases. If $s \geqslant n$, then we have $\mathfrak{m}^s\Gamma_{\mathfrak{m}}(L) = 0$ because $\mathfrak{m}^n\Gamma_{\mathfrak{m}}(L) = 0$, and the isomorphism is evident. If $s < n$, then we have $n > s \geqslant t$, so $\mathfrak{m}^tA = \mathfrak{m}^sA = \mathfrak{m}^nA$ since $\mathfrak{m}^tA = \mathfrak{m}^{t+1}A$; it follows that $\mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^nA, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA, \Gamma_{\mathfrak{m}}(L))$. □

For the next result, the example $\mathrm{Hom}_R(E, E) \cong \widehat{R}$ shows that the condition $\mathfrak{m}^n\Gamma_{\mathfrak{m}}(L) = 0$ is necessary.

Theorem 2.11. *Let A and L be R-modules such that A is artinian and $\mathfrak{m}^n\Gamma_{\mathfrak{m}}(L) = 0$ for some $n \geqslant 1$. Fix an index $t \geqslant 0$ such that $\mathfrak{m}^tA = \mathfrak{m}^{t+1}A$, and let s be an integer such that $s \geqslant \min(n, t)$. Then there is an inequality*

$$\mathrm{len}_R(\mathrm{Hom}_R(A, L)) \leqslant \beta^R_0(A)\,\mathrm{len}_R(0 :_L \mathfrak{m}^s).$$

Here, we use the convention $0 \cdot \infty = 0$.

Proof. We deal with the degenerate case first. If $\beta^R_0(A) = 0$, then $A/\mathfrak{m}A = 0$, so

$$\mathrm{Hom}_R(A, L) \cong \mathrm{Hom}_R(A/\mathfrak{m}A, L) = \mathrm{Hom}_R(0, L) = 0$$

by Lemma 2.10. So, we assume for the rest of the proof that $\beta^R_0(A) \neq 0$. We also assume without loss of generality that $\mathrm{len}_R(0 :_L \mathfrak{m}^s) < \infty$.

Lemma 2.10 explains the first step in the following sequence:

$$\begin{aligned}\mathrm{len}_R(\mathrm{Hom}_R(A,L)) &= \mathrm{len}_R(\mathrm{Hom}_R(A/\mathfrak{m}^s A,(0:_L \mathfrak{m}^s)))\\ &\leqslant \beta_0^R(A/\mathfrak{m}^s A)\,\mathrm{len}_R(0:_L \mathfrak{m}^s)\\ &= \beta_0^R(A)\,\mathrm{len}_R(0:_L \mathfrak{m}^s).\end{aligned}$$

The second step can be proved by induction on $\beta_0^R(A/\mathfrak{m}^s A)$ and $\mathrm{len}_R(0:_L \mathfrak{m}^s)$. □

The next result can also be obtained as a corollary to [8, Proposition 6.1]. Example 6.3 shows that $\mathrm{len}_R(\mathrm{Ext}^i_R(A,N))$ can be infinite when $i \geqslant 1$.

Corollary 2.12. *If A and N are R-modules such that A is artinian and N is noetherian, then* $\mathrm{len}_R(\mathrm{Hom}_R(A,N)) < \infty$.

Proof. Apply Theorem 2.11 and Lemma 1.19. □

3. Properties of $\mathrm{Tor}_i^R(M,-)$

This section focuses on properties of the functors $\mathrm{Tor}_i^R(M,-)$ where M is a mini-max R-module.

Artinianness of $\mathrm{Tor}_i^R(A,L)$

The next result contains part of Theorem 1 from the introduction. Recall that a module is artinian over R if and only if it is artinian over $\widehat{R}$; see Lemma 1.14.

Theorem 3.1. *Let A and L be R-modules such that A is artinian. For each index $i \geqslant 0$ such that $\beta_i^R(L) < \infty$, the module $\mathrm{Tor}_i^R(A,L)$ is artinian.*

Proof. Lemma 1.13(b) implies that $\mu_R^i(L^\vee) = \beta_i^R(L) < \infty$. By Remark 1.9, we have $\mathrm{Ext}^i_R(A,L^\vee) \cong \mathrm{Tor}_i^R(A,L)^\vee$. Thus, $\mathrm{Tor}_i^R(A,L)^\vee$ is a noetherian $\widehat{R}$-module by Theorem 2.2, and we conclude that $\mathrm{Tor}_i^R(A,L)$ is artinian by Lemma 1.15(b). □

For the next result, the example $E \otimes_R R \cong E$ shows that $\mathrm{Tor}_i^R(A,L)$ is not necessarily noetherian over R or $\widehat{R}$.

Corollary 3.2. *Let A and M be R-modules such that A is artinian and M mini-max. For each index $i \geqslant 0$, the module $\mathrm{Tor}_i^R(A,M)$ is artinian.*

Proof. Apply Theorem 3.1 and Lemma 1.19. □

The proofs of the next two results are similar to those of Corollaries 2.4 and 2.5. The first result contains part of Theorem 3 from the introduction.

Corollary 3.3. *Let A and L be R-modules such that $R/(\mathrm{Ann}_R(A)+\mathrm{Ann}_R(L))$ is complete and A is artinian. For each index $i \geqslant 0$ such that $\beta_i^R(L) < \infty$, the module $\mathrm{Tor}_i^R(A,L)$ is artinian and Matlis reflexive over R and $\widehat{R}$.*

Corollary 3.4. *Let A and L be R-modules such that $R/(\mathrm{Ann}_R(A)+\mathrm{Ann}_R(L))$ is artinian and A is artinian. Given an index $i \geqslant 0$ such that $\beta_i^R(L) < \infty$, one has $\mathrm{len}_R(\mathrm{Tor}_i^R(A,L)) < \infty$.*

$\mathrm{Tor}_i^R(M,M')$ *is Mini-max*

Theorem 3.5. *Let M and M′ be mini-max R-modules, and fix an index $i \geqslant 0$.*

(a) *The R-module $\mathrm{Tor}_i^R(M,M')$ is mini-max over R.*
(b) *If $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(M'))$ is complete, then $\mathrm{Tor}_i^R(M,M')$ is Matlis reflexive over R and $\widehat{R}$.*
(c) *If $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(M'))$ is artinian, then $\mathrm{Tor}_i^R(M,M')$ has finite length.*

Proof. (a) Choose a noetherian submodule $N \subseteq M$ such that M/N is artinian. Lemmas 1.23 and 1.24(c) say that $\mathrm{Tor}_i^R(N,M')$ is mini-max. Corollary 3.2 implies that $\mathrm{Tor}_i^R(M/N,M')$ mini-max, so $\mathrm{Tor}_i^R(M,M')$ is mini-max by Lemma 1.25(c).
Parts (b) and (c) now follow from Lemmas 1.20 and 1.21. □

A special case of the next result is contained in [3, Theorem 3].

Corollary 3.6. *Let M and M′ be R-modules such that M is mini-max and M′ is Matlis reflexive. For each index $i \geqslant 0$, the module $\mathrm{Tor}_i^R(M,M')$ is Matlis reflexive over R and $\widehat{R}$.*

Proof. Apply Theorem 3.5(b) and Fact 1.18. □

Length Bounds for $A \otimes_R L$

Lemma 3.7. *Let A be an artinian module, and let $\mathfrak{a}$ be a proper ideal of R. Fix an integer $t \geqslant 0$ such that $\mathfrak{a}^t A = \mathfrak{a}^{t+1} A$. Given an $\mathfrak{a}$-torsion R-module L, one has*

$$A \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L).$$

Proof. The isomorphism $(A/\mathfrak{a}^t A) \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L)$ is from the following:

$$\begin{aligned}(A/\mathfrak{a}^t A) \otimes_R L &\cong [(A/\mathfrak{a}^t A) \otimes_R (R/\mathfrak{a}^t)] \otimes_R L \\ &\cong (A/\mathfrak{a}^t A) \otimes_R [(R/\mathfrak{a}^t) \otimes_R L] \\ &\cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L).\end{aligned}$$

For the isomorphism $A \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R L$, consider the exact sequence:

$$0 \to \mathfrak{a}^t A \to A \to A/\mathfrak{a}^t A \to 0.$$

The exact sequence induced by $- \otimes_R L$ has the form

$$(\mathfrak{a}^t A) \otimes_R L \to A \otimes_R L \to (A/\mathfrak{a}^t A) \otimes_R L \to 0. \tag{3.7.1}$$

The fact that L is $\mathfrak{a}$-torsion and $\mathfrak{a}^t A = \mathfrak{a}^{t+i} A$ for all $i \geqslant 1$ implies that $(\mathfrak{a}^t A) \otimes_R L = 0$, so the sequence (3.7.1) yields the desired isomorphism. $\square$

The example $E \otimes_R R \cong R$ shows that the $\mathfrak{m}$-torsion assumption on L is necessary in the next result.

Theorem 3.8. *Let A be an artinian R-module, and let L be an $\mathfrak{m}$-torsion R-module. Fix an integer $t \geqslant 0$ such that $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$. Then there are inequalities*

$$\operatorname{len}_R(A \otimes_R L) \leqslant \operatorname{len}_R\left(A/\mathfrak{m}^t A\right) \beta_0^R(L) \tag{3.8.1}$$

$$\operatorname{len}_R(A \otimes_R L) \leqslant \beta_0^R(A) \operatorname{len}_R\left(L/\mathfrak{m}^t L\right). \tag{3.8.2}$$

Here we use the convention $0 \cdot \infty = 0$.

Proof. From Lemma 3.7 we have

$$A \otimes_R L \cong (A/\mathfrak{m}^t A) \otimes_R (L/\mathfrak{m}^t L). \tag{3.8.3}$$

Lemmas 1.19 and 1.21 imply that $\operatorname{len}_R(A/\mathfrak{m}^t A) < \infty$ and $\beta_0^R(A) < \infty$.

For the degenerate cases, first note that $\operatorname{len}_R(A/\mathfrak{m}^t A) = 0$ if and only if $\beta_0^R(A) = 0$. When $\operatorname{len}_R(A/\mathfrak{m}^t A) = 0$, the isomorphism (3.8.3) implies that $A \otimes_R L = 0$; hence the desired inequalities. Thus, we assume without loss of generality that $1 \leqslant \beta_0^R(A) \leqslant \operatorname{len}_R(A/\mathfrak{m}^t A)$. Further, we assume that $\beta_0^R(L) < \infty$.

The isomorphism (3.8.3) provides the first step in the next sequence:

$$\operatorname{len}_R(A \otimes_R L) = \operatorname{len}_R((A/\mathfrak{m}^t A) \otimes_R (L/\mathfrak{m}^t L)) \leqslant \operatorname{len}_R(A/\mathfrak{m}^t A)\beta_0^R(L).$$

The second step in this sequence can be verified by induction on $\operatorname{len}_R(A/\mathfrak{m}^t A)$ and $\beta_0^R(L)$. This explains the inequality (3.8.1), and (3.8.2) is verified similarly. $\square$

The next corollary recovers [8, Proposition 6.1]. Note that Example 6.4 shows that $\operatorname{len}_R(\operatorname{Tor}_i^R(A, A'))$ can be infinite when $i \geqslant 1$.

Corollary 3.9. *If A and A' are artinian R-modules, then $\operatorname{len}_R(A \otimes_R A') < \infty$.*

Proof. Apply Theorem 3.8 and Lemmas 1.19 and 1.21. (Alternatively, apply Corollary 2.12 and Matlis duality.) $\square$

4. The Matlis dual of $\operatorname{Ext}_R^i(L, L')$

This section contains the proof of Theorem 4 from the introduction; see Corollary 4.11. Most of the section is devoted to technical results for use in the proof.

Lemma 4.1. *Let L be an R-module. If I is an R-injective resolution of L, and J is an $\widehat{R}$-injective resolution of $\widehat{R} \otimes_R L$, then there is a homotopy equivalence $\Gamma_{\mathfrak{m}}(I) \xrightarrow{\sim} \Gamma_{\mathfrak{m}}(J) = \Gamma_{\mathfrak{m}\widehat{R}}(J)$.*

Proof. Each injective $\widehat{R}$-module J' is injective over R; this follows from the isomorphism $\operatorname{Hom}_R(-, J') \cong \operatorname{Hom}_R(-, \operatorname{Hom}_{\widehat{R}}(\widehat{R}, J')) \cong \operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R -, J')$ since $\widehat{R}$ is flat over R. Hence, there is a lift $f : I \to J$ of the natural map $\xi : L \to \widehat{R} \otimes_R L$. This lift is a chain map of R-complexes.

We show that the induced map $\Gamma_{\mathfrak{m}}(f)\colon \Gamma_{\mathfrak{m}}(I) \to \Gamma_{\mathfrak{m}}(J) = \Gamma_{\mathfrak{m}\widehat{R}}(J)$ is a homotopy equivalence. As $\Gamma_{\mathfrak{m}}(I)$ and $\Gamma_{\mathfrak{m}}(J)$ are bounded above complexes of injective R-modules, it suffices to show that $\Gamma_{\mathfrak{m}}(f)$ induces an isomorphism on homology in each degree. The induced map on homology is compatible with the following sequence:

$$\mathrm{H}^i(\Gamma_{\mathfrak{m}}(I)) \cong \mathrm{H}^i_{\mathfrak{m}}(L) \xrightarrow[\cong]{\mathrm{H}^i_{\mathfrak{m}}(\xi)} \mathrm{H}^i_{\mathfrak{m}}(\widehat{R}\otimes_R L) \cong \mathrm{H}^i(\Gamma_{\mathfrak{m}}(J)).$$

The map $\mathrm{H}^i_{\mathfrak{m}}(\xi)\colon \mathrm{H}^i_{\mathfrak{m}}(L) \to \mathrm{H}^i_{\mathfrak{m}}(\widehat{R}\otimes_R L)$ is an isomorphism (see the proof of [6, Proposition 3.5.4(d)]) so we have the desired homotopy equivalence. □

Lemma 4.2. *Let L and L' be R-modules such that L is $\mathfrak{m}$-torsion. Then for each index $i \geqslant 0$, there are $\widehat{R}$-module isomorphisms*

$$\mathrm{Ext}^i_R(L, L') \cong \mathrm{Ext}^i_R(L, \widehat{R}\otimes_R L') \cong \mathrm{Ext}^i_{\widehat{R}}(L, \widehat{R}\otimes_R L').$$

Proof. Let I be an R-injective resolution of L', and let J be an $\widehat{R}$-injective resolution of $\widehat{R}\otimes_R L'$. Because L is $\mathfrak{m}$-torsion, Lemma 1.5(b) explains the first, third and sixth steps in the next display:

$$\mathrm{Hom}_R(L, I) \cong \mathrm{Hom}_R(L, \Gamma_{\mathfrak{m}}(I)) \sim \mathrm{Hom}_R(L, \Gamma_{\mathfrak{m}}(J)) \cong \mathrm{Hom}_R(L, J)$$
$$\mathrm{Hom}_R(L, \Gamma_{\mathfrak{m}}(J)) = \mathrm{Hom}_R(L, \Gamma_{\mathfrak{m}\widehat{R}}(J)) = \mathrm{Hom}_{\widehat{R}}(L, \Gamma_{\mathfrak{m}\widehat{R}}(J)) \cong \mathrm{Hom}_{\widehat{R}}(L, J).$$

The homotopy equivalence in the second step is from Lemma 4.1. The fifth step is from Lemma 1.5(a). Since L is $\mathfrak{m}$-torsion, it is an $\widehat{R}$-module, so the isomorphisms and the homotopy equivalence in this sequence are $\widehat{R}$-linear. In particular, the complexes $\mathrm{Hom}_R(L, I)$ and $\mathrm{Hom}_R(L, J)$ and $\mathrm{Hom}_{\widehat{R}}(L, J)$ have isomorphic cohomology over $\widehat{R}$, so one has the desired isomorphisms. □

The next result contains Theorem 2 from the introduction. It shows, for instance, that given artinian R-modules A and A', there are noetherian $\widehat{R}$-modules N and N' such that $\mathrm{Ext}^i_R(A, A') \cong \mathrm{Ext}^i_{\widehat{R}}(N, N')$; thus, it provides an alternate proof of Corollary 2.3.

Theorem 4.3. *Let A and M be R-modules such that A is artinian and M is mini-max. Then for each index $i \geqslant 0$, we have $\mathrm{Ext}^i_R(A, M) \cong \mathrm{Ext}^i_{\widehat{R}}(M^\vee, A^\vee)$.*

Proof. Case 1: R is complete. Let F be a free resolution of A. It follows that each F_i is flat, so the complex $F^\vee$ is an injective resolution of $A^\vee$; see [7, Theorem 3.2.9]. We obtain the isomorphism $\mathrm{Ext}^i_R(A, M) \cong \mathrm{Ext}^i_R(M^\vee, A^\vee)$ by taking cohomology in the next sequence:

$$\mathrm{Hom}_R(F, M) \cong \mathrm{Hom}_R(F, M^{\vee\vee}) \cong \mathrm{Hom}_R(M^\vee, F^\vee).$$

The first step follows from the fact that M is Matlis reflexive; see Fact 1.18. The second step is from Hom–tensor adjointness

Case 2: the general case. The first step below is from Lemma 4.2:

$$\mathrm{Ext}^i_R(A, M) \cong \mathrm{Ext}^i_{\widehat{R}}(A, \widehat{R}\otimes_R M) \cong \mathrm{Ext}^i_{\widehat{R}}((\widehat{R}\otimes_R M)^v, A^v) \cong \mathrm{Ext}^i_{\widehat{R}}(M^\vee, A^\vee).$$

Here $(-)^v = \mathrm{Hom}_{\widehat{R}}(-, E)$. Since M is mini-max, it follows that $\widehat{R}\otimes_R M$ is mini-max over $\widehat{R}$. Thus, the second step is from Case 1. For the third step use Hom–tensor adjointness and Lemma 1.5(a) to see that $(\widehat{R}\otimes_R M)^v \cong M^\vee$ and $A^v \cong A^\vee$. □

Fact 4.4. Let L and L' be R-modules, and fix an index $i \geqslant 0$. Then the following diagram commutes, where the unlabeled isomorphism is from Remark 1.9:

$$\begin{tikzcd}
\mathrm{Ext}^i_R(L', L) \arrow[r, "\delta_{\mathrm{Ext}^i_R(L',L)}"] \arrow[d, "\mathrm{Ext}^i_R(L',\delta_L)"'] & \mathrm{Ext}^i_R(L', L)^{\vee\vee} \arrow[d, "(\Theta^i_{L'L})^\vee"] \\
\mathrm{Ext}^i_R(L', L^{\vee\vee}) \arrow[r, "\cong"] & \mathrm{Tor}_i^R(L', L^\vee)^\vee.
\end{tikzcd}$$

Lemma 4.5. *Let L be an R-module, and fix an index $i \geqslant 0$. If $\mu^i_R(L) < \infty$, then the map $\mathrm{Ext}^i_R(k, \delta_L)\colon \mathrm{Ext}^i_R(k, L) \to \mathrm{Ext}^i_R(k, L^{\vee\vee})$ is an isomorphism.*

Proof. The assumption $\mu^i_R(L) < \infty$ says that $\mathrm{Ext}^i_R(k, L)$ is a finite dimensional k-vector space, so it is Matlis reflexive over R; that is, the map

$$\delta_{\mathrm{Ext}^i_R(k,L)}\colon \mathrm{Ext}^i_R(k, L) \to \mathrm{Ext}^i_R(k, L)^{\vee\vee}$$

is an isomorphism. Since k is finitely generated, Remark 1.9 implies that

$$\Theta^i_{kL}\colon \mathrm{Tor}_i^R(k, L^\vee) \to \mathrm{Ext}^i_R(k, L)^\vee$$

is an isomorphism. Hence $(\Theta^i_{kL})^\vee$ is also an isomorphism. Using Fact 4.4 with $L' = k$, we conclude that $\mathrm{Ext}^i_R(k, \delta_L)$ is an isomorphism, as desired. □

Lemma 4.6. *Let A and L be R-modules such that A is artinian. Fix an index $i \geqslant 0$ such that $\mu_R^{i-1}(L)$, $\mu_R^i(L)$ and $\mu_R^{i+1}(L)$ are finite. Then the map*

$$\mathrm{Ext}_R^i(A, \delta_L) \colon \mathrm{Ext}_R^i(A, L) \to \mathrm{Ext}_R^i(A, L^{\vee\vee})$$

is an isomorphism.

Proof. Lemma 4.5 implies that for $t = i-1, i, i+1$ the maps

$$\mathrm{Ext}_R^t(k, \delta_L) \colon \mathrm{Ext}_R^t(k, L) \to \mathrm{Ext}_R^t(k, L^{\vee\vee})$$

are isomorphisms. As the biduality map δ_L is injective, we have an exact sequence

$$0 \to L \xrightarrow{\delta_L} L^{\vee\vee} \to \mathrm{Coker}\,\delta_L \to 0. \tag{4.6.1}$$

Using the long exact sequence associated to $\mathrm{Ext}_R(k, -)$, we conclude that for $t = i-1, i$ we have $\mathrm{Ext}_R^t(k, \mathrm{Coker}\,\delta_L) = 0$. In other words, we have $\mu_R^t(\mathrm{Coker}\,\delta_L) = 0$.

Let J be a minimal injective resolution of $\mathrm{Coker}\,\delta_L$. The previous paragraph shows that for $t = i-1, i$ the module J^t does not have E as a summand by Remark 1.11(a). That is, we have $\Gamma_{\mathfrak{m}}(J^t) = 0$, so Lemma 1.5(b) implies that

$$\mathrm{Hom}_R(A, J^t) \cong \mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(J^t)) = 0.$$

It follows that $\mathrm{Ext}_R^t(A, \mathrm{Coker}(\delta_L)) = 0$ for $t = i-1, i$. From the long exact sequence associated to $\mathrm{Ext}_R(A, -)$ with respect to (4.6.1), it follows that $\mathrm{Ext}_R^i(A, \delta_L)$ is an isomorphism, as desired. $\square$

We are now ready to tackle the main results of this section.

Theorem 4.7. *Let A and L be R-modules such that A is artinian. Fix an index $i \geqslant 0$ such that $\mu_R^{i-1}(L)$, $\mu_R^i(L)$ and $\mu_R^{i+1}(L)$ are finite.*

(a) *There is an R-module isomorphism $\mathrm{Ext}_R^i(A, L)^v \cong \mathrm{Tor}_i^R(A, L^\vee)$ where $(-)^v = \mathrm{Hom}_{\widehat{R}}(-, E)$.*
(b) *If $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(L))$ is complete, then Θ_{AL}^i provides an isomorphism $\mathrm{Tor}_i^R(A, L^\vee) \cong \mathrm{Ext}_R^i(A, L)^\vee$.*

Proof. (b) Corollary 2.4 and Lemma 4.6 show that the maps

$$\delta_{\mathrm{Ext}_R^i(A,L)} \colon \mathrm{Ext}_R^i(A, L) \to \mathrm{Ext}_R^i(A, L)^{\vee\vee}$$
$$\mathrm{Ext}_R^i(A, \delta_L) \colon \mathrm{Ext}_R^i(A, L) \to \mathrm{Ext}_R^i(A, L^{\vee\vee})$$

are isomorphisms. Fact 4.4 implies that $(\Theta_{AL}^i)^\vee$ is an isomorphism, so we conclude that Θ_{AL}^i is also an isomorphism.

(a) Lemma 4.2 explains the first step in the next sequence:

$$\begin{aligned}
\mathrm{Ext}_R^i(A, L)^v &\cong \mathrm{Ext}_{\widehat{R}}^i(A, \widehat{R} \otimes_R L)^v \\
&\cong \mathrm{Tor}_i^{\widehat{R}}(A, (\widehat{R} \otimes_R L)^v) \\
&\cong \mathrm{Tor}_i^R(A, (\widehat{R} \otimes_R L)^v) \\
&\cong \mathrm{Tor}_i^R(A, L^\vee).
\end{aligned}$$

The second step is from part (b), as $\widehat{R}$ is complete and $\mu_{\widehat{R}}^t(\widehat{R} \otimes_R L) = \mu_R^t(L) < \infty$ for $t = i-1, i, i+1$. The fourth step is from Hom–tensor adjointness. For the third step, let P be a projective resolution of A over R. Since $\widehat{R}$ is flat over R, the complex $\widehat{R} \otimes_R P$ is a projective resolution of $\widehat{R} \otimes_R A \cong A$ over $\widehat{R}$; see Lemma 1.4(a). Thus, the third step follows from the isomorphism $(\widehat{R} \otimes_R P) \otimes_{\widehat{R}} (\widehat{R} \otimes_R L)^v \cong P \otimes_R (\widehat{R} \otimes_R L)^v$. $\square$

Question 4.8. Do the conclusions of Lemma 4.6 and Theorem 4.7 hold when one only assumes that $\mu_R^i(L)$ is finite?

Corollary 4.9. *Let A and M be R-modules such that A is artinian and M is mini-max. For each index $i \geqslant 0$, one has $\mathrm{Ext}_R^i(A, M)^v \cong \mathrm{Tor}_i^R(A, M^\vee)$, where $(-)^v = \mathrm{Hom}_{\widehat{R}}(-, E)$.*

Proof. Apply Theorem 4.7(a) and Lemma 1.19. $\square$

Theorem 4.10. *Let M and M' be mini-max R-modules, and fix an index $i \geqslant 0$. If $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(M'))$ is complete, then $\Theta_{MM'}^i$ is an isomorphism, so*

$$\mathrm{Ext}_R^i(M, M')^v = \mathrm{Ext}_R^i(M, M')^\vee \cong \mathrm{Tor}_i^R(M, M'^\vee)$$

where $(-)^v = \mathrm{Hom}_{\widehat{R}}(-, E)$.

Proof. Theorem 2.8(a) implies that $\mathrm{Ext}^i_R(M, M')$ is Matlis reflexive over R, so Lemma 1.5(a) and Fact 1.18 imply that $\mathrm{Ext}^i_R(M, M')^v = \mathrm{Ext}^i_R(M, M')^\vee$. Thus, it remains to show that $\Theta^i_{MM'}$ is an isomorphism.

Case 1: M is noetherian. In the next sequence, the first and last steps are from Hom–tensor adjointness. The second step is standard since M is noetherian:

$$\begin{aligned}\mathrm{Ext}^i_R(M, M')^\vee &\cong (\widehat{R} \otimes_R \mathrm{Ext}^i_R(M, M'))^v\\ &\cong \mathrm{Ext}^i_{\widehat{R}}(\widehat{R} \otimes_R M, \widehat{R} \otimes_R M')^v\\ &\cong \mathrm{Tor}^{\widehat{R}}_i(\widehat{R} \otimes_R M, (\widehat{R} \otimes_R M')^v)\\ &\cong \mathrm{Tor}^R_i(M, (\widehat{R} \otimes_R M')^v)\\ &\cong \mathrm{Tor}^R_i(M, M'^\vee).\end{aligned}$$

Since M and M' are mini-max over R, the modules $\widehat{R} \otimes_R M$ and $\widehat{R} \otimes_R M'$ are Matlis reflexive over $\widehat{R}$; see Fact 1.18. Thus [2, Theorem 4(c)] explains the third step. The fourth step is from the fact that $\widehat{R}$ is flat over R. Since these isomorphisms are compatible with $\Theta^i_{MM'}$, it follows that $\Theta^i_{MM'}$ is an isomorphism.

Case 2: the general case. Since M is mini-max over R, there is an exact sequence of R-modules homomorphisms $0 \to N \to M \to A \to 0$ such that N is noetherian and A is artinian. The long exact sequences associated to $\mathrm{Tor}^R(-, M'^\vee)$ and $\mathrm{Ext}_R(-, M')$ fit into the following commutative diagram:

$$\begin{tikzcd}\cdots \arrow[r] & \mathrm{Tor}^R_i(N, M'^\vee) \arrow[r] \arrow[d, "\Theta^i_{NM'}"] & \mathrm{Tor}^R_i(M, M'^\vee) \arrow[r] \arrow[d, "\Theta^i_{MM'}"] & \mathrm{Tor}^R_i(A, M'^\vee) \arrow[r] \arrow[d, "\Theta^i_{AM'}"] & \cdots\\ \cdots \arrow[r] & \mathrm{Ext}^i_R(N, M')^\vee \arrow[r] & \mathrm{Ext}^i_R(M, M')^\vee \arrow[r] & \mathrm{Ext}^i_R(A, M')^\vee \arrow[r] & \cdots .\end{tikzcd}$$

Case 1 shows that $\Theta^i_{NM'}$ and $\Theta^{i-1}_{NM'}$ are isomorphisms. Theorem 4.7(b) implies that $\Theta^i_{AM'}$ and $\Theta^{i+1}_{AM'}$ are isomorphisms. Hence, the Five Lemma shows that $\Theta^i_{MM'}$ is an isomorphism. □

The next result contains Theorem 4 from the introduction. A special case of it can be found in [3, Theorem 3].

Corollary 4.11. *Let M and M' be mini-max R-modules, and fix an index $i \geqslant 0$. If either M or M' is Matlis reflexive, then $\Theta^i_{MM'}$ is an isomorphism, so one has $\mathrm{Ext}^i_R(M, M')^v = \mathrm{Ext}^i_R(M, M')^\vee \cong \mathrm{Tor}^R_i(M, M'^\vee)$, where $(-)^v = \mathrm{Hom}_{\widehat{R}}(-, E)$.*

Proof. Apply Theorem 4.10 and Fact 1.18. □

The next example shows that the modules $\mathrm{Ext}^i_R(L, L')^\vee$ and $\mathrm{Tor}^R_i(L, L'^\vee)$ are not isomorphic in general.

Example 4.12. Assume that R is not complete. We have $\mathrm{Ann}_R(E) = 0$, so the ring $R/\mathrm{Ann}_R(E) \cong R$ is not complete, by assumption. Thus, Fact 1.18 implies that E is not Matlis reflexive, that is, the biduality map $\delta_E : E \hookrightarrow E^{\vee\vee}$ is not an isomorphism. Since $E^{\vee\vee}$ is injective, we have $E^{\vee\vee} \cong E \oplus J$ for some non-zero injective R-module J. The uniqueness of direct sum decompositions of injective R-modules implies that $E^{\vee\vee} \ncong E$. This provides the second step below:

$$\mathrm{Hom}_R(E, E)^\vee \cong E^{\vee\vee} \ncong E \cong E \otimes_R \widehat{R} \cong E \otimes_R E^\vee.$$

The third step is from Lemma 1.4(a), and the remaining steps are standard.

5. Vanishing of Ext and Tor

In this section we describe the sets of associated primes of $\mathrm{Hom}_R(A, M)$ and attached primes of $A \otimes_R M$ over $\widehat{R}$. The section concludes with some results on the related topic of vanishing for $\mathrm{Ext}^i_R(A, M)$ and $\mathrm{Tor}^R_i(A, M)$.

Associated and attached primes

The following is dual to the notion of associated primes of noetherian modules; see, e.g., [13] or [14, Appendix to §6] or [16].

Definition 5.1. Let A be an artinian R-module. A prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ is *attached* to A if there is a submodule $A' \subseteq A$ such that $\mathfrak{p} = \mathrm{Ann}_R(A/A')$. We let $\mathrm{Att}_R(A)$ denote the set of prime ideals attached to A.

Lemma 5.2. *Let A be an artinian R-module such that $R/\mathrm{Ann}_R(A)$ is complete, and let N be a noetherian R-module. There are equalities*

$$\mathrm{Supp}_R(A^\vee) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(A^\vee)} V(\mathfrak{p}) = \bigcup_{\mathfrak{p} \in \mathrm{Att}_R(A)} V(\mathfrak{p})$$

$$\mathrm{Att}_R(N^\vee) = \mathrm{Ass}_R(N)$$

$$\mathrm{Att}_R(A) = \mathrm{Ass}_R(A^\vee).$$

Proof. The R-module $A^\vee$ is noetherian by Lemma 1.15(c), so the first equality is standard, and the second equality follows from the fourth one. The third equality is from [18, (2.3) Theorem]. This also explains the second step in the next sequence

$$\mathrm{Att}_R(A) = \mathrm{Att}_R(A^{\vee\vee}) = \mathrm{Ass}_R(A^\vee)$$

since $A^\vee$ is noetherian. The first step in this sequence follows from the fact that A is Matlis reflexive; see Fact 1.18. □

The next proposition can also be deduced from a result of Melkersson and Schenzel [15, Proposition 5.2].

Proposition 5.3. *Let A and L be R-modules such that $\mu_R^0(L) < \infty$ and A is artinian. Then*

$$\mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, L)) = \mathrm{Ass}_{\widehat{R}}(A^\vee) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee) = \mathrm{Att}_{\widehat{R}}(A) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee).$$

Proof. The assumption $\mu_R^0(L) < \infty$ implies that $\Gamma_{\mathfrak{m}}(L)$ is artinian. This implies that $\Gamma_{\mathfrak{m}}(L)^\vee$ is a noetherian $\widehat{R}$-module, so a result of Bourbaki [5, IV 1.4 Proposition 10] provides the third equality in the next sequence; see also [6, Exercise 1.2.27]:

$$\begin{aligned}\mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, L)) &= \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)))\\ &= \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee, A^\vee))\\ &= \mathrm{Ass}_{\widehat{R}}(A^\vee) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee)\\ &= \mathrm{Att}_{\widehat{R}}(A) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee).\end{aligned}$$

The remaining equalities are from Lemmas 1.5(b), 2.1(a) and 5.2, respectively. □

Corollary 5.4. *Let M and M' be mini-max R-modules such that the quotient $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(M'))$ is complete.*

(a) *For each index $i \geqslant 0$, one has $\mathrm{Ext}_R^i(M, M') \cong \mathrm{Ext}_R^i(M'^\vee, M^\vee)$.*
(b) *If M' is noetherian, then*

$$\mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(M, M')) = \mathrm{Att}_{\widehat{R}}(M'^\vee) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(M^\vee)^\vee).$$

Proof. (a) The first step in the next sequence comes from Theorem 2.8(a):

$$\mathrm{Ext}_R^i(M, M') \cong \mathrm{Ext}_R^i(M, M')^{\vee\vee} \cong (\mathrm{Tor}_i^R(M, M'^\vee))^\vee \cong \mathrm{Ext}_R^i(M'^\vee, M^\vee).$$

The remaining steps are from Theorem 4.10 and Remark 1.9, respectively.
(b) This follows from the case $i = 0$ in part (a) because of Proposition 5.3. □

Proposition 5.5. *Let A and L be R-modules such that A is artinian and $\beta_0^R(L) < \infty$. Then*

$$\mathrm{Att}_{\widehat{R}}(A \otimes_R L) = \mathrm{Ass}_{\widehat{R}}(A^\vee) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee) = \mathrm{Att}_{\widehat{R}}(A) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee).$$

Proof. Theorem 3.1 implies that $A \otimes_R L$ is artinian. Hence, we have

$$\mathrm{Hom}_{\widehat{R}}(A \otimes_R L, E) \cong \mathrm{Hom}_R(A \otimes_R L, E) \cong \mathrm{Hom}_R(A, L^\vee)$$

by Lemma 1.5(a), and this explains the second step in the next sequence:

$$\mathrm{Att}_{\widehat{R}}(A \otimes_R L) = \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(A \otimes_R L, E)) = \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, L^\vee)).$$

The first step is from Lemma 5.2. Since $\mu_R^0(L^\vee) < \infty$ by Lemma 1.13(b), we obtain the desired equalities from Proposition 5.3. □

Next, we give an alternate description of the module $\Gamma_{\mathfrak{m}}(L)^\vee$ from the previous results. See Lemma 5.2 for a description of its support.

Remark 5.6. Let L be an R-module. There is an isomorphism $\Gamma_{\mathfrak{m}}(L)^\vee \cong \widehat{L^\vee}$. In particular, given a noetherian R-module N, one has $\Gamma_{\mathfrak{m}}(N^\vee)^\vee \cong \widehat{R} \otimes_R N$. When R is Cohen–Macaulay with a dualizing module D, Grothendieck's local duality theorem implies that $\Gamma_{\mathfrak{m}}(N)^\vee \cong \widehat{R} \otimes_R \mathrm{Ext}_R^{\dim(R)}(N, D)$; see, e.g., [6, Theorem 3.5.8]. A similar description is available when R is not Cohen–Macaulay, provided that it has a dualizing complex; see [10, Chapter V, §6].

Vanishing of Hom and Tensor product

For the next result note that if L is noetherian, then the conditions on $\mu_R^0(L)$ and $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$ are automatically satisfied. Also, the example $\mathrm{Hom}_R(E, E) \cong R$ when R is complete shows the necessity of the condition on $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$.

Proposition 5.7. *Let A be an artinian R-module. Let L be an R-module such that $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$ is artinian and $\mu_R^0(L) < \infty$. Then $\mathrm{Hom}_R(A, L) = 0$ if and only if $A = \mathfrak{m}A$ or $\Gamma_{\mathfrak{m}}(L) = 0$.*

Proof. If $\Gamma_{\mathfrak{m}}(L) = 0$, then we are done by Lemma 1.5(b), so assume that $\Gamma_{\mathfrak{m}}(L) \neq 0$. Theorem 2.2 and Lemma 1.21 show that $\mathrm{Hom}_R(A, L)$ has finite length. Thus Proposition 5.3 implies that $\mathrm{Hom}_R(A, L) \neq 0$ if and only if $\mathfrak{m}\widehat{R} \in \mathrm{Ass}_{\widehat{R}}(A^\vee)$, that is, if and only if $\mathrm{depth}_{\widehat{R}}(A^\vee) = 0$. Lemma 1.13(c) shows that $\mathrm{depth}_{\widehat{R}}(A^\vee) = 0$ if and only if $\mathfrak{m}\widehat{R}A \neq A$, that is, if and only if $\mathfrak{m}A \neq A$. $\square$

For the next result note that the conditions on L are satisfied when L is artinian.

Proposition 5.8. *Let A be an artinian R-module, and let L be an $\mathfrak{m}$-torsion R-module. The following conditions are equivalent:*

(i) $A \otimes_R L = 0$;
(ii) *either* $A = \mathfrak{m}A$ *or* $L = \mathfrak{m}L$; *and*
(iii) *either* $\mathrm{depth}_R(A^\vee) > 0$ *or* $\mathrm{depth}_R(L^\vee) > 0$.

Proof. (i) $\iff$ (ii) If $A \otimes_R L = 0$, then we have

$$0 = \mathrm{len}_R(A \otimes_R L) \geqslant \beta_0^R(A)\beta_0^R(L)$$

so either $\beta_0^R(A) = 0$ or $\beta_0^R(L) = 0$, that is $A/\mathfrak{m}A = 0$ or $L/\mathfrak{m}L = 0$. Conversely, if $A/\mathfrak{m}A = 0$ or $L/\mathfrak{m}L = 0$, then we have either $\beta_0^R(A) = 0$ or $\beta_0^R(L) = 0$, so Theorem 3.8 implies that $\mathrm{len}_R(A \otimes_R L) = 0$.

The implication (ii) $\iff$ (iii) is from Lemma 1.13(c). $\square$

The next result becomes simpler when L is artinian, as $\Gamma_{\mathfrak{m}}(L) = L$ in this case.

Theorem 5.9. *Let A and L be R-modules such that A is artinian and $\mu_R^0(L) < \infty$. The following conditions are equivalent:*

(i) $\mathrm{Hom}_R(A, L) = 0$;
(ii) $\mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) = 0$;
(iii) $\mathrm{Hom}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee, A^\vee) = 0$;
(iv) *there is an element* $x \in \mathrm{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))$ *such that* $A = xA$;
(v) $\mathrm{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))A = A$;
(vi) $\mathrm{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))$ *contains a non-zero-divisor for* $A^\vee$; *and*
(vii) $\mathrm{Att}_{\widehat{R}}(A) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee) = \emptyset$.

Proof. The equivalence (i) $\iff$ (ii) is from Lemma 1.5(b). The equivalence (ii) $\iff$ (vii) follows from Proposition 5.3, and the equivalence (ii) $\iff$ (iii) follows from Lemma 2.1(a). The equivalence (iv) $\iff$ (vi) follows from the fact that the map $A \xrightarrow{x} A$ is surjective if and only if the map $A^\vee \xrightarrow{x} A^\vee$ is injective. The equivalence (v) $\iff$ (vi) follows from Lemma 1.13, parts (c) and (e).

The module $\Gamma_{\mathfrak{m}}(L)$ is artinian as $\mu_R^0(L) < \infty$. Since $A^\vee$ and $\Gamma_{\mathfrak{m}}(L)^\vee$ are noetherian over $\widehat{R}$, the equivalence (iii) $\iff$ (vi) is standard; see [6, Proposition 1.2.3]. $\square$

As with Theorem 5.9, the next result simplifies when L is noetherian. Also, see Remark 5.6 for some perspective on the module $\Gamma_{\mathfrak{m}}(L^\vee)^\vee$.

Corollary 5.10. *Let A be a non-zero artinian R-module, and let L be an R-module such that $\beta_0^R(L) < \infty$. The following conditions are equivalent:*

(i) $A \otimes_R L = 0$;
(ii) $\mathrm{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee))A = A$;
(iii) *there is an element* $x \in \mathrm{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee))$ *such that* $xA = A$;
(iv) $\mathrm{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee))$ *contains a non-zero-divisor for* $A^\vee$; *and*
(v) $\mathrm{Att}_{\widehat{R}}(A) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee) = \emptyset$.

Proof. For an artinian R-module A', one has $\mathrm{Att}_{\widehat{R}}(A') = \emptyset$ if and only if $A' = 0$ by Lemma 5.2. Thus, Proposition 5.5 explains the equivalence (i) $\iff$ (v); see [16, Corollary 2.3]. Since one has $A \otimes_R L = 0$ if and only if $(A \otimes_R L)^\vee = 0$, the isomorphism $(A \otimes_R L)^\vee \cong \mathrm{Hom}_R(A, L^\vee)$ from Remark 1.9 in conjunction with Theorem 5.9 shows that the conditions (i)–(iv) are equivalent. $\square$

Depth and vanishing

Proposition 5.11. *Let A and L be R-modules such that A is artinian. Then $\mathrm{Ext}_R^i(A, L) = 0$ for all $i < \mathrm{depth}_R(L)$.*

Proof. Let J be a minimal R-injective resolution of L, and let $i < \operatorname{depth}_R(L)$. It follows that $\operatorname{Ext}^i_R(k, L) = 0$, that is $\mu^i_R(L) = 0$, so the module E does not appear as a summand of J^i. As in the proof of Theorem 2.2, this implies that $\operatorname{Hom}_R(A, J)^i = 0$, so $\operatorname{Ext}^i_R(A, L) = 0$. $\square$

The next example shows that, in Proposition 5.11 one may have $\operatorname{Ext}^i_R(A, L) = 0$ when $i = \operatorname{depth}_R(L)$. See also Eq. (5.14.1).

Example 5.12. Assume that $\operatorname{depth}(R) \geqslant 1$. Then $\mathfrak{m}E = E$ by Lemma 1.13(c), so Lemma 2.10 implies that

$$\operatorname{Ext}^0_R(E, k) \cong \operatorname{Hom}_R(E, k) \cong \operatorname{Hom}_R(E/\mathfrak{m}E, k) = 0$$

even though $\operatorname{depth}_R(k) = 0$.

Proposition 5.13. *Let A and L be R-modules such that A is artinian. Then for all $i < \operatorname{depth}_R(L^\vee)$ one has $\operatorname{Tor}^R_i(A, L) = 0$.*

Proof. When $i < \operatorname{depth}_R(L^\vee)$, one has $\operatorname{Tor}^R_i(A, L)^\vee \cong \operatorname{Ext}^i_R(A, L^\vee) = 0$ by Remark 1.9 and Proposition 5.11, so $\operatorname{Tor}^R_i(A, L) = 0$. $\square$

Theorem 5.14. *Let A and A' be artinian R-modules, and let N and N' be noetherian R-modules. Then one has*

$$\operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Ext}^i_R(A, A') \neq 0\} \tag{5.14.1}$$

$$\operatorname{depth}_R(\operatorname{Ann}_R(N'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Ext}^i_R(A, N'^\vee) \neq 0\} \tag{5.14.2}$$

$$\operatorname{depth}_R(\operatorname{Ann}_R(N'); N) = \inf\{i \geqslant 0 \mid \operatorname{Ext}^i_R(N^\vee, N'^\vee) \neq 0\}. \tag{5.14.3}$$

Proof. We verify Eq. (5.14.1) first. For each index i, Theorem 4.3 implies that

$$\operatorname{Ext}^i_R(A, A') \cong \operatorname{Ext}^i_{\widehat{R}}(A'^\vee, A^\vee).$$

Since $A^\vee$ and $A'^\vee$ are noetherian over $\widehat{R}$, this explains the first equality below:

$$\inf\{i \geqslant 0 \mid \operatorname{Ext}^i_R(A, A') \neq 0\} = \operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'^\vee); A^\vee) = \operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'); A^\vee).$$

The second equality is standard since $A'^\vee = \operatorname{Hom}_{\widehat{R}}(A', E)$ by Lemma 1.5(a).

Next, we verify Eq. (5.14.2). Since $N'^\vee$ is artinian, Eq. (5.14.1) shows that we need only verify that

$$\operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(N'^\vee); A^\vee) = \operatorname{depth}_R(\operatorname{Ann}_R(N'); A^\vee). \tag{5.14.4}$$

For this, we compute as follows:

$$\widehat{R} \otimes_R N' \overset{(1)}{\cong} \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R N', E), E) \overset{(2)}{\cong} \operatorname{Hom}_{\widehat{R}}(N'^\vee, E).$$

Step (1) follows from the fact that $\widehat{R} \otimes_R N'$ is noetherian (hence, Matlis reflexive) over $\widehat{R}$, and step (2) is from Hom–tensor adjointness. This explains step (4) below:

$$\operatorname{Ann}_{\widehat{R}}(N'^\vee) \overset{(3)}{=} \operatorname{Ann}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(N'^\vee, E)) \overset{(4)}{=} \operatorname{Ann}_{\widehat{R}}(\widehat{R} \otimes_R N') \overset{(5)}{=} \operatorname{Ann}_R(N')\widehat{R}.$$

Steps (3) and (5) are standard. This explains step (6) in the next sequence:

$$\operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(N'^\vee); A^\vee) \overset{(6)}{=} \operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_R(N')\widehat{R}; A^\vee) \overset{(7)}{=} \operatorname{depth}_R(\operatorname{Ann}_R(N'); A^\vee).$$

Step (7) is explained by the following, where step (8) is standard, and step (9) is a consequence of Hom–tensor adjointness:

$$\begin{aligned} \operatorname{Ext}^i_{\widehat{R}}(\widehat{R}/\operatorname{Ann}_R(N')\widehat{R}, A^\vee) &\overset{(8)}{\cong} \operatorname{Ext}^i_{\widehat{R}}(\widehat{R} \otimes_R (R/\operatorname{Ann}_R(N')), A^\vee) \\ &\overset{(9)}{\cong} \operatorname{Ext}^i_R(R/\operatorname{Ann}_R(N'), A^\vee). \end{aligned}$$

This establishes Eq. (5.14.4) and thus Eq. (5.14.2).

Eq. (5.14.3) follows from (5.14.2) because we have

$$\operatorname{depth}_R(\operatorname{Ann}_R(N'); N^{\vee\vee}) = \operatorname{width}_R(\operatorname{Ann}_R(N'); N^\vee) = \operatorname{depth}_R(\operatorname{Ann}_R(N'); N)$$

by Lemma 1.13(a). $\square$

Corollary 5.15. *Let A and A' be artinian R-modules, and let N and N' be noetherian R-modules. Then*

$$\operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Tor}^R_i(A, A'^\vee) \neq 0\} \tag{5.15.1}$$

$$\operatorname{depth}_R(\operatorname{Ann}_R(N'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Tor}^R_i(A, N') \neq 0\} \tag{5.15.2}$$

$$\operatorname{depth}_R(\operatorname{Ann}_R(N'); N) = \inf\{i \geqslant 0 \mid \operatorname{Tor}^R_i(N^\vee, N') \neq 0\}. \tag{5.15.3}$$

Proof. We verify Eq. (5.15.1); the others are verified similarly.

Since $\mathrm{Ext}^i_R(A, A') \neq 0$ if and only if $\mathrm{Hom}_{\widehat{R}}(\mathrm{Ext}^i_R(A, A'), E) \neq 0$, the isomorphism $\mathrm{Hom}_{\widehat{R}}(\mathrm{Ext}^i_R(A, A'), E) \cong \mathrm{Tor}^R_i(A, A'^{\vee})$ from Corollary 4.9 shows that

$$\inf\{i \geqslant 0 \mid \mathrm{Ext}^i_R(A, A') \neq 0\} = \inf\{i \geqslant 0 \mid \mathrm{Tor}^R_i(A, A'^{\vee}) \neq 0\}.$$

Thus Eq. (5.15.1) follows from (5.14.1). $\square$

6. Examples

This section contains some explicit computations of Ext and Tor for the classes of modules discussed in this paper. Our first example shows that $\mathrm{Ext}^i_R(A, A')$ need not be mini-max over R.

Example 6.1. Let k be a field, and set $R = k[X_1, \ldots, X_d]_{(X_1,\ldots,X_d)}$. We show that $\mathrm{Hom}_R(E, E) \cong \widehat{R}$ is not mini-max over R. Note that R is countably generated over k, and $\widehat{R} \cong k[[X_1, \ldots, X_d]]$ is not countably generated over k. So, $\widehat{R}$ is not countably generated over R. Also, every artinian R-module A is a countable union of the finite length submodules $(0 :_A \mathfrak{m}^n)$, so A is countably generated. It follows that every mini-max R-module is also countably generated. Since $\widehat{R}$ is not countably generated, it is not mini-max over R.

Our next example describes $\mathrm{Ext}^i_R(A, A')$ for some special cases.

Example 6.2. Assume that $\mathrm{depth}(R) \geqslant 1$, and let A be an artinian R-module. Let $x \in \mathfrak{m}$ be an R-regular element. The map $E \xrightarrow{x} E$ is surjective since E is divisible, and the kernel $(0 :_E x)$ is artinian, being a submodule of E. Using the injective resolution $0 \to E \xrightarrow{x} E \to 0$ for $(0 :_E x)$, one can check that

$$\mathrm{Ext}^i_R(A, (0 :_E x)) \cong \begin{cases} (0 :_{A^{\vee}} x) & \text{if } i = 0 \\ A^{\vee}/xA^{\vee} & \text{if } i = 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

For instance, in the case $A = (0 :_E x)$, the isomorphism $(0 :_E x)^{\vee} \cong \widehat{R}/x\widehat{R}$ implies

$$\mathrm{Ext}^i_R((0 :_E x), (0 :_E x)) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = 0, 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

On the other hand, if x, y is an R-regular sequence, then $(0 :_E y)^{\vee} \cong \widehat{R}/y\widehat{R}$; it follows that x is $(0 :_E y)^{\vee}$-regular, so one has

$$\mathrm{Ext}^i_R((0 :_E y), (0 :_E x)) \cong \begin{cases} \widehat{R}/(x, y)\widehat{R} & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

The next example shows that $\mathrm{Ext}^i_R(A, N)$ need not be mini-max over R.

Example 6.3. Assume that R is Cohen–Macaulay with $d = \dim(R)$, and let A be an artinian R-module. Assume that R admits a dualizing (i.e., canonical) module D. (For instance, this is so when R is Gorenstein, in which case $D = R$.) A minimal injective resolution of D has the form

$$J = \quad 0 \to \coprod_{\mathrm{ht}(\mathfrak{p})=0} E_R(R/\mathfrak{p}) \to \cdots \to \coprod_{\mathrm{ht}(\mathfrak{p})=d-1} E_R(R/\mathfrak{p}) \to E \to 0.$$

In particular, we have $\Gamma_{\mathfrak{m}}(J) = (0 \to 0 \to 0 \to \cdots \to 0 \to E \to 0)$ where the copy of E occurs in degree d. Since $\mathrm{Hom}_R(A, J) \cong \mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(J))$, it follows that

$$\mathrm{Ext}^i_R(A, D) \cong \begin{cases} A^{\vee} & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Assume that $d \geqslant 1$, and let $x \in \mathfrak{m}$ be an R-regular element. It follows that the map $D \xrightarrow{x} D$ is injective, and the cokernel D/xD is noetherian. Consider the exact sequence $0 \to D \xrightarrow{x} D \to D/xD \to 0$. The long exact sequence associated to $\mathrm{Ext}^i_R(A, -)$ shows that

$$\mathrm{Ext}^i_R(A, D/xD) \cong \begin{cases} (0 :_{A^{\vee}} x) & \text{if } i = d-1 \\ A^{\vee}/xA^{\vee} & \text{if } i = d \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

As in Example 6.2, we have $(0 :_E x)^{\vee} \cong \widehat{R}/x\widehat{R}$ and

$$\mathrm{Ext}^i_R((0 :_E x), D/xD) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = d-1, d \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

Also, if x, y is an R-regular sequence, then $(0 :_E y)^\vee \cong \widehat{R}/y\widehat{R}$ and

$$\mathrm{Ext}^i_R((0 :_E y), D/xD) \cong \begin{cases} \widehat{R}/(x, y)\widehat{R} & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Next, we show that $\mathrm{Tor}^R_i(A, A')$ need not be noetherian over R or $\widehat{R}$.

Example 6.4. Assume that R is Gorenstein and complete with $d = \dim(R)$. (Hence $D = R$ is a dualizing R-module.) Given two artinian R-modules A and A', Theorem 3.1 implies that $\mathrm{Tor}^R_i(A, A')$ is artinian, hence Matlis reflexive for each index i, since R is complete. This explains the first isomorphism below, and Remark 1.9 provides the second isomorphism:

$$\mathrm{Tor}^R_i(A, E) \cong \mathrm{Tor}^R_i(A, E)^{\vee\vee} \cong \mathrm{Ext}^i_R(A, E^\vee)^\vee \cong \mathrm{Ext}^i_R(A, R)^\vee \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Example 6.3 explains the fourth isomorphism. Assume that $d \geqslant 1$, and let $x \in \mathfrak{m}$ be an R-regular element. Then $(0 :_E x)^\vee \cong R/xR$, so Example 6.3 implies that

$$\mathrm{Tor}^R_i(A, (0 :_E x)) \cong \mathrm{Ext}^i_R(A, (0 :_E x)^\vee)^\vee \cong \begin{cases} A/xA & \text{if } i = d - 1 \\ (0 :_A x) & \text{if } i = d \\ 0 & \text{if } i \neq d - 1, d \end{cases}$$

$$\mathrm{Tor}^R_i((0 :_E x), (0 :_E x)) \cong \begin{cases} (0 :_E x) & \text{if } i = d - 1, d \\ 0 & \text{if } i \neq d - 1, d. \end{cases}$$

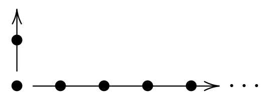
On the other hand, if x, y is an R-regular sequence, then

$$\mathrm{Tor}^R_i((0 :_E y), (0 :_E x)) \cong \begin{cases} (R/(x, y)R)^\vee \cong E_{R/(x,y)R}(k) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Lastly, we provide an explicit computation of $E \otimes_R E$.

Example 6.5. Let k be a field and set $R = k[\![X, Y]\!]/(XY, Y^2)$. This is the completion of the multi-graded ring $R' = k[X, Y]/(XY, Y^2)$ with homogeneous maximal ideal $\mathfrak{m}' = (X, Y)R'$. The multi-graded structure on R' is represented in the following diagram:

R'

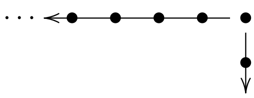


where each bullet represents the corresponding monomial in R'. It follows that $E \cong E_{R'}(k) \cong k[X^{-1}] \oplus kY^{-1}$ with graded module structure given by the formulas

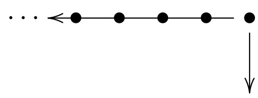
$$X \cdot 1 = 0 \qquad X \cdot X^{-n} = X^{1-n} \qquad X \cdot Y^{-1} = 0$$
$$Y \cdot 1 = 0 \qquad Y \cdot Y^{-1} = 1 \qquad Y \cdot X^{-n} = 0$$

for $n \geqslant 1$. Using this grading, one can show that $\mathfrak{m}E = \mathfrak{m}'E \cong k[X^{-1}]$ and $\mathfrak{m}^2E = \mathfrak{m}E$. These modules are represented in the next diagrams:

E



$\mathfrak{m}E$



It follows that $E/\mathfrak{m}E \cong k$, so Lemma 3.7 implies that

$$E \otimes_R E \cong (E/\mathfrak{m}E) \otimes_R (E/\mathfrak{m}E) \cong k \otimes_R k \cong k.$$

A similar computation shows the following: Fix positive integers a, b, c such that $c > b$, and consider the ring $S = k[\![X, Y]\!]/(X^aY^b, Y^c)$ with maximal ideal $\mathfrak{n}$ and $E_S = E_S(k)$. Then $\mathfrak{n}^{c-b}E_S = \mathfrak{n}^{c-b+1}E_S$ and we get the following:

$$E_S/\mathfrak{n}^{c-b}E_S \cong S/(X^a, Y^{c-b})S \cong k[X, Y]/(X^a, Y^{c-b})$$
$$E_S \otimes_S E_S \cong (E_S/\mathfrak{n}^{c-b}E_S) \otimes_S (E_S/\mathfrak{n}^{c-b}E_S) \cong S/(X^a, Y^{c-b})S.$$

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