RESEARCH STATEMENT
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> The human mind has never invented a labor-saving device equal to algebra.
>
> J. Willard Gibbs

1. Overview

My research is in commutative algebra, that is, the study of commutative rings. Historically, this area is heavily influenced by arithmetic and geometry, as follows.

In the oldest sense, arithmetic is the study of operations on integers: addition, subtraction, multiplication, and (when defined) division. A somewhat more modern perspective is that arithmetic subsumes large parts of number theory, which includes other domains where such operations are defined, e.g., in fields like $\mathbb{Q}$ and $\mathbb{C}$ or subsets of these fields like the set $\mathbb{Z}[i]$ of Gaussian integers. For instance, such rings (as they are now called) were introduced, in part, as tools for studying problems like Fermat's Last Theorem. An important lesson here is that sometimes to solve a problem about integers, you might have to work in a more general setting, in some sense, reevaluating what you mean by "integer".

On the geometric side, specifically in algebraic geometry, the fundamental objects of interest are the solution sets of systems of polynomial equations. For instance, this includes the familiar lines and conics in $\mathbb{R}^2$ and surfaces like spheres and hyperboloids in $\mathbb{R}^3$. Given that each of these geometric objects is given in terms of the vanishing of a set of polynomials, which are algebraic objects, we can study the geometric objects algebraically. This is done by considering the algebraic object obtained by formally setting these polynomials equal to zero. For example, the unit sphere S^2 in $\mathbb{R}^3$ is described by the polynomial equation $x^2 + y^2 + z^2 - 1 = 0$, so the corresponding algebraic object is the ring quotient $\mathbb{R}[S^2] = \mathbb{R}[x, y, x]/\langle x^2 + y^2 + z^2 - 1\rangle$. Nice properties of the sphere can be seen from this ring, and vice versa. For instance, the ring $\mathbb{R}[S^2]$ can show you why S^2 is smooth, and the fact that S^2 has no non-vanishing continuous tangent vector field can be used to show that $\mathbb{R}[S^2]$ has a non-free finitely generated projective module.

So, commutative algebra is the study of these commutative rings, which are everywhere in mathematics. My research in particular is focused primarily on understanding a ring by studying its modules. These are the objects that the ring acts on by scalar multiplication, like a field acts on a vector space. For instance, the ring $\mathbb{R}[S^2]$ is somehow fundamentally different from the polynomial ring $\mathbb{R}[x, y]$, even though they are very similar in many ways, e.g., because $\mathbb{R}[x, y]$ does not have a non-free finitely generated projective module. This is a fundamental principle of module theory: understanding a ring is the same as understanding its modules. If your ring only has modules that are very simple, then your ring is somehow simple, and conversely.

In the sections that follow, I describe a few aspects of how this plays out in my research. These sections focus on some of my favorite results, usually not in full generality, and omitting many results. Hopefully they give you a taste of what I do and what I see myself doing in the future.

2. Semidualizing Modules

A standard way to study a mathematical object is by looking at certain sets of functions it describes. For instance, each element of the ring $\mathbb{R}[S^2]$ describes a well-defined function $S^2 \to \mathbb{R}$. Homotopy groups and cohomology groups of topological spaces are described by sets of functions, as are duals of Hilbert spaces, for instance.

The largest portion of my research (twenty-six of fifty-five papers) focuses on semidualizing modules. These were introduced by Foxby [22], but seem to have been rediscovered independently by others, including Golod [27], Vasconcelos [57], and Wakamatsu [59]. These are modules over a commutative noetherian ring R that are particularly well-suited for describing dualities. For the purpose of this document, the *duality* described by an R-module C is the operation that takes a module M and associates the set of R-module homomorphisms $\mathrm{Hom}_R(M, C)$. I am more generally interested in dualities on derived categories; while that is the most natural place to study these objects, it is more general than is suitable for this document.

A finitely generated R-module C is *semidualizing* provided that $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$. Loosely speaking, the first condition says that C is big enough to see all R-modules, but small enough so that some modules M can be *C-reflexive*, that is, such that $M \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$. The second condition ensures that C is small enough homologically for a good notion of reflexivity. For instance, R is a semidualizing R-module, so this recovers the classical duality with respect to R. This also recovers part of Grothendieck's Local Duality.

Vasconcelos' Finiteness Question.

In joint work with my former PhD student Saeed Nasseh, I recently answered one of the major open questions about semidualizing modules, posed by Vasconcelos [57] in 1974.

Theorem 2.1 ([40, Theorem A])**.** *A commutative noetherian local ring has only finitely many semidualizing modules, up to isomorphism.*

I proved a preliminary version of this result with Christensen [12, Theorem 1] where the ring contains a field. Nasseh and I use a fusion of techniques from homotopy theory and representation theory to answer the question in general. Specifically, we prove versions of results of Happel [30] and Voigt [58] that allow us to use algebraic geometry techniques to study modules over differential graded (DG) algebras. An important part of this proof is a version of a lifting result of Auslander, Ding, and Solberg for DG algebras, which we prove separately [42, Main Theorem], motivated in part by a paper I wrote with Christensen [13]. Note that this follows a theme from the introduction: to solve a problem about rings, we needed to work in a more general setting, in some sense, reevaluating what we mean by "ring". This paper has been recommended to publication in Adv. Math.

Nasseh and I are working to take the techniques we developed to answer Vasconcelos' question and apply them to other questions. For instance, Avramov [4] has used these techniques to completely describe the sequences of Bass numbers for all local rings of embedding codepth 3. (See also the subsection below on a question of Huneke.) Nasseh and I have successfully exploited these ideas to explicitly describe all the semidualizing modules for such rings.

Theorem 2.2 ([41, Theorem A]). *A local ring of embedding codepth at most 3 has at most 2 semidualizing modules, up to isomorphism.*

As part of the proof of this result, Nasseh and I defined the notion of syzygies for DG modules. While this is a standard (and extremely powerful) idea for rings, it had apparently not been extended successfully to this context before.

In Summer 2012, Kristen Beck and I gave a short course on some of these topics, entitled "Differential Graded Commutative Algebra" as part of the Workshop on Connections Between Algebra and Geometry held at the University of Regina. The notes from this [8] have appeared as part of a Springer volume. In addition, Beck and I [7] have applied commutative algebra techniques to the study of DG algebras. Specifically, we show that Krull dimension of complexes (à la Foxby and Christensen) agrees with a more naive notion when applied to DG algebras.

Takahashi and White's Gorenstein Question. Another open question I have solved recently (with one of my current PhD students, Jonathan Totushek) is from work of Takahashi and White [55]. It is based on a famous result of Foxby [23], which states that if a local ring has a module of finite depth, finite injective dimension, and finite flat dimension, then the ring must be Gorenstein. Takahashi and White's question adds a semidualizing module to the mix in a non-trivial way with surprising consequences for the ring.

Theorem 2.3 ([52, Theorem 1.1]). *Let R be a local ring and C a semidualizing R-module. If R has a module of finite depth, finite C-injective dimension, and finite C-flat dimension, then R must be Gorenstein.*

Yassemi and I [54] made some initial progress on this question a few years ago. The final solution uses non-trivial techniques that Frankild and I developed in [24], along with ideas from Totushek's solo paper [56].

Application to a Question of Avramov and Foxby.

One of my main reasons for studying semidualizing modules is found in work of Avramov and Foxby [5] on the homological properties of ring homomorphisms $\varphi\colon R \to S$. They extend work of Auslander and Bridger [3] based on duality with respect to R to investigate the *G-dimension* of φ. A question that Avramov and Foxby were not able to answer in general was the following: must the composition of two local ring homomorphisms of finite G-dimension also have finite G-dimension? In other words, is the class of local ring homomorphisms of finite G-dimension closed under composition?

Avramov and Foxby show that semidualizing modules may hold the key to understanding the composition question. Essentially, the point is that they can translate questions about the composition to a similar question for duality with respect to the "relative dualizing module" which is a specific semidualizing module. Avramov and Foxby answer some special cases of this question in their original paper using the relative dualizing module, most notably the case where one of the homomorphisms has finite flat dimension. In joint work with Iyengar [33], I address this question for local endomorphisms. The only other real progress on this question has been the following.

Theorem 2.4 ([45, Theorem A]). *Let $\varphi\colon R \to S$ and $\psi\colon S \to T$ be local homomorphisms. If φ has finite CI-dimension and ψ has finite G-dimension, then the composition $\psi \circ \varphi$ has finite G-dimension.*

Note that this result subsumes the aforementioned result of Avramov and Foxby. Finite CI-dimension is another homological condition on a local ring homomorphism. It is built from the CI-dimension of Avramov, Gasharov, and Peeva [6] in the same way that Avramov and Foxby built their notion from Auslander and Bridger's. It is significantly weaker than finite flat dimension, but significantly stronger than finite G-dimension.

Application to a Question of Huneke.

It is not an exaggeration to say that most of my work on semidualizing modules has been in an effort to answer the composition question discussed above. However, another application of these gadgets is found in their ability to force the growth of the *Bass numbers* of a local ring. Technically, the ith Bass number of a local ring R with residue field k is $\mu^i(R) := \operatorname{rank}_k(\operatorname{Ext}^i_R(k,R))$ for $i \geqslant 0$. Morally speaking, these numbers measure how far R is from being Gorenstein: the ring R is Gorenstein if and only if $\mu^i(R) = 0$ for $i \gg 0$. The Gorenstein rings are the ones for which duality with respect to R is particularly well-behaved. (For instance, the "G" in G-dimension is short for "Gorenstein".) For semidualizing modules, the Gorenstein condition is quite restrictive: if R is Gorenstein, then the only semidualizing R-module is R, up to isomorphism.

In 2000, Huneke asked the following: if the sequence of Bass numbers of R is bounded, must R be Gorenstein? In general this question is wide open, though some special cases have been treated in [15, 34]. My application of semidualizing modules to this question is the following.

Theorem 2.5 ([46, Theorem B]). *Let R be a commutative noetherian local ring. If R has at least three distinct semidualizing modules, up to isomorphism, then the sequence $\{\mu^i(R)\}$ is unbounded.*

It follows that the only remaining case of Huneke's question is the case where R has only two distinct semidualizing modules, up to isomorphism.

Semidualizing Modules and Topology. Another surprising fact about semidualizing modules is found in the next result, which shows that they have the ability to detect non-trivial topological properties about a ring, via the connectedness of its prime spectrum.

Theorem 2.6 ([48, Theorem 1.3]). *Let A be a non-zero finitely generated R-module that is totally A-reflexive and not semidualizing. Then there are commutative noetherian rings $R_1, R_2 \neq 0$ with identity such that $R \cong R_1 \times R_2$, and there is a* *semidualizing R_1-module A_1 such that $A \cong A_1 \times 0$. In particular,* $\operatorname{Spec}(R)$ *is disconnected.*

Finding Semidualizing Modules.

Given the status of Huneke's question, it would be valuable to be able to tell how many semidualizing modules a given local ring has. Unfortunately, this is a very hard question in general. (Recall from the above discussion, for instance, that it took almost 40 years to prove that R has only finitely many distinct semidualizing modules.) Currently, the best we can do is to look at particular well-studied classes of rings to see if we can say how many semidualizing modules they have. One tool I have used successfully in this endeavor is the divisor class group, via the following.

Theorem 2.7 ([44, Proposition 3.4])**.** *Let R be a normal integral domain. Then the set of isomorphism classes of semidualizing R-modules is naturally a subset of the divisor class group of R.*

In most of the non-trivial cases where we have an explicit description of all the semidualizing modules over a ring R, this theorem has been an important tool.

3. Other Topics

G-Injective Dimension.

Gorenstein homological algebra is a sub-field of homological algebra that also has its roots in Auslander and Bridger's work [3]. Despite its power, this work had two deficiencies that made it less pleasant to work with than other homological invariants. First, the G-dimension was only defined for finitely generated modules. Second, there was no resolution-free characterization of the G-dimension, unlike the characterization of projective dimension in terms of Ext-vanishing. The first of these deficiencies was remedied with the introduction of the G-projective dimension, the G-injective dimension, and the G-flat dimension by Enochs, Jenda, and Torrecillas [18, 19]. The second deficiency was addressed for all but the G-injective dimension by Christensen, Frankild, and Holm [11], Esmkhani and Tousi [20], and Yassemi [60]. Christensen and I answered the final question in this area with the following result.

Theorem 3.1 ([14, Theorem A])**.** *Let R be a local ring with completion $\widehat{R}$. An R-module M has finite G-injective dimension if and only if $\mathbf{R}\mathrm{Hom}_R(\widehat{R}, M)$ is in the Bass class $\mathcal{B}(\widehat{R})$.*

An important feature of this result is in the use of the derived Hom-complex $\mathbf{R}\mathrm{Hom}_R(\widehat{R}, M)$. Specifically, we show that the resolution-free characterization of the G-injective dimension must involve the derived category. Again, this follows our theme from the introduction: in order to characterize this property, you must enlarge your perspective from modules to complexes.

Another vexing open question for G-injective dimension is based on work of Matlis [39], who shows that every injective module over a noetherian ring has a canonical direct sum composition. A similar result for G-injective modules is expected to hold, but we seem to be quite far from understanding even what the statement should be.

Enochs and Huang [17] recently shed some light on this by showing that, over a Gorenstein ring, every G-injective module has a "canonical filtration" by special G-injective modules. Feickert and I [21] have improved this result significantly in two ways: by giving a direct sum decomposition (stronger than a filtration), and doing so over Cohen-Macaulay rings that have a dualizing module.

Theorem 3.2 ([21, Theorem A]). *Let R be a d-dimensional Cohen-Macaulay ring with a dualizing module D. If G is a Gorenstein injective R-module, then G has a direct sum decomposition*

$$G \cong \oplus_{\mathfrak{p}\in\operatorname{supp}_R(G)} G_{(\mathfrak{p})}$$

such that each module $G_{(\mathfrak{p})} = \operatorname{Tor}_k^R(\operatorname{Hom}_R(D, E_R(R/\mathfrak{p})), G)$ is Gorenstein injective and satisfies $t(\mathfrak{p})$.

It is worth noting that we also have a C-Gorenstein injective version of this result, where C is semidualizing.

Ascent of Module-Structures.

Given a flat local ring homomorphism $\varphi\colon R \to S$, the rings R and S are intimately related via the "closed fibre" of φ, i.e., the quotient ring $S/\mathfrak{m}S$ where $\mathfrak{m}$ is the maximal ideal of R. For instance, S is Gorenstein if and only if R and $S/\mathfrak{m}S$ are Gorenstein. One can use this to one's advantage, for instance, by "passing to the completion" $\widehat{R}$, which comes equipped with a flat local ring homomorphism $R \to \widehat{R}$ whose closed fibre is isomorphic to the residue field $R/\mathfrak{m}$. In some sense, this is the point of Theorem 3.1: to understand the R-module M, replace it with the $\widehat{R}$-complex $\mathbf{R}\operatorname{Hom}_R(\widehat{R}, M)$ which is easier to understand in some respects.

The ith cohomology module of the complex $\mathbf{R}\operatorname{Hom}_R(\widehat{R}, M)$ is $\operatorname{Ext}^i_R(\widehat{R}, M)$. So, in some sense, the work of understanding M reduces to a need to understand $\operatorname{Ext}^i_R(\widehat{R}, M)$. In general, this is actually somewhat difficult. For instance, with the relatively simple ring $R = k[X,Y]_{(X,Y)}$, we do not know if $\operatorname{Ext}^i_R(\widehat{R}, R)$ vanishes or not for $i = 1$ or 2. We do know that at least one of these Ext-modules is non-zero, however, thanks to the special case $M = R$ of the following result.

Theorem 3.3 ([2, Theorem 1.5]). *Let R be a local ring with completion $\widehat{R}$. Given a finitely generated R-module M, the following conditions are equivalent.*

(i) *M is $\mathfrak{m}$-adically complete,*
(ii) *$\operatorname{Ext}^i_R(\widehat{R}, M) = 0$ for all $i \geqslant 1$,*
(iii) *$\operatorname{Ext}^i_R(\widehat{R}, M)$ is finitely generated over R (or over $\widehat{R}$) for $i = 1, \dots, \dim_R(M)$,*
(iv) *$\operatorname{Ext}^i_R(\widehat{R}, M)$ satisfies Nakayama's Lemma for $i = 1, \dots, \dim_R(M)$.*

This result is actually the culmination of three papers [2, 25, 26] written with Frankild, R. Wiegand, and my former PhD student Benjamin Anderson. Each of these papers in sequence contains a stronger version of this result with a simpler proof. The proof in the final paper boils down to relatively straightforward properties of Koszul homology. However, as with some of the results described above, the proof follows our theme from the introduction: in order to accomplish this, you must enlarge your perspective from modules to complexes, namely, via Koszul complexes.

Recently, I have improved this further by involving a second module in the mix.

Theorem 3.4 ([47, Theorem 3])**.** *Let R be a local ring with completion $\widehat{R}$, and let M, N be finitely generated R-modules. Then the following conditions are equivalent:*

(i) *$M \otimes_R N$ is $\mathfrak{m}$-adically complete,*
(ii) *$\mathrm{Tor}_i^R(M,N)$ is $\mathfrak{m}$-adically complete for all $i \geqslant 0$,*
(iii) *$\mathrm{Ext}_R^i(M,N)$ is $\mathfrak{m}$-adically complete for all $i \geqslant 0$,*
(iv) *$\mathrm{Ext}_R^i(M,N)$ is $\mathfrak{m}$-adically complete for $i = 0, \ldots, \dim_R(N) - 1$,*
(v) *$\mathrm{Ext}_R^i(S \otimes_R M, N)$ is finitely generated over R for all $i \geqslant 1$.*

Finiteness Properties of Cohomology.

Part of Bethany Kubik's dissertation work under my supervision involved an extension of the semidualizing property from finitely generated modules to artinian modules. The point is that duality with respect to a semidualizing module is nice, but it doesn't recover other important dualities, e.g., Matlis duality. At one point, she realized that she needed to answer the following: If R is noetherian, A and A' are artinian R-modules, and N is a noetherian R-module, what finiteness properties do $\mathrm{Ext}_R^i(A,A')$, $\mathrm{Ext}_R^i(A,N)$, and $\mathrm{Tor}_i^R(A,A')$ satisfy? (If A is noetherian instead of artinian, it is well-known that $\mathrm{Ext}_R^i(A,A')$ and $\mathrm{Tor}_i^R(A,A')$ are artinian and that $\mathrm{Ext}_R^i(A,N)$ is noetherian over R.)

We answer this question in collaboration with Leamer [36, 37]. I state only the local case here, to avoid worrying about certain technicalities.

Theorem 3.5 ([36, Theorem 1])**.** *Let R be a local noetherian ring with completion $\widehat{R}$. Let A and A' be artinian R-modules and let N be a noetherian R-module. Then $\mathrm{Ext}_R^i(A,A')$ and $\mathrm{Ext}_R^i(A,N)$ are noetherian $\widehat{R}$-modules, and $\mathrm{Tor}_i^R(A,A')$ is an artinian R-module, for each i.*

It is worth noting that Kubik was able to use this result in her other research [35] to answer several of the questions that motivated the investigation with Leamer.

My former PhD student Richard Wicklein and I [53] have been investigating another finiteness condition, which we call "adic finiteness". This notion has its roots in work of Hartshorne [31] on local cohomology. The $\mathfrak{a}$-adically finite modules are those that satisfy the equivalent conditions of the next result along with a support condition.

Theorem 3.6 ([53, Theorem 1.3])**.** *Let M be an R-module, and let $\mathfrak{a}$ be an ideal of R. Then the following conditions are equivalent:*

(i) *The Koszul homology modules $\mathrm{H}_i(\underline{x}), M)$ are finitely generated over R for all i, for some (equivalently for every) generating sequence $\underline{x}$ of $\mathfrak{a}$,*
(ii) *The Tor modules $\mathrm{Tor}_i^R(R/\mathfrak{a}, M)$ are finitely generated over R for all i,*
(iii) *The Ext modules $\mathrm{Ext}_R^i(R/\mathfrak{a}, M)$ are finitely generated over R for all i,*
(iv) *The local homology modules $\mathrm{H}_i^{\mathfrak{a}}(M)$ are finitely generated over $\widehat{R}^{\mathfrak{a}}$ for all i.*

We also show that these modules behave like finitely generated ones in many surprising ways. For instance, we prove that one can detect isomorphisms between such modules using tests that are much simpler than those needed in general.

Combinatorial Commutative Algebra.

Recently, I have started conducting some research on the combinatorial side of commutative algebra. In this area, one considers commutative rings determined by some combinatorial object, and works to understand properties of the rings in terms of the combinatorics of the initial object.

For instance, given a finite simple graph G on a vertex set $V = \{v_1, \ldots, v_d\}$, one considers the associated *edge ideal* which is essentially generated by the edges in G. Specifically, in the polynomial ring $k[X_1, \ldots, X_d]$ over a field k, the edge ideal associated to G is the ideal generated by the set of monomials of the form X_iX_j such that G includes the edge v_iv_j between v_i and v_j. Much work in this area focuses on characterizing the graphs G for which the ideal $I(G)$ is Cohen-Macaulay.

In joint work with my former undergraduate research student Chelsey Paulsen, I introduce and study a version of this construction for *weighted* graphs. These are finite simple graphs such that every edge e comes equipped with a positive integer weight $\omega(e)$. Such a weighted graph is denoted G_ω. For instance, if the edges of the graph represent the wires in an electrical network, then the weights might represent the capacity of the wires. Paulsen and I give an explicit description of the minimal irreducible decomposition of $I(G_\omega)$ in terms of what we call the "weighted vertex covers" of G_ω. This mirrors the decomposition of $I(G)$ in terms of vertex covers of G. We also characterize the Cohen-Macaulayness of the weighted cycles and weighted trees. For instance, we prove the following.

Theorem 3.7 ([43, Theorem A])**.** *Consider a weighted d-cycle C_ω^d.*

(a) *If $I(C_\omega^d)$ is Cohen-Macaulay, then $d \in \{3, 5\}$.*
(b) *$I(C_\omega^3)$ is always Cohen-Macaulay.*
(c) *$I(C_\omega^5)$ is Cohen-Macaulay if and only if it can be written in the form*

$$\begin{tikzcd}
v_1 \arrow[r, dash, "a"] \arrow[d, dash, "a"'] & v_2 \arrow[r, dash, "b"] & v_3 \arrow[dl, dash, "c"] \\
v_5 \arrow[r, dash, "d"'] & v_4 &
\end{tikzcd}$$

such that $a \leqslant b \geqslant c \leqslant d \geqslant e$.

My former PhD student Bethany Kubik and I [38] have recently souped up this construction to also include "r-path ideals" of weighted graphs. This ideal $I_r(G_\omega)$ subsumes the unweighted construction of Conca and De Negri [16]. Several aspects of this construction make it more challenging to work with than its progenitors. As an example of this, given an n-clique, i.e, a complete graph K^n, it is straightforward to show that the edge ideal $I(K_\omega^n)$ is always Cohen-Macaulay, since it is unmixed of dimension 1. On the other hand, the case of $I_r(G_\omega)$ with $r \geqslant 2$ is more complicated. However, we have the following result.

Theorem 3.8 ([38, Theorem C])**.** *Assume that $n \geqslant 3$, and let K_ω^n be a weighted n-clique. Then the ideal $I_2(K_\omega^n)$ is Cohen-Macaulay if and only if every induced weighted sub-3-clique $K_{\omega'}^3$ of K_ω^n has $I_2(K_{\omega'}^3)$ Cohen-Macaulay.*

Another facet of my work on the interactions between graph theory and ring theory is joint with Spiroff [49]. Our motivation for this paper comes from our earlier work [50], where we were forced to use a version of integral closure called the S_2-ification of a ring. Essentially, because of results of Hochster and Huneke [32], we were able to use this construction to prove that a construction of Griffith and Weston is local. To understand the S_2-ification, Hochster and Huneke associate a graph Γ_R to the ring R, and the point of our work [49] is to further understand the interaction between R and Γ_R. For instance, we improve on a result from [32] by showing that the number of connected components of Γ_R is equal to the number of maximal ideals of the S_2-ification of R. Furthermore, we introduce a labelling protocol for graphs such that, if a graph G admits such a labelling, then there is a complete, local ring R such that Γ_R is graph-isomorphic to G.

Divisor Class Groups.

If R is a unique factorization domain, then it is normal, i.e., integrally closed in its field of fractions. The converse fails in general, and the failure of this converse is measured in a sense by the divisor class group $\mathrm{Cl}(R)$: a normal domain R is a unique factorization domain if and only $\mathrm{Cl}(R) = 0$.

Sandra Sprioff and I have written two papers about induced maps on divisor class groups. In the first of these, we unify some previous results on the subject by proving the following .

Theorem 3.9 ([51, Theorem A])**.** *A ring homomorphism $R \to S$ of finite flat dimension between noetherian normal domains induces a well-defined group homomorphism $\mathrm{Cl}(R) \to \mathrm{Cl}(S)$.*

In [50, 51], we study properties of the kernels of such group homomorphisms. For instance, motivated by work of Griffith and Spiroff, we give conditions (1) guaranteeing that $\cap_n \mathrm{Ker}(\mathrm{Cl}(R) \to \mathrm{Cl}(R_n)) = 0$, and (2) limiting the torsion in $\mathrm{Ker}(\mathrm{Cl}(R) \to \mathrm{Cl}(S))$. In particular, item (2) fixes a mild oversight from a paper of Griffith and Weston [29].

4. Some Future Work

I am still working to understand Avramov and Foxby's composition question and Huneke's question about Bass numbers. It is a surprising fact (to me, at least) that these questions are deeply related to each other: I have shown that an affirmative answer to the composition question implies an affirmative answer to Huneke's question. This is the topic of a project that I am currently completing.

Now that Vasconcelos' finiteness question for semidualizing modules over a local ring has been resolved, I am focused on other related questions. First, there is a version of Vasconcelos' question for non-local rings. In general, the set $\mathfrak{S}_0(R)$ of isomorphism classes of semidualizing modules over a non-local ring R is not finite. However, there is an action of the Picard group on $\mathfrak{S}_0(R)$, and the orbit space under this action is conjectured to be finite. Unfortunately, the techniques that Nasseh and I developed to treat the local case do not help in general, though we were able to deal with this question in the semi-local case. Second, there is the question of computing the cardinalities of these sets. All evidence suggests that when R is local we have $|\mathfrak{S}_0(R)| = 2^n$ for some integer n; see, e.g., some of the subsections above.

When R is complete and not Gorenstein, we know that $|\mathfrak{S}_0(R)|$ is even by my joint work with Frankild [24]. Outside of this and the few cases where we can explicitly describe $\mathfrak{S}_0(R)$, very little is known about this cardinality.

Kubik's dissertation work [35] on quasidualizing modules has created more questions than it has answered. For instance, these modules are only defined for local rings, and the classes that they define have markedly different behavior from their semidualizing counterparts. Wicklein and I have recently shown how to define a class of objects that recovers both the semidualizing modules and quasidualizing modules as special cases. Using these gadgets, we have been able to address several of the question raised by Kubik's work, and we are working to answer more of them.

On the topic of G-dimensions, O. Celikbas are working to understand modules that detect finite G-dimension. These modules are similar to the residue field k of a local ring R, which has the following property: If M is a finitely generated R-module such that $\operatorname{Tor}_i^R(k, M) = 0$ for $i \gg 0$, then M has finite projective dimension. As an application of this investigation we have considerably improved a result of Goto and Hayasaka [28].

Related to the semidualizing modules are the *spherical* modules. These objects come from derived algebraic geometry, and their existence provides a certain amount of symmetry in derived categories of projective algebraic varieties. They are defined in part in terms of Ext, in a way that is similar to semidualizing modules. My initial work on these objects indicates that they are a purely projective phenomenon, that is, that they almost never exist in the local setting, including the local DG setting.

Further on the subject of DG algebra, I have developed a theory of support and co-support for DG modules which compliments constructions of Benson, Iyengar, and Krause [9, 10] and Foxby [23]. As an application of this theory, I have showed that semidualizing DG modules are faithful. This result is a key to some of the dissertation research of one of my current PhD students, Hannah Altmann [1].

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