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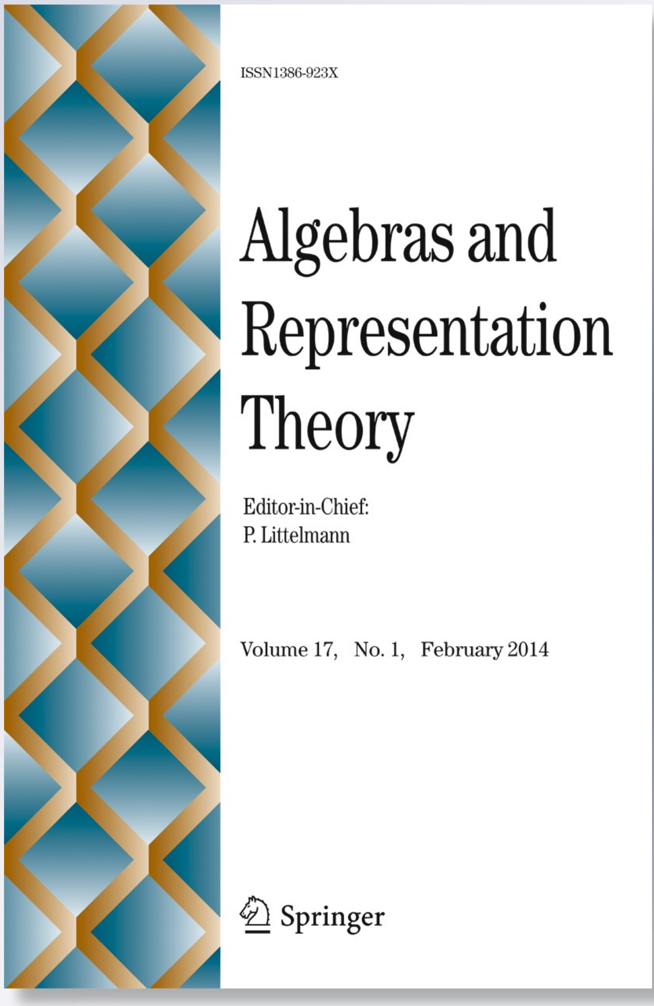

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Relative Tor Functors with Respect to a Semidualizing Module

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Abstract Let C be a semidualizing module over a commutative noetherian ring R. We exhibit an isomorphism $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(-,-) \cong \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(-,-)$ between the bifunctors defined via C-flat and C-projective resolutions. We show how the vanishing of these functors characterizes the finiteness of $\mathcal{F}_C$- pd, and use this to give a relation between the $\mathcal{F}_C$- pd of a module and of a pure submodule. On the other hand, we show that other isomorphisms force C to be trivial.

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M. Salimi · E. Tavasoli
Department of Mathematics, East Tehran Branch, Islamic Azad University, Tehran, Iran

M. Salimi
e-mail: maryamsalimi@ipm.ir

E. Tavasoli
e-mail: elhamtavasoli@ipm.ir

S. Sather-Wagstaff (✉)
Department of Mathematics, North Dakota State University Dept. # 2750,
P.O. Box 6050, Fargo, ND 58108-6050, USA
e-mail: Sean.Sather-Wagstaff@ndsu.edu
URL: http://www.ndsu.edu/pubweb/~ssatherw/

S. Yassemi
Department of Mathematics, University of Tehran, Tehran, Iran

S. Yassemi
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
e-mail: yassemi@ipm.ir
URL: http://math.ipm.ac.ir/yassemi/

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1 Introduction

For our purposes, relative homological algebra is the study of non-traditional resolutions and the (co)homology theories (i.e., relative derived functors) that they define. By “non-traditional” we mean that these resolutions are not given directly by projective, injective, or flat modules, as they are in “absolute” homological algebra. This idea goes back to Butler and Horrocks [2] and Eilenberg and Moore [5], and was reinvigorated by Enochs and Jenda [6] and Avramov and Martsinkovsky [1].

Much of the recent work [1, 15, 17] on the derived functors that arise in this context focuses on relative Ext functors. The point of this paper is to treat some properties of relative Tor. The relative homology functors that arise in this context come from resolutions that model projective and flat resolutions using semidualizing modules. (See Sections 2 and 3 for terminology, notation, and foundational results.)

Certain relations between the relative Tor functors defined by a semidualizing module C over a commutative noetherian ring R are obvious. For instance, commutativity of tensor product implies that $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_C}(N, M)$ and $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(N, M)$. Other relations are not obvious. For instance, it is well-known that $\mathrm{Tor}_i^R(M, N)$ can be computed using a projective resolution of M or a flat resolution of M. The corresponding result for relative Tor is our first main theorem, stated next. It is contained in Theorem 3.10.

Theorem A *Let C be a semidualizing R-module, and let M and N be R-modules. For each i, there is a natural isomorphism* $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$.

This result allows for a certain amount of flexibility for proving results about relative Tor, as in the absolute case. This is the subject of the rest of Section 3.

On the other hand, many properties of absolute Tor do not pass to the relative setting. These differences are the subject of Section 4. For instance, in Example 4.1 we show that in general we have, for instance, $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) \ncong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N)$. The remainder of this section focuses on two areas. First, our results Propositions 4.2–4.5 provide classes of modules M, N such that these non-isomorphisms are isomorphisms. Second, we show that the only way that these non-isomorphisms are always isomorphisms is in the trivial case. For instance, the next result is proved in Proof of Theorem B.

Theorem B *Assume that $(R, \mathfrak{m}, k)$ is local, and let B and C be semidualizing R-modules. Then the following conditions are equivalent:*

(i) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N)$ *for all $i \geqslant 0$ and for all R-modules M, N.*
(ii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B, k) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B, k)$ *for $i = 0$ and some $i > 0$.*
(iii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(k, C) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C)$ *for $i = 0$ and some $i > 0$.*
(iv) $B \cong R \cong C$.

Section 5 discusses $\mathcal{F}_C$- pd, the homological dimension obtained from bounded proper $\mathcal{F}_C$-resolutions, and its relation to relative Tor. For instance, we characterize finiteness of $\mathcal{F}_C$- pd in terms of vanishing of relative Tor. The paper concludes with the following version of [6, Lemma 9.1.4] and [12, Lemma 5.2(a)], proved in Proof of Theorem C.

Theorem C *Let C be a semidualizing R-module, and let $M' \subseteq M$ be a pure submodule. Then one has* $\mathcal{F}_C\text{- pd}_R(M) \geqslant \sup\{\mathcal{F}_C\text{- pd}_R(M'), \mathcal{F}_C\text{- pd}_R(M/M') - 1\}$.

2 Background Material

Convention 2.1 Throughout this paper R and S are commutative noetherian rings, and $\mathcal{M}(R)$ is the category of R-modules. Also, $\mathcal{X}$ is a full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ closed under isomorphisms. Write $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$ for the subcategories of projective, flat and injective R-modules. Write m-Spec(R) for the set of maximal ideals of R.

Definition 2.2 We index R-complexes homologically:

$$Y = \cdots \xrightarrow{\partial^Y_{n+1}} Y_n \xrightarrow{\partial^Y_n} Y_{n-1} \xrightarrow{\partial^Y_{n-1}} \cdots .$$

An R-complex Y is $\mathrm{Hom}_R(\mathcal{X}, -)$*-exact* if for each X in $\mathcal{X}$, the complex $\mathrm{Hom}_R(X, Y)$ is exact, and similarly for $\mathrm{Hom}_R(-, \mathcal{X})$*-exact*. Two R-complexes Y and Z are *quasiisomorphic*, written $Y \simeq Z$, provided that there is a sequence of quasiisomorphisms $Y \leftarrow Y^1 \rightarrow Y^2 \leftarrow \cdots \leftarrow Y^m \rightarrow Z$ for some integer m.

We build resolutions from precovers and preenvelopes, defined next; see, e.g., [6].

Definition 2.3 An $\mathcal{X}$*-precover* of an R-module M is an R-module homomorphism $X \xrightarrow{\varphi} M$, where $X \in \mathcal{X}$, and such that the map $\mathrm{Hom}_R(X', \varphi)$ is surjective for every $X' \in \mathcal{X}$. If every R-module admits $\mathcal{X}$-precover, then the class $\mathcal{X}$ is *precovering*.

Assume that $\mathcal{X}$ is precovering. Then each R-module M has an *augmented proper* $\mathcal{X}$*-resolution*, that is, an $\mathrm{Hom}_R(\mathcal{C}, -)$-exact R-complex

$$X^+ = \cdots \xrightarrow{\partial^X_2} X_1 \xrightarrow{\partial^X_1} X_0 \xrightarrow{\tau} M \longrightarrow 0.$$

The truncated complex $X = \cdots \xrightarrow{\partial^X_2} X_1 \xrightarrow{\partial^X_1} X_0 \longrightarrow 0$ is a *proper* $\mathcal{X}$*-resolution* of M. The $\mathcal{X}$*-projective dimension* of M is

$$\mathcal{X}\text{- pd}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is a proper } \mathcal{X}\text{-resolution of } M\}.$$

The terms *preenveloping*, *proper* $\mathcal{X}$*-coresolution* and $\mathcal{X}$- id are defined dually.

When $\mathcal{X}$ is the class of projective R-modules, we write $\mathrm{pd}_R(M)$ for the associated homological dimension and call it the *projective dimension* of M. Similarly, the flat and injective dimensions of M are denoted $\mathrm{fd}_R(M)$ and $\mathrm{id}_R(M)$.

Remark 2.4 Assume that $\mathcal{X}$ is precovering. We note explicitly that augmented proper $\mathcal{X}$-resolutions need not be exact. According to our definitions, we have

$\mathcal{X}\text{-}\mathrm{pd}_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension zero are the non-zero modules in $\mathcal{X}$. Note that projective resolutions (in the usual sense) are automatically proper, and that augmented proper flat resolutions are automatically exact.

The next result is a straightforward consequence of Remark 2.4.

Lemma 2.5 *Let N be a module such that there exists an exact sequence*

$$0 \to G_n \to \cdots \to G_0 \to N \to 0$$

where each G_i is flat. Let F be a proper flat resolution of N. Then the truncation

$$\widetilde{F}^+ = \left(0 \to \mathrm{Im}\left(\partial^F_{n+1}\right) \to F_{n-1} \xrightarrow{\partial^F_{n-1}} \cdots \xrightarrow{\partial^F_1} F_0 \to N \to 0\right)$$

is also a proper flat resolution of N.

Remark 2.6 The difference between flat resolutions (in the usual sense) and proper flat resolutions is subtle. For instance, the next example shows that bounded flat resolutions need not be proper. On the other hand, Lemma 2.5 shows that the classical flat dimension of N is the same as $\mathrm{fd}_R(N)$.

Example 2.7 Assume that $(R, \mathfrak{m}, k)$ is a local, non-complete, Gorenstein domain of dimension 1. The augmented minimal injective resolution of R (over itself) has the form $X = (0 \to R \to Q \xrightarrow{\alpha} E \to 0)$ where Q is the field of fractions of R and E is the injective hull of k. Of course, R and Q are flat. By [9, Theorem 2.5], we have $\mathrm{Ext}^1_R(\widehat{R}, R) \neq 0$ where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R. It follows that $\mathrm{Hom}_R(\widehat{R}, X)$ is not exact, so X is not an augmented proper flat resolution.

Semidualizing modules, defined by Foxby [7], yield our categories of interest.

Definition 2.8 A finitely generated R-module C is *semidualizing* if $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}^i_R(C, C) = 0$ for $i \geqslant 1$. Following [10, 12], we set

$$\mathcal{P}_C(R) = \text{the subcategory of modules } M \cong P \otimes_R C \text{ for some } P \in \mathcal{P}(R)$$

$$\mathcal{F}_C(R) = \text{the subcategory of modules } M \cong F \otimes_R C \text{ for some } F \in \mathcal{F}(R)$$

$$\mathcal{I}_C(R) = \text{the subcategory of modules } M \cong \mathrm{Hom}_R(C, I) \text{ for some } I \in \mathcal{I}(R).$$

A *dualizing* module for R is a semidualizing module of finite injective dimension.

Remark 2.9 Let C be semidualizing over R. Then $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are precovering and closed under coproducts, and $\mathcal{I}_C(R)$ is preenveloping, by [12]. As R is noetherian and C is finitely generated, $\mathcal{F}_C(R)$ is closed under products.

Remark 2.10 Let C be semidualizing over R. Then C is cyclic if and only if it is free, if and only if $C \cong R$. Also, $\mathrm{pd}_R(C) < \infty$ if and only if C is projective (of rank 1). If R is Gorenstein local, then $C \cong R$. If $R \to S$ is a flat homomorphism, then $S \otimes_R C$ is semidualizing over S. See [14, Section 1] and [13, Chapter 2].

The next classes were also introduced by Vasconcelos [18].

Definition 2.11 Let C be a semidualizing R-module. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that:

(i) $\operatorname{Tor}_i^R(C, M) = 0 = \operatorname{Ext}_R^i(C, C \otimes_R M)$ for all $i \geqslant 1$, and
(ii) the natural map $M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that:

(i) $\operatorname{Ext}_R^i(C, M) = 0 = \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M))$ for all $i \geqslant 1$, and
(ii) the natural evaluation map $C \otimes_R \operatorname{Hom}_R(C, M) \xrightarrow{\xi_M^C} M$ is an isomorphism.

Remark 2.12 Let C be a semidualizing R-module. Given an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-module homomorphisms, if two of the M_i are in $\mathcal{A}_C(R)$ or in $\mathcal{B}_C(R)$, then so is the third M_i; see [12, Corollary 6.3]. The class $\mathcal{A}_C(R)$ contains all flat R-modules and all modules from $\mathcal{I}_C(R)$, and $\mathcal{B}_C(R)$ contains all injective R-modules and all modules from $\mathcal{F}_C(R)$; see [12, Corollary 6.1] and [17, 1.9]. Foxby equivalence ([3, Theorem 4.6] and [17, Theorem 2.8]) states:

(a) An R-module M is in $\mathcal{B}_C(R)$ if and only if $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C(R)$.
(b) An R-module M is in $\mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(R)$.

The next result follows readily from the previous remark.

Lemma 2.13 *Let C be a semidualizing R-module, and let X, Y be R-complexes such that $X_i, Y_i \in \mathcal{A}_C(R)$ for each index i. Assume that X and Y are both either bounded above or bounded below.*

(a) *If X is exact, then so is $C \otimes_R X$.*
(b) *If $f\colon X \to Y$ is a quasiisomorphism, then so is $C \otimes_R f$.*
(c) *If X and Y are bounded below and $X \simeq Y$, then $C \otimes_R X \simeq C \otimes_R Y$.*

The following functors are studied in [15, 17].

Definition 2.14 Let C be a semidualizing R-module, and let M and N be R-modules. Let L be a proper $\mathcal{P}_C$-resolution of M, and let J be a proper $\mathcal{I}_C$-coresolution of N. For each i, set

$$\operatorname{Ext}^i_{\mathcal{P}_C\mathcal{M}}(M, N) := \operatorname{H}_{-i}(\operatorname{Hom}_R(L, N))$$
$$\operatorname{Ext}^i_{\mathcal{M}\mathcal{I}_C}(M, N) := \operatorname{H}_{-i}(\operatorname{Hom}_R(M, J)).$$

Lemma 2.15 *Assume that R is local, and let C be a semidualizing R-module. Then $C \cong R$ if and only if $C \otimes_R C$ is free.*

Proof The forward implication is straightforward. For the converse, assume that $C \otimes_R C$ is free, and let β be the minimal number of generators of C. By Nakayama's Lemma, the module $C \otimes_R C$ is minimally generated by β^2 many elements, so $C \otimes_R C \cong R^{\beta^2}$. On the other hand, the surjection $R^\beta \twoheadrightarrow C$ gives a surjection $C^\beta \twoheadrightarrow$

$C \otimes_R C \cong R^{\beta^2}$ by right exactness of tensor product. This splits, so R^{β^2} is a direct summand of C^β. Taking endomorphism rings, we conclude that $\mathrm{End}(R^{\beta^2}) \cong R^{\beta^4}$ is a direct summand of $\mathrm{End}_R(C^\beta) \cong R^{\beta^2}$. In particular, this implies that $\beta^4 \leqslant \beta^2$, which implies that $\beta = 1$. It follows that C is cyclic, so $C \cong R$ by Remark 2.10. □

Lemma 2.16 *Let k be a field, and set $R = k[X, Y]/(X, Y)^2$. If C is a non-free semidualizing R-module, then C is dualizing for R and $C \otimes_R C \cong k^4$.*

Proof The ring R is local with maximal ideal $\mathfrak{m} = (X, Y)R$ such that $\mathfrak{m}^2 = 0$. As C is not free, it follows that C is dualizing for R, so $C \cong E_R(k)$. As a k-vector space, we have $C \cong k \cdot X^{-1} \oplus k \cdot Y^{-1} \oplus k \cdot 1$ with R-module structure given by

$$X \cdot 1 = 0 \qquad X \cdot X^{-1} = 1 \qquad X \cdot Y^{-1} = 0$$

$$Y \cdot 1 = 0 \qquad Y \cdot Y^{-1} = 1 \qquad Y \cdot X^{-1} = 0.$$

Using this, one can show that $C \supseteq RY^{-1} \cong R/XR$ and $C/RY^{-1} \cong k$. In particular, there is an exact sequence $0 \to R/XR \to C \to k \to 0$. Also, we see that $XC = k \cdot 1$, so $C/XC \cong k^2$.

We claim that $\mathrm{len}_R(C \otimes_R C) \leqslant 4$. For this, we apply $C \otimes_R -$ to the previous exact sequence to obtain the exact sequence $C/XC \to C \otimes_R C \to C \otimes_R k \to 0$. As we noted above, we have $C/XC \cong k^2 \cong C \otimes_R k$, so additivity of length implies that $\mathrm{len}_R(C \otimes_R C) \leqslant 4$.

We next claim that $\mathrm{len}_R(C \otimes_R C) = 4$. To check this, consider the epimorphism $C \to k^2$ coming from the fact that C is minimally generated by X^{-1}, Y^{-1}. The right exactness of $C \otimes_R -$ implies that the map $C \otimes_R C \to C \otimes_R k^2$ is surjective. Since $C \otimes_R k \cong k^2$, it follows that $4 \leqslant \mathrm{len}_R(C \otimes_R C)$. The previous claim implies that $\mathrm{len}_R(C \otimes_R C) = 4$.

Nakayama's Lemma implies that $C \otimes_R C$ is minimally generated by four elements. That is, the modules $C \otimes_R C$ and $(C \otimes_R C)/\mathfrak{m}(C \otimes_R C)$ both have length 4. Since $(C \otimes_R C)/\mathfrak{m}(C \otimes_R C)$ is a homomorphic image of $C \otimes_R C$, it follows that $C \otimes_R C \cong (C \otimes_R C)/\mathfrak{m}(C \otimes_R C) \cong k^4$ as desired. □

3 Proper Resolutions and Relative Homology

In this section, C is a semidualizing R-module, and M and N are R-modules.

The results of this section document some properties of proper $\mathcal{F}_C$-resolutions and proper $\mathcal{P}_C$-resolutions, beginning with two lemmas that are implicit in [17].

Lemma 3.1

(a) *If F is a proper flat resolution of* $\mathrm{Hom}_R(C, M)$*, then $C \otimes_R F$ is a proper $\mathcal{F}_C$-resolution of M.*
(b) *If G is a proper $\mathcal{F}_C$-resolution of M, then* $\mathrm{Hom}_R(C, G)$ *is a proper flat resolution of* $\mathrm{Hom}_R(C, M)$.
(c) *If P is a projective resolution of* $\mathrm{Hom}_R(C, M)$*, then $C \otimes_R P$ is a proper $\mathcal{P}_C$-resolution of M.*
(d) *If Q is a proper $\mathcal{P}_C$-resolution of M, then* $\mathrm{Hom}_R(C, Q)$ *is a projective resolution of* $\mathrm{Hom}_R(C, M)$.

Lemma 3.2 *Let X be an R-complex. Then X is $\mathrm{Hom}_R(\mathcal{P}_C, -)$-exact if and only if $\mathrm{Hom}_R(C, X)$ is exact.*

The next result is a routine consequence of Lemma 3.2. We do not know if the corresponding result for proper $\mathcal{F}_C$-resolutions holds. See, however, Corollary 5.3.

Lemma 3.3 *Let $R \to S$ be a flat ring homomorphism. If L is a proper $\mathcal{P}_C$-resolution of M over R, then $S \otimes_R L$ is a proper $\mathcal{P}_{S\otimes_R C}$-resolution of $S \otimes_R M$ over S.*

Lemma 3.4 *Let $\{M_j\}_{j\in J}$ be a set of R-modules. For each $j \in J$, let X_j be a proper $\mathcal{F}_C$-resolution of M_j, and let Y_j be a proper $\mathcal{P}_C$-resolution of M_j.*

(a) *The product $\prod_j X_j$ is a proper $\mathcal{F}_C$-resolution of $\prod_j M_j$.*
(b) *The coproduct $\coprod_j Y_j$ is a proper $\mathcal{P}_C$-resolution of $\coprod_j M_j$.*

In our setting, there are four different relative Tor-modules to consider.

Definition 3.5 Let Q be a proper $\mathcal{P}_C$-resolution of M, and let G be a proper $\mathcal{F}_C$-resolution of M. For each $i \geqslant 0$, set

$$\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) := \mathrm{H}_i(Q \otimes_R N) \cong \mathrm{H}_i(N \otimes_R Q) =: \mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_C}(N, M)$$
$$\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) := \mathrm{H}_i(G \otimes_R N) \cong \mathrm{H}_i(N \otimes_R G) =: \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(N, M).$$

Remark 3.6 The properness assumption on the resolutions in Definition 3.5 guarantee that these relative Tor constructions are independent of the choice of resolutions and functorial in both arguments. See [6, Section 8.2]. Also, there are natural transformations of bifunctors

$$\mathrm{Tor}_0^{\mathcal{P}_C\mathcal{M}}(-,-) \to - \otimes_R - \qquad \mathrm{Tor}_0^{\mathcal{M}\mathcal{P}_C}(-,-) \to - \otimes_R -$$
$$\mathrm{Tor}_0^{\mathcal{F}_C\mathcal{M}}(-,-) \to - \otimes_R - \qquad \mathrm{Tor}_0^{\mathcal{M}\mathcal{F}_C}(-,-) \to - \otimes_R -.$$

In general, these are not isomorphisms, as we see in Example 4.1 below.

Example 3.7 In the trivial case $C = R$, we have $\mathcal{F}_R(R) = \mathcal{F}(R)$ and $\mathcal{P}_R(R) = \mathcal{P}(R)$, and the relative Tors are the same as the absolute Tors.

$$\mathrm{Tor}_i^{\mathcal{P}_R\mathcal{M}}(-,-) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_R}(-,-) \cong \mathrm{Tor}_i^{\mathcal{F}_R\mathcal{M}}(-,-) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_R}(-,-) \cong \mathrm{Tor}_i^{R}(-,-)$$

The following long exact sequences come from [6, Theorem 8.2.3].

Lemma 3.8 *Let $\mathbb{L} = (0 \to L' \to L \to L'' \to 0)$ be a complex of R-modules.*

(a) *If $\mathbb{L}$ is $\mathrm{Hom}_R(\mathcal{P}_C, -)$-exact (i.e., if $\mathrm{Hom}_R(C, \mathbb{L})$ is exact, e.g., if $L' \in \mathcal{B}_C(R)$), then there is a long exact sequence*

$$\cdots \mathrm{Tor}_1^{\mathcal{P}_C\mathcal{M}}(L'', N) \to \mathrm{Tor}_0^{\mathcal{P}_C\mathcal{M}}(L', N) \to \mathrm{Tor}_0^{\mathcal{P}_C\mathcal{M}}(L, N) \to \mathrm{Tor}_0^{\mathcal{P}_C\mathcal{M}}(L'', N) \to 0$$

that is natural in $\mathbb{L}$ and N.

(b) *If* $\mathbb{L}$ *is* $\operatorname{Hom}_R(\mathcal{F}_C, -)$*-exact, then there is a long exact sequence*

$$\cdots \operatorname{Tor}_1^{\mathcal{F}_C\mathcal{M}}(L'', N) \to \operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(L', N) \to \operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(L, N) \to \operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(L'', N) \to 0$$

that is natural in $\mathbb{L}$ *and* N*.*

Construction 3.9 For each i, there is a natural transformation of bifunctors $\varrho_i \colon \operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(-,-) \to \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(-,-)$. To construct ϱ_i, let Q be a proper $\mathcal{P}_C$-resolution of M, and let G be a proper $\mathcal{F}_C$-resolution of M. The containment $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$ implies that the augmented resolution G^+ is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact. It follows that there is a chain map $Q^+ \to G^+$ that is an isomorphism in degree -1. The induced morphism $Q \otimes_R N \to G \otimes_R N$ gives rise to ϱ_i by taking homology.

The next result compares to [17, Theorem 4.1] which has similar formulas for relative Ext. This contains Theorem A from the introduction.

Theorem 3.10 *For each i, there are natural isomorphisms*

$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \xrightarrow{\cong} \operatorname{Tor}_i^R(\operatorname{Hom}_R(C, M), C \otimes_R N) \xleftarrow{\cong} \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$$

and the morphism $\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(-,-) \xrightarrow{\varrho_i} \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(-,-)$ *is an isomorphism.*

Proof Let F be a proper flat resolution of $\operatorname{Hom}_R(C, M)$. Lemma 3.1(a) implies that $C \otimes_R F$ is a proper $\mathcal{F}_C$-resolution of M, so we have

$$\begin{aligned} \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) &\cong \mathrm{H}_i((C \otimes_R F) \otimes_R N) \\ &\cong \mathrm{H}_i(F \otimes_R (C \otimes_R N)) \\ &\cong \operatorname{Tor}_i^R(\operatorname{Hom}_R(C, M), C \otimes_R N). \end{aligned}$$

The naturality of this isomorphism comes from the naturality of the constructions, and similarly for $\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N)$.

Let $(C \otimes_R F)^{\pm}$ denote the following complex

$$(C \otimes_R F)^{\pm} = \cdots \xrightarrow{C \otimes \partial_2^F} C \otimes_R F_1 \xrightarrow{C \otimes \partial_1^F} C \otimes_R F_0 \xrightarrow{\xi_M^C \circ (C \otimes_R \tau)} M \to 0$$

and similarly for $(C \otimes_R P)^{\pm}$. Let $P \to F$ be a lift of the identity map on $\operatorname{Hom}_R(C, M)$. Then the induced map $(C \otimes_R P)^{\pm} \to (C \otimes_R F)^{\pm}$ is of the form $Q^+ \to G^+$, as in Construction 3.9. It follows that $\varrho_i(M, N)$ is gotten by taking homology in the map $(C \otimes_R P) \otimes_R N \to (C \otimes_R F) \otimes_R N$. Of course, this is equivalent to taking homology in the map $P \otimes_R (C \otimes_R N) \to F \otimes_R (C \otimes_R N)$. The fact that $\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, M), C \otimes_R N)$ can be computed using P or F implies that the induced maps on homology are isomorphisms, as desired. □

Theorem 3.10 allows for a certain amount of flexibility for relative Tor, in the same way that flat and projective resolutions give flexibility for absolute Tor. For instance, the next two results follow from Theorem 3.10, using Lemmas 3.3 and 3.4.

Corollary 3.11 *Let* $\{N_j\}_{j\in J}$ *be a set of R-modules.*

(a) *For each i, there are isomorphisms*

$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, \textstyle\coprod_j N_j) \cong \coprod_j \operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N_j)$$
$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(\textstyle\coprod_j N_j, M) \cong \coprod_j \operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(N_j, M)$$

and similarly for $\operatorname{Tor}^{\mathcal{F}_C\mathcal{M}}$.

(b) *If M is finitely generated, then for each i, there are isomorphisms*

$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, \textstyle\prod_j N_j) \cong \prod_j \operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N_j)$$
$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(\textstyle\prod_j N_j, M) \cong \prod_j \operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(N_j, M)$$

and similarly for $\operatorname{Tor}^{\mathcal{F}_C\mathcal{M}}$.

Corollary 3.12 *Let* $R \to S$ *be a flat ring homomorphism. Then for all i there are S-module isomorphisms*

$$\operatorname{Tor}_i^{\mathcal{P}_{S\otimes_R C}\mathcal{M}}(S \otimes_R M, S \otimes_R N) \cong S \otimes_R \operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N)$$
$$\operatorname{Tor}_i^{\mathcal{F}_{S\otimes_R C}\mathcal{M}}(S \otimes_R M, S \otimes_R N) \cong S \otimes_R \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N).$$

The last result of this section is a relative version of Hom-tensor adjointness.

Corollary 3.13 *Let I be an injective R-module. For all* $i \geqslant 0$ *one has*

$$\operatorname{Ext}^i_{\mathcal{P}_C\mathcal{M}}(M, \operatorname{Hom}_R(N, I)) \cong \operatorname{Hom}_R(\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N), I) \tag{3.13.1}$$

$$\operatorname{Ext}^i_{\mathcal{M}\mathcal{I}_C}(M, \operatorname{Hom}_R(N, I)) \cong \operatorname{Hom}_R(\operatorname{Tor}_i^{\mathcal{M}\mathcal{P}_C}(M, N), I). \tag{3.13.2}$$

Proof The first isomorphism in the next sequence follows from [17, Theorem 4.1]

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{P}_C\mathcal{M}}(M, \operatorname{Hom}_R(N, I)) &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, \operatorname{Hom}_R(N, I)))\\ &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C \otimes_R N, I))\\ &\cong \operatorname{Hom}_R(\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, M), C \otimes_R N), I)\\ &\cong \operatorname{Hom}_R(\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N), I).\end{aligned}$$

The second isomorphism is by Hom-tensor adjointness, and the remaining steps follow from [6, Theorem 3.2.1] and Theorem 3.10. This explains Eqs. 3.13.1, and 3.13.2 is established similarly. □

4 Comparison of Relative Homologies

In this section, B, C are semidualizing R-modules, and M, N are R-modules.

The next example shows that relative Tors do not satisfy a naive version of balance, they are not commutative, and they do not agree with absolute Tor in general.

Example 4.1 Assume that $(R, \mathfrak{m}, k)$ is local and that C is not free, that is, that C is not cyclic. We show that

$$\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, k) \ncong \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C) \ncong \operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C) \tag{4.1.1}$$

$$\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C) \ncong \operatorname{Tor}_i^{R}(k, C) \tag{4.1.2}$$

for all i, at least in a specific example.

Let $\beta \geqslant 2$ be the minimum number of generators for C. It is straightforward to show that $\operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C) = 0$ for all $i \geqslant 1$ and that $\operatorname{Tor}_0^{\mathcal{M}\mathcal{F}_C}(k, C) \cong k \otimes_R C \cong k^\beta$; see also Proposition 4.2 and Theorem 5.4. From Theorem 3.10, we have

$$\operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(k, C) \cong \operatorname{Hom}_R(C, k) \otimes_R (C \otimes_R C) \cong k^\beta \otimes_R (C \otimes_R C) \cong k^{\beta^3}.$$

This is not isomorphic to

$$\operatorname{Tor}_0^{R}(C, k) \cong \operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(C, k) \cong k^\beta \cong \operatorname{Tor}_0^{\mathcal{M}\mathcal{F}_C}(k, C)$$

as $\beta \geqslant 2$. This explains Eqs. 4.1.1 and 4.1.2 for $i = 0$.

Again using Theorem 3.10, for $i \geqslant 1$ we have

$$\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C) \cong \operatorname{Tor}_i^{R}(\operatorname{Hom}_R(C, k), C \otimes_R C) \cong \operatorname{Tor}_i^{R}(k, C \otimes_R C)^\beta \tag{4.1.3}$$

and $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, k) = 0 = \operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C)$. Thus, to show Eq. 4.1.1 in general, it suffices to find an example such that $\operatorname{Tor}_i^{R}(k, C \otimes_R C) \neq 0$ for all $i \geqslant 1$, that is, such that $\operatorname{pd}_R(C \otimes_R C) = \infty$.[1] This is supplied by Lemma 2.15 and the Auslander-Buchsbaum formula, assuming that R is artinian.

Finally, we give a specific example where $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C) \ncong \operatorname{Tor}_i^{R}(k, C)$ for all i. Note that Eq. 4.1.3 shows that $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C) \cong k^{\beta \cdot \beta_i(C \otimes_R C)}$; here we use β_i for the ith Betti number. As $\operatorname{Tor}_i^{R}(k, C) \cong k^{\beta_i(C)}$, it suffices to provide an example where

$$\beta \cdot \beta_i(C \otimes_R C) > \beta_i(C) \tag{4.1.4}$$

for all $i \geqslant 1$. To this end, set $R = k[X, Y]/(X, Y)^2$, so we have $\mathfrak{m}^2 = 0$. Let C be dualizing for R, so $\beta = 2$. Lemma 2.16 implies that $C \otimes_R C \cong k^4$, so we have $\beta_i(C \otimes_R C) = 4\beta_i(k) = 4 \cdot 2^i = 2^{i+2}$ for all $i \geqslant 0$. From [4, (1.5) Example], we have $\beta_i(C) = 3 \cdot 2^{i-1}$ for all $i \geqslant 1$. From these equalities, one easily deduces the inequality (4.1.4) for all $i \geqslant 1$, since $\beta = 2$.

We continue this section by giving some special cases where some naive properties do hold for relative homology.

Proposition 4.2 *If the natural map $C \otimes_R \operatorname{Hom}_R(C, M) \to M$ is an isomorphism (e.g., if $M \in \mathcal{B}_C(R)$), then $\operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(M, -) \cong M \otimes_R -$.*

[1] We believe that this is true in general, under the assumption that C is not free; see [8].

Proof Again, by Theorem 3.10 we have

$$\mathrm{Tor}_0^{\mathcal{F}_C\mathcal{M}}(M,N) \cong \mathrm{Hom}_R(C,M)\otimes_R (C\otimes_R N) \cong M\otimes_R N$$

where the last isomorphism is from the assumption $C\otimes_R \mathrm{Hom}_R(C,M)\cong M$. □

Example 4.1 shows that we can have $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M,-)\ncong \mathrm{Tor}_i^R(M,-)$, even when $M\in\mathcal{B}_C(R)$. The next result shows that this fails when $N\in\mathcal{A}_C(R)$.

Proposition 4.3 *If $M\in\mathcal{B}_C(R)$ and $N\in\mathcal{A}_C(R)$, then for each i one has*

$$\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M,N)\cong \mathrm{Tor}_i^R(M,N) \qquad \text{and} \qquad \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M,N)\cong \mathrm{Tor}_i^R(M,N).$$

Proof Let P be a projective resolution of $\mathrm{Hom}_R(C,M)$, and let Q be a projective resolution of N. Lemma 3.1(c) implies that $C\otimes_R P$ is a proper $\mathcal{P}_C$-resolution of M.

Since Q is a bounded below complex of projective R-modules, it respects quasiisomorphisms. This explains the second isomorphism in the next sequence:

$$\begin{aligned}\mathrm{Tor}_i^R(M,N) &\cong \mathrm{H}_i(M\otimes_R Q)\\ &\cong \mathrm{H}_i((C\otimes_R P)\otimes_R Q)\\ &\cong \mathrm{H}_i((C\otimes_R P)\otimes_R N)\\ &\cong \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M,N).\end{aligned}$$

The first isomorphism is from the balance of Tor. The fourth isomorphism is by definition. It remains to explain the third isomorphism.

Since Q is a projective resolution of N, there is a quasiisomorphism $Q\xrightarrow{\simeq} N$. Since P is a bounded below complex of projective R-modules, the functor $P\otimes_R -$ respects quasiisomorphisms. So there is a quasiisomorphism $P\otimes_R Q\xrightarrow{\simeq} P\otimes_R N$. Lemma 2.13(b) implies that the induced map $C\otimes_R P\otimes_R Q\xrightarrow{\simeq} C\otimes_R P\otimes_R N$ is also a quasiisomorphism, so $(C\otimes_R P)\otimes_R Q\simeq (C\otimes_R P)\otimes_R N$, as desired. □

The best results we know for balance and commutativity are the following two corollaries of Proposition 4.3.

Corollary 4.4 *If $M\in\mathcal{B}_B(R)\cap\mathcal{A}_C(R)$ and $N\in\mathcal{B}_C(R)\cap\mathcal{A}_B(R)$, then one has $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(M,N)\cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M,N)$ for all $i\geqslant 0$.*

Corollary 4.5 *If $M,N\in\mathcal{B}_C(R)\cap\mathcal{A}_C(R)$, then for all indices $i\geqslant 0$ one has $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M,N)\cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(N,M)$.*

The next example provides modules satisfying the hypotheses of these results.

Example 4.6 Assume that $C\in\mathcal{A}_B(R)$, e.g., if D is a dualizing R-module and $B=C^\dagger := \mathrm{Hom}_R(C,D)$. By [8, Corollary 3.8], it follows that $B\in\mathcal{A}_C(R)$. Then $B\in\mathcal{B}_B(R)\cap\mathcal{A}_C(R)$, so $\mathcal{F}_B(R)\subseteq\mathcal{B}_B(R)\cap\mathcal{A}_C(R)$. By Remark 2.12, every module of finite $\mathcal{F}_B$-projective dimension is in $\mathcal{B}_B(R)\cap\mathcal{A}_C(R)$. Similarly, every module of finite $\mathcal{F}_C$-projective dimension is in $\mathcal{B}_C(R)\cap\mathcal{A}_B(R)$.

For another example, assume that R has a dualizing module D. Then every module of finite $\mathcal{G}(\mathcal{P}_C)$-projective dimension or finite $\mathcal{G}(\mathcal{I}_{C^\dagger})$-injective dimension is in $\mathcal{B}_C(R) \cap \mathcal{A}_{C^\dagger}(R)$, and every module of finite $\mathcal{G}(\mathcal{P}_{C^\dagger})$-projective dimension or finite $\mathcal{G}(\mathcal{I}_C)$-injective dimension is in $\mathcal{B}_{C^\dagger}(R) \cap \mathcal{A}_C(R)$. See [15, Fact 3.13] for details.

Finding modules in $\mathcal{A}_C(R) \cap \mathcal{B}_C(R)$ is more difficult. If R is a domain, then the quotient field $Q(R)$ is flat and injective, so it is in $\mathcal{A}_C(R) \cap \mathcal{B}_C(R)$.

Proof of Theorem B We verify the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i). The implications (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) are verified similarly. The implication (i) $\Longrightarrow$ (ii) is trivial, and the implication (iv) $\Longrightarrow$ (i) is from Example 3.7.

(ii) $\Longrightarrow$ (iv) We exploit Theorem 3.10:

$$\begin{aligned}\operatorname{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B,k) &\cong \operatorname{Tor}_i^R(\operatorname{Hom}_R(B,B), B\otimes_R k) \cong \operatorname{Tor}_i^R(R, k^{\beta_0(B)})\\ &\cong \begin{cases} k^{\beta_0(B)} & i=0\\ 0 & i\neq 0\end{cases}\\ \operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B,k) &\cong \operatorname{Tor}_i^R(C\otimes_R B, \operatorname{Hom}_R(C,k)) \cong \operatorname{Tor}_i^R(C\otimes_R B, k^{\beta_0(C)})\\ &\cong k^{\beta_i(C\otimes_R B)\beta_0(C)}.\end{aligned}$$

Assuming that $\operatorname{Tor}_0^{\mathcal{F}_B\mathcal{M}}(B,k) \cong \operatorname{Tor}_0^{\mathcal{M}\mathcal{F}_C}(B,k)$, we conclude that

$$\beta_0(B) = \beta_0(C\otimes_R B)\beta_0(C) = \beta_0(B)\beta_0(C)^2.$$

Since $\beta_0(B) \neq 0$, it follows that $\beta_0(C) = 1$. So C is cyclic, and therefore $C \cong R$ by Remark 2.10. Assuming $\operatorname{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B,k) \cong \operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B,k)$ for some $i \geqslant 1$, we have

$$0 = \beta_i(C\otimes_R B)\beta_0(C) = \beta_i(B).$$

It follows that $\operatorname{pd}_R(B) < \infty$, so Remark 2.10 implies that $B \cong R$, as desired. □

The next results follow from Theorem B with $B = C$ and $B = R$, respectively.

Corollary 4.7 *Assume that $(R, \mathfrak{m}, k)$ is local. The following are equivalent:*

(i) $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(X,Y) \cong \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(Y,X)$ *for all $i \geqslant 0$ and for all R-modules X, Y.*
(ii) $\operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X,Y) \cong \operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(Y,X)$ *for all $i \geqslant 0$ and for all R-modules X, Y.*
(iii) $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C,k) \cong \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k,C)$ *for $i = 0$ and some $i \geqslant 1$.*
(iv) $\operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(C,k) \cong \operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k,C)$ *for $i = 0$ and some $i \geqslant 1$.*
(v) $C \cong R$.

Corollary 4.8 *Assume that $(R, \mathfrak{m}, k)$ is local. The following are equivalent:*

(i) $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(X,Y) \cong \operatorname{Tor}_i^R(X,Y)$ *for all $i \geqslant 0$ and for all R-modules X, Y.*
(ii) $\operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X,Y) \cong \operatorname{Tor}_i^R(X,Y)$ *for all $i \geqslant 0$ and for all R-modules X, Y.*
(iii) $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C,k) \cong \operatorname{Tor}_i^R(C,k)$ *for some $i \geqslant 1$.*
(iv) $\operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k,C) \cong \operatorname{Tor}_i^R(k,C)$ *for some $i \geqslant 1$.*
(v) $C \cong R$.
(vi) $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k,k) \cong \operatorname{Tor}_i^R(k,k)$ *for some $i \geqslant 0$.*
(vii) $\operatorname{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k,k) \cong \operatorname{Tor}_i^R(k,k)$ *for some $i \geqslant 0$.*

Remark 4.9 Note that Theorem B and Corollary 4.7 do not contain versions of the conditions (iv) and (vii) of Corollary 4.8. Indeed, for Corollary 4.7 this is because we always have $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k,k) \cong k^{\beta_0(C)^2\beta_i(k)} \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k,k)$. Similarly, if one assumes in Theorem B that $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(k,k) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k,k)$, then the only conclusion one would be able to draw from this is that $\beta_0(B) = \beta_0(C)$, which is not enough to guarantee that B and C are isomorphic, let alone isomorphic to R.

For the non-local versions of the above results, we require a bit more technology.

Definition 4.10 Let $\mathrm{Pic}(R)$ denote the Picard group of R. The elements of $\mathrm{Pic}(R)$ are the isomorphism classes $[P]$ of finitely generated rank 1 projective R-modules P. The group structure on $\mathrm{Pic}(R)$ is given by tensor product $[P][Q] = [P \otimes_R Q]$. Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes $[C]$ of semidualizing R-modules.

Fact 4.11 Remark 2.10 implies that $\mathrm{Pic}(R) \subseteq \mathfrak{S}_0(R)$. Also, there is an action of $\mathrm{Pic}(R)$ on $\mathfrak{S}_0(R)$ given by $[P][C] = [P \otimes_R C]$; see [8].

Definition 4.12 The equivalence relation defined by the $\mathrm{Pic}(R)$-action on $\mathfrak{S}_0(R)$ is denoted $\approx$: given $[B], [C] \in \mathfrak{S}_0(R)$ we have $[B] \approx [C]$ if there is an element $[P] \in \mathrm{Pic}(R)$ such that $C \cong P \otimes_R B$. Write $B \approx C$ when $[B] \approx [C]$.

Fact 4.13 For semidualizing R-modules B, C, one has $B \approx C$ if and only if $B_\mathfrak{m} \cong C_\mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subset R$, by [8, Proposition 5.1] and [13, Chapter 2].

The next result is routine, and the corollary follows using Example 3.7.

Proposition 4.14 *Assume that $B \approx C$. Then one has $\mathcal{P}_B(R) = \mathcal{P}_C(R)$ and $\mathcal{F}_B(R) = \mathcal{F}_C(R)$. Thus, for all i, there are natural isomorphisms*

$$\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M,N) \cong \mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(M,N) \quad \text{and} \quad \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M,N) \cong \mathrm{Tor}_i^{\mathcal{P}_B\mathcal{M}}(M,N).$$

Corollary 4.15 *Let $[C] \in \mathrm{Pic}(R)$. For each i there are isomorphisms*

$$\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M,N) \cong \mathrm{Tor}_i^{R}(M,N) \quad \text{and} \quad \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M,N) \cong \mathrm{Tor}_i^{R}(M,N).$$

Corollary 4.16 *The following conditions are equivalent:*

(i) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(X,Y) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X,Y)$ *for all $i \geqslant 0$ and for all R-modules X, Y.*
(ii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B,R/\mathfrak{m}) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B,R/\mathfrak{m})$ *for $i = 0$, for some $i \geqslant 1$, and for all $\mathfrak{m} \in$ m-Spec(R).*
(iii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(R/\mathfrak{m},C) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(R/\mathfrak{m},C)$ *for $i = 0$, for some $i \geqslant 1$, and for all $\mathfrak{m} \in$ m-Spec(R).*
(iv) *$B \approx R \approx C$, i.e., $[B], [C] \in \mathrm{Pic}(R)$.*

Proof As in the proof of Theorem B, we verify the implications (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i). The implication (iv) $\Longrightarrow$ (i) is from Corollary 4.15.

(ii) $\Longrightarrow$ (iv) Assume that $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(R/\mathfrak{m}, C) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(R/\mathfrak{m}, C)$ for all $i \geqslant 0$, for all $\mathfrak{m} \in$ m-Spec(R). Corollary 3.12 then implies that

$$\begin{aligned}\mathrm{Tor}_i^{\mathcal{F}_{B_\mathfrak{m}}\mathcal{M}}(R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}, C_\mathfrak{m}) &\cong \mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(R/\mathfrak{m}, C)_\mathfrak{m} \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(R/\mathfrak{m}, C)_\mathfrak{m} \\ &\cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_{C_\mathfrak{m}}}(R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}, C_\mathfrak{m}).\end{aligned}$$

Theorem B implies that $B_\mathfrak{m} \cong R_\mathfrak{m} \cong C_\mathfrak{m}$ for all $\mathfrak{m}$, so $B \approx R \approx C$ by Fact 4.13. □

The next two results are proved similarly.

Corollary 4.17 *The following conditions are equivalent:*

(i) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(X, Y) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(Y, X)$ *for all* $i \geqslant 0$ *and for all R-modules* X, Y.
(ii) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X, Y) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(Y, X)$ *for all* $i \geqslant 0$ *and for all R-modules* X, Y.
(iii) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, R/\mathfrak{m}) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(R/\mathfrak{m}, C)$ *for* $i = 0$, *for some* $i \geqslant 1$, *and for all* $\mathfrak{m} \in$ m-Spec(R).
(iv) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(C, R/\mathfrak{m}) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(R/\mathfrak{m}, C)$ *for* $i = 0$, *for some* $i \geqslant 1$, *and for all* $\mathfrak{m} \in$ m-Spec(R).
(v) $C \approx R$.

Corollary 4.18 *The following conditions are equivalent:*

(i) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(X, Y) \cong \mathrm{Tor}_i^{R}(X, Y)$ *for all* $i \geqslant 0$ *and for all R-modules* X, Y.
(ii) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X, Y) \cong \mathrm{Tor}_i^{R}(X, Y)$ *for all* $i \geqslant 0$ *and for all R-modules* X, Y.
(iii) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, R/\mathfrak{m}) \cong \mathrm{Tor}_i^{R}(C, R/\mathfrak{m})$ *for some* $i \geqslant 1$, *for all* $\mathfrak{m} \in$ m-Spec(R).
(iv) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(R/\mathfrak{m}, C) \cong \mathrm{Tor}_i^{R}(R/\mathfrak{m}, C)$ *for some* $i \geqslant 1$, *for all* $\mathfrak{m} \in$ m-Spec(R).
(v) $C \approx R$.
(vi) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(R/\mathfrak{m}, R/\mathfrak{m}) \cong \mathrm{Tor}_i^{R}(R/\mathfrak{m}, R/\mathfrak{m})$ *for some* $i \geqslant 0$, *and for all* $\mathfrak{m} \in$ m-Spec(R).
(vii) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(R/\mathfrak{m}, R/\mathfrak{m}) \cong \mathrm{Tor}_i^{R}(R/\mathfrak{m}, R/\mathfrak{m})$ *for some* $i \geqslant 0$, *and for all* $\mathfrak{m} \in$ m-Spec(R).

5 $\mathcal{F}_C$-Projective Dimension and Vanishing of Relative Homology

In this section, C is a semidualizing R-module, and M and N are R-modules.

We begin this section with a counterpoint to Example 2.7: the example says that bounded and exact resolutions are not necessarily proper, while the following lemma says that bounded and proper resolutions are exact.

Lemma 5.1 *Assume that* $\mathcal{F}_C\text{-}\mathrm{pd}_R(M) \leqslant n$ *and let* L *be a proper* $\mathcal{F}_C$*-resolution of* M *such that* $L_i = 0$ *for* $i > n$. *Then* L^+ *is exact and we have* $M \in \mathcal{B}_C(R)$.

Proof Lemma 3.1(b) implies that the complex $\mathrm{Hom}_R(C, L)$ is a proper flat resolution of $\mathrm{Hom}_R(C, M)$ such that $\mathrm{Hom}_R(C, L)_i = 0$ for $i > n$. In particular, we have $\mathrm{fd}_R(\mathrm{Hom}_R(C, M)) \leqslant n$, so $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C(R)$. Remark 2.12(a) shows that $M \in \mathcal{B}_C(R)$, so $M \cong C \otimes_R \mathrm{Hom}_R(C, M)$. The conditions $L \cong C \otimes_R \mathrm{Hom}_R(C, L)$ and $M \cong C \otimes_R \mathrm{Hom}_R(C, M)$ imply that $L^+ \cong C \otimes_R \mathrm{Hom}_R(C, L)^+$, so Lemma 2.13(a) implies that L^+ is exact. Since each L_i is in $\mathcal{B}_C(R)$, the condition $M \in \mathcal{B}_C(R)$ follows from Remark 2.12. □

Proposition 5.2

(a) *One has* $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant n$ *if and only if there is an exact sequence* $0 \to L_n \to \cdots \to L_0 \to M \to 0$ *such that each* $L_i \in \mathcal{F}_C(R)$.
(b) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) = \operatorname{fd}_R(\operatorname{Hom}_R(C, M))$.
(c) $\mathcal{F}_C\text{-}\operatorname{pd}_R(C \otimes_R M) = \operatorname{fd}_R(M)$.
(d) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(M)$.

Proof Argue as in [17, Corollary 2.10(a)], using Lemma 5.1. □

Corollary 5.3 *If* $\varphi\colon R \to S$ *is a flat homomorphism, then there is an inequality* $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \geqslant \mathcal{F}_{S\otimes_R C}\text{-}\operatorname{pd}_S(S \otimes_R M)$ *with equality when* φ *is faithfully flat.*

Theorem 5.4 *Given an integer* $n \geqslant 0$, *the following conditions are equivalent:*

(i) $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, -) = 0$ *for all* $i > n$;
(ii) $\operatorname{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, -) = 0$; *and*
(iii) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant n$.

Proof Let E be a faithfully injective R-module, and set $(-)^\vee = \operatorname{Hom}_R(-, E)$. Theorem 3.10 and Corollary 3.13 imply that $\operatorname{Ext}^i_{\mathcal{M}\mathcal{I}_C}(-, M^\vee) \cong \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, -)^\vee$, so Condition (i) is equivalent to the following:

(i′) $\operatorname{Ext}^i_{\mathcal{M}\mathcal{I}_C}(-, M^\vee) = 0$ for all $i > n$.

Similarly, conditions (ii) and (iii) are (respectively) equivalent to the following, by the proof of [16, Lemma 4.2(a)]:

(ii′) $\operatorname{Ext}^{n+1}_{\mathcal{M}\mathcal{I}_C}(-, M^\vee) = 0$.
(iii′) $\mathcal{I}_C\text{-}\operatorname{id}_R(M^\vee) \leqslant n$.

Finally, conditions (i′)–(iii′) are equivalent by [17, Theorem 3.2(b)]. □

Theorem 5.5 *Assume that* M *is finitely generated over* R. *Given an integer* $n \geqslant 0$, *the following conditions are equivalent:*

(i) $\operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) = 0$ *for all* $i > n$ *and for each* $\mathfrak{m} \in \text{m-Spec}(R)$;
(ii) $\operatorname{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) = 0$ *for each* $\mathfrak{m} \in \text{m-Spec}(R)$;
(iii) $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) \leqslant n$; *and*
(iv) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant n$.

In particular, one has $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) = \mathcal{P}_C\text{-}\operatorname{pd}_R(M)$.

Proof The implication (i) $\Longrightarrow$ (ii) is trivial, and the implications (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) are from Proposition 5.2(d) and Theorem 5.4.

(ii) $\Longrightarrow$ (iii) Assume that $\operatorname{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) = 0$ for each $\mathfrak{m} \in \text{m-Spec}(R)$. The module $C_\mathfrak{m}$ is a semidualizing for $R_\mathfrak{m}$, so it is non-zero and finitely generated and

$$C \otimes_R R/\mathfrak{m} \cong C/\mathfrak{m}C \cong C_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}C_\mathfrak{m} \cong (R/\mathfrak{m})^{\beta_0(\mathfrak{m};C)} \tag{5.5.1}$$

where $\beta_0(\mathfrak{m}; C) \neq 0$. The second step in the next sequence is by Theorem 3.10:

$$\begin{aligned} 0 &= \operatorname{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) \\ &\cong \operatorname{Tor}_{n+1}^{R}(\operatorname{Hom}_R(C, M), (C \otimes_R R/\mathfrak{m})) \\ &\cong \operatorname{Tor}_{n+1}^{R}(\operatorname{Hom}_R(C, M), R/\mathfrak{m})^{\beta_0(\mathfrak{m};C)}. \end{aligned}$$

The third step is by Eq. 5.5.1. Since $\beta_0(\mathfrak{m}; C) \neq 0$, we conclude that

$$\operatorname{Tor}_{n+1}^{R}(\operatorname{Hom}_R(C, M), R/\mathfrak{m}) = 0$$

for each $\mathfrak{m}$. Thus, Proposition 5.2(b) explains the first step in the next display

$$\mathcal{F}_C\text{-}\operatorname{pd}_R(M) = \operatorname{fd}_R(\operatorname{Hom}_R(C, M)) = \operatorname{pd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$$

and the other steps follow from the fact that $\operatorname{Hom}_R(C, M)$ is finitely generated. □

Corollary 5.6 *Given a set $\{N_j\}_{j\in J}$ of R-modules, one has*

$$\mathcal{F}_C\text{-}\operatorname{pd}_R(\textstyle\coprod_j N_j) = \sup\{\mathcal{F}_C\text{-}\operatorname{pd}_R(N_j) \mid j \in J\}.$$

Proof Apply Theorem 3.10, Corollary 3.11(a), and Theorem 5.4. □

Next, we present a two-of-three result.

Corollary 5.7 *Given an exact sequence $\mathbb{M} = (0 \to M_1 \to M_2 \to M_3 \to 0)$ of R-module homomorphisms, one has*

$$\begin{aligned} \mathcal{F}_C\text{-}\operatorname{pd}_R(M_2) &\leqslant \sup\{\mathcal{F}_C\text{-}\operatorname{pd}_R(M_1), \mathcal{F}_C\text{-}\operatorname{pd}_R(M_3)\} \\ \mathcal{F}_C\text{-}\operatorname{pd}_R(M_1) &\leqslant \sup\{\mathcal{F}_C\text{-}\operatorname{pd}_R(M_2), \mathcal{F}_C\text{-}\operatorname{pd}_R(M_3) - 1\} \\ \mathcal{F}_C\text{-}\operatorname{pd}_R(M_3) &\leqslant \sup\{\mathcal{F}_C\text{-}\operatorname{pd}_R(M_2), \mathcal{F}_C\text{-}\operatorname{pd}_R(M_1) + 1\}. \end{aligned}$$

In particular, if two of the M_i have finite $\mathcal{F}_C$- pd*, then so does the third one.*

Proof For each inequality, one can assume without loss of generality that two of the modules in the sequence have finite $\mathcal{F}_C$-projective dimension. In particular, these modules are in $\mathcal{B}_C(R)$, so Remark 2.12 implies that all three modules are in $\mathcal{B}_C(R)$. In particular, we have $\operatorname{Ext}_R^1(C, M_1) = 0$, so Lemma 3.2 implies that $\mathbb{M}$ is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact. The desired conclusion follows from Theorem 5.4, using the long exact sequence from Lemma 3.8(a) and Theorem 3.10. □

We conclude with Theorem C from the introduction.

Definition 5.8 An R-submodule $M' \subseteq M$ is *pure* if for every R-module N the induced map $N \otimes_R M' \to N \otimes_R M$ is 1-1, i.e., if for each finitely generated R-module N, the map $\operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(N, M'')$ is onto; see [19, Proposition 3].

Proof of Theorem C Assume without loss of generality that $\mathcal{F}_C\text{-}\mathrm{pd}_R(M) = n < \infty$. It follows that $M \in \mathcal{B}_C(R)$, and from [11, Proposition 2.4(a) and Theorem 3.1] we know that M' and $M'' := M/M'$ are in $\mathcal{B}_C(R)$. In particular, the sequence

$$0 \to M' \to M \to M'' \to 0 \tag{5.9.1}$$

is $\mathrm{Hom}_R(\mathcal{P}_C, -)$-exact by Lemma 3.2. By adjointness $\mathrm{Hom}_R(N, \mathrm{Hom}_R(C, -)) \cong \mathrm{Hom}_R(N \otimes_R C, -)$, the submodule $\mathrm{Hom}_R(C, M') \subseteq \mathrm{Hom}_R(C, M)$ is pure.

We prove that $\mathcal{F}_C\text{-}\mathrm{pd}_R(M') \leqslant n$. It suffices by Theorem 5.4 to show that $\mathrm{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M', -) = 0$. Let G be an R-module. Theorems 3.10 and 5.4 imply that $\mathrm{Tor}_{n+1}^{R}(\mathrm{Hom}_R(C, M), C \otimes_R G) \cong \mathrm{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, G) = 0$. Let

$$0 \to K_{n+1} \to P_n \to \cdots \to P_0 \to C \otimes_R G \to 0,$$

be a truncation of a projective resolution of $C \otimes_R G$. In the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & K_{n+1} \otimes_R \mathrm{Hom}_R(C, M') & \longrightarrow & P_n \otimes_R \mathrm{Hom}_R(C, M') \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_{n+1} \otimes_R \mathrm{Hom}_R(C, M) & \longrightarrow & P_n \otimes_R \mathrm{Hom}_R(C, M)
\end{array}$$

the bottom row is exact, as $\mathrm{Tor}_{n+1}^{R}(\mathrm{Hom}_R(C, M), C \otimes_R G) = 0$. The vertical arrows are injective, since $\mathrm{Hom}_R(C, M') \subseteq \mathrm{Hom}_R(C, M)$ is pure. Hence, the top row of the diagram is exact, so we have $\mathrm{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M', G) \cong \mathrm{Tor}_{n+1}^{R}(\mathrm{Hom}_R(C, M'), C \otimes_R G) = 0$ by Theorem 3.10, as desired.

To complete the proof, note that $\mathcal{F}_C\text{-}\mathrm{pd}_R(M'') - 1 \leqslant n$ by Corollary 5.7. □

Example 5.9 Let M', M'' be R-modules with $\mathcal{F}_C\text{-}\mathrm{pd}_R(M') < \mathcal{F}_C\text{-}\mathrm{pd}_R(M'') < \infty$. The split inclusion $M' \subseteq M' \oplus M''$ is pure, but

$$\mathcal{F}_C\text{-}\mathrm{pd}_R(M' \oplus M'') = \mathcal{F}_C\text{-}\mathrm{pd}_R(M'') > \sup\{\mathcal{F}_C\text{-}\mathrm{pd}_R(M'), \mathcal{F}_C\text{-}\mathrm{pd}_R(M'') - 1\}$$

by Proposition 5.6. Thus, we can have strict inequality in Theorem C.

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