Illinois Journal of Mathematics
Volume 51, Number 1, Spring 2007, Pages 255–285
S 0019-2082

SEMIDUALIZING MODULES AND THE DIVISOR CLASS GROUP

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Dedicated to Phillip Griffith on the occasion of his retirement

Abstract. Among the finitely generated modules over a Noetherian ring R, the semidualizing modules have been singled out due to their particularly nice duality properties. When R is a normal domain, we exhibit a natural inclusion of the set of isomorphism classes of semidualizing R-modules into the divisor class group of R. After a description of the basic properties of this inclusion, it is employed to investigate the structure of the set of isomorphism classes of semidualizing R-modules. In particular, this set is described completely for determinantal rings over normal domains.

1. Introduction

Semidualizing modules arise naturally in the investigations of various duality theories in commutative algebra. One instance of this is Grothendieck and Hartshorne's local duality wherein a dualizing module, or more generally a dualizing complex, is employed to study local cohomology [30], [31]. Another instance is Auslander and Bridger's methodical study of duality properties with respect to a rank 1 free module that gives rise to the Gorenstein dimension [3], [4]. A free module of rank 1 and a dualizing module are both examples of semidualizing modules.

Let R be a Noetherian ring. A finitely generated R-module C is *semidualizing* if the natural homothety map $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}^i_R(C,C)=0$ for each integer $i>0$. The study of such modules in the abstract was initiated by Foxby [21] and Golod [28], where they were called "suitable" modules, and has been continued recently by others; see, for example, [12], [24], [26], [27].

Received November 17, 2005; received in final form December 20, 2006.
2000 *Mathematics Subject Classification.* Primary 13C05, 13C13, 13C20. Secondary 13C40, 13D05, 13D25.
This research was conducted in part while the author was an NSF Mathematical Sciences Postdoctoral Research Fellow and a visitor at the University of Nebraska-Lincoln.

The semidualizing modules and more generally the semidualizing *complexes* are useful, for example, in identifying local homomorphisms of finite Gorenstein dimension with particularly nice properties as in [7], [22]. This utility, along with our desire to expand upon it, motivates our investigation of the basic properties of such modules and the structure of the entire set of isomorphism classes of semidualizing R-modules, which we denote $\mathfrak{S}_0(R)$. Surprisingly little is known about his set. For instance, researchers in this subject have been grappling with the following open question for several years now; see [13, (1)] for recent progress.

QUESTION 1.1. When R is local, must $\mathfrak{S}_0(R)$ be finite?

This work is part of a research effort focused on determining the overall structure of $\mathfrak{S}_0(R)$. Much of the ground work for this effort, motivated by [12], [26], is found in [23]. Initial evidence of the richness of the structure of this set is found in the fact that it admits an ordering described in terms of a reflexivity relation; see 2.4. Further structure is uncovered in [24], where numerical data from this ordering is used to build a nontrivial metric on $\mathfrak{S}_0(R)$. While the existence of a metric does not itself provide answers to any of the open questions about the structure of $\mathfrak{S}_0(R)$, it represents a new perspective from which to view this set. This perspective has proved particularly useful for identifying questions that we would not have otherwise thought to ask. For instance, our investigation into the nontriviality of the metric led us to the fact that, when R is Cohen-Macaulay and $\mathfrak{S}_0(R)$ is nontrivial, there exist elements of $\mathfrak{S}_0(R)$ that are incomparable under the reflexivity ordering; see [24, (3.5)].

In the current paper, we forward another new perspective from which to investigate the set $\mathfrak{S}_0(R)$. It is motivated by Bruns' work [9], wherein the divisor class group is used to describe the dualizing module of certain Cohen-Macaulay normal domains. Accordingly, when R is a normal domain, we exhibit a natural inclusion $\mathfrak{S}_0(R) \hookrightarrow \mathrm{Cl}(R)$ that behaves well with respect to standard operations. This inclusion allows us to exploit the known behavior of $\mathrm{Cl}(R)$ to gain insight into the structure of $\mathfrak{S}_0(R)$. For instance, if $\mathrm{Cl}(R)$ is finite, then $\mathfrak{S}_0(R)$ is also finite. The basic results from this analysis are presented in Section 3.

Section 4 contains the meat of this investigation and demonstrates the power of this new perspective. It consists of analyses showing how the divisor class group can be used to give a complete description of the set of semidualizing modules for certain classes of rings. We recount here three such situations, focusing our attention on the finiteness and size of $\mathfrak{S}_0(R)$. First, taking our cues from [9], we describe the semidualizing modules over a determinantal ring in Theorem 4.5.

THEOREM 1.2. *Let A be a normal domain and m, n, r nonnegative integers such that $r < \min\{m, n\}$. With $\mathbf{X} = \{X_{ij}\}$ an $m \times n$ matrix of variables, set*

$R = A[\mathbf{X}]/I_{r+1}(\mathbf{X})$, where $I_{r+1}(\mathbf{X})$ is the ideal generated by the minors of $\mathbf{X}$ of size $r+1$. The set $\mathfrak{S}_0(R)$ is finite if and only if $\mathfrak{S}_0(A)$ is so. More specifically, one has the following cases.

(a) *If $r = 0$ or $m = n$, then there is a bijection $\mathfrak{S}_0(R) \approx \mathfrak{S}_0(A)$.*
(b) *If $r > 0$ and $m \neq n$, then there is a bijection $\mathfrak{S}_0(R) \approx \mathfrak{S}_0(A) \times \{0, 1\}$.*

When A is a graded Cohen-Macaulay (super-)normal domain with A_0 local (and complete), this result extends to the localization (and to the completion) of R at its graded maximal ideal; see Corollaries 4.7 and 4.8.

The next two results demonstrate how this technique yields information about rings that are themselves not normal domains; they are contained in Corollary 4.11. Theorem 1.4 is new even when B is a field.

THEOREM 1.3. *Let $A = \coprod_{i \geq 0} A_i$ be a graded super-normal domain with A_0 local and complete. Let $\mathfrak{n}$ be the graded maximal ideal of A and $\widehat{A}$ the $\mathfrak{n}$-adic completion of A. Let $\mathbf{y} = y_1, \ldots, y_q \in \mathfrak{n}A_\mathfrak{n}$ be an $A_\mathfrak{n}$-sequence and fix an integer $m \geq 1$. There are bijections*

$$\mathfrak{S}_0(\widehat{A}/(\mathbf{y})^m) \approx \mathfrak{S}_0(A_\mathfrak{n}/(\mathbf{y})^m) \approx \begin{cases} \mathfrak{S}_0(A_\mathfrak{n}) & \text{if } m = 1 \text{ or } q = 1 \\ \mathfrak{S}_0(A_\mathfrak{n}) \times \{0, 1\} & \text{if } m, q > 1. \end{cases}$$

THEOREM 1.4. *With A as in Theorem 1.3, let B denote either $A_\mathfrak{n}$ or $\widehat{A}$, and let t be a positive integer. For $l = 1, \ldots, t$ fix a positive integer q_l and set*

$$S = (B \ltimes B^{q_1}) \otimes_B \cdots \otimes_B (B \ltimes B^{q_t}).$$

If s is the number of indices l with $q_l > 1$, then there is a bijection

$$\mathfrak{S}_0(B) \times \{0, 1\}^s \approx \mathfrak{S}_0(S).$$

The statements of our main results are module-theoretic in nature. However, we often employ tools from the derived category. We include a summary of the relevant notions in Section 2 along with basic facts about semidualizing modules and the divisor class group.

2. Background

In this paper, the term "ring" is used for a commutative Noetherian ring with identity, and "module" is used for a unital module. Let R be a ring.

2.1. An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

with $\partial^X_i \partial^X_{i+1} = 0$ for each i. We work occasionally in the derived category $\mathsf{D}(R)$ whose objects are the R-complexes; references on the subject include [25], [30], [36], [37]. The category of R-modules $\mathrm{Mod}(R)$ is naturally identified with the full subcategory of $\mathsf{D}(R)$ whose objects are the complexes homologically concentrated in degree 0. For R-complexes X and Y the left derived tensor product complex is denoted $X \otimes^{\mathbf{L}}_R Y$ and the right derived homomorphism complex is $\mathbf{R}\operatorname{Hom}_R(X,Y)$. For an integer n, the nth *shift* or *suspension* of X is denoted $\mathsf{\Sigma}^n X$, where $(\mathsf{\Sigma}^n X)_i = X_{i-n}$ and $\partial_i^{\mathsf{\Sigma}^n X} = (-1)^n \partial_{i-n}^X$. The symbol "$\simeq$" indicates an isomorphism in $\mathsf{D}(R)$, and "$\sim$" is isomorphism up to shift.

A complex X is *homologically finite*, respectively *homologically degreewise finite*, if its total homology module $\mathrm{H}(X)$, respectively each individual homology module $\mathrm{H}_i(X)$, is a finite R-module. The *infimum*, *supremum*, and *amplitude* of X are

$$\begin{aligned}
\inf(X) &= \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\},\\
\sup(X) &= \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\},\\
\operatorname{amp}(X) &= \sup(X) - \inf(X),
\end{aligned}$$

respectively, with the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. When R is local with residue field k, the *depth* of a homologically finite complex X is

$$\operatorname{depth}_R X = -\sup(\mathbf{R}\operatorname{Hom}_R(k,X)).$$

The *Bass series* of X is the formal Laurent series $I_R^X(t) = \sum_i \mu_R^i(X) t^i$, where

$$\mu_R^i(X) = \operatorname{rank}_k \mathrm{H}_{-i}(\mathbf{R}\operatorname{Hom}_R(k,X))$$

for each integer i. From Foxby [20, (13.11)] the quantity $\operatorname{id}_R X$ is finite if and only if $I_R^X(t)$ is a Laurent polynomial.

2.2. Associated to a complex K is a natural homothety morphism

$$\chi_K^R \colon R \to \mathbf{R}\operatorname{Hom}_R(K,K).$$

When K is homologically finite, it is *semidualizing* if χ_K^R is an isomorphism. A complex D is *dualizing* if it is semidualizing and has finite injective dimension. The set of shift-isomorphism classes of semidualizing R-complexes is denoted $\mathfrak{S}(R)$, and the class of a semidualizing complex K in $\mathfrak{S}(R)$ is denoted $[K]_R$ or simply $[K]$ when there is no danger of confusion. The ring R is $\mathfrak{S}$*-finite* if $\mathfrak{S}(R)$ is a finite set.

A finitely generated R-module C is *semidualizing* if the natural homothety map $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}_R^{\geqslant 1}(C,C) = 0$. The module R is semidualizing. When R is Cohen-Macaulay, a *dualizing module* (or *canonical module*) is a semidualizing module of finite injective dimension. The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$. The identification of $\operatorname{Mod}(R)$ with a subcategory of $\mathsf{D}(R)$ provides a natural inclusion $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$, and we identify $\mathfrak{S}_0(R)$ with its image in $\mathfrak{S}(R)$. In particular, the class of a semidualizing module C in $\mathfrak{S}_0(R)$ is denoted $[C]_R$ or $[C]$. The ring R is $\mathfrak{S}_0$*-finite* if $\mathfrak{S}_0(R)$ is a finite set.

Some of our favorite ring theoretic properties have characterizations in terms of semidualizing objects. If R is Cohen-Macaulay local, then $\mathfrak{S}(R) = \mathfrak{S}_0(R)$. If R is Gorenstein local, then $\mathfrak{S}(R) = \{[R]\}$. The converses hold when R admits a dualizing complex; see Christensen [12, (3.7),(8.6)].

2.3. Let K be a semidualizing complex. A homologically finite complex X is *K-reflexive* if $\mathbf{R}\operatorname{Hom}_R(X, K)$ is homologically bounded and the natural biduality morphism

$$\delta_X^K : X \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X, K), K)$$

is an isomorphism. For instance, the complexes R and K are both K-reflexive. When $\dim(R)$ is finite, the complex K is dualizing if and only if every homologically finite R-complex is K-reflexive. The G_K*-dimension* of X is

$$\mathrm{G}_K\text{-}\dim_R X = \begin{cases} \inf K - \inf \mathbf{R}\operatorname{Hom}_R(X, K) & \text{when } X \text{ is } K\text{-reflexive} \\ \infty & \text{otherwise.} \end{cases}$$

If R is local and X is K-reflexive, then the AB formula [12, (3.14)] reads

$$\mathrm{G}_K\text{-}\dim_R X = \operatorname{depth} R - \operatorname{depth}_R X.$$

When C is a semidualizing module, the G_C-dimension of a finitely generated R-module M can be described in terms of resolutions. We first describe the modules used in the resolutions. A finitely generated R-module G is *totally C-reflexive* if the natural biduality map $G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G, C), C)$ is bijective, and $\operatorname{Ext}_R^{\geqslant 1}(G, C) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(G, C), C)$. A finitely generated R-module M then has finite G_C-dimension if and only if it admits a resolution

$$0 \to G_g \to \cdots \to G_0 \to M \to 0$$

with each G_i totally C-reflexive; the G_C-dimension of M is then the minimum integer g for which M admits such a resolution.

2.4. The above notion of reflexivity gives rise to orderings on the sets $\mathfrak{S}_0(R)$ and $\mathfrak{S}(R)$: write $[C] \trianglelefteq [C']$ whenever C' is C-reflexive. This ordering is trivially reflexive: $[C] \trianglelefteq [C]$. Also, when R is local, the ordering is antisymmetric: if $[C] \trianglelefteq [C']$ and $[C'] \trianglelefteq [C]$, then $[C] = [C']$; see [2, (5.3)]. The question of the transitivity of this ordering has been of interest in this area for some time:

QUESTION 2.5. If $[C] \trianglelefteq [C']$ and $[C'] \trianglelefteq [C'']$, then must one have $[C] \trianglelefteq [C'']$?

2.6. For $1, \ldots, n$ let U_i be a set with a relation $\trianglelefteq_i$. The Cartesian product $U_1 \times \cdots \times U_n$ is endowed with the *product relation*: $(u_1, \ldots, u_n) \trianglelefteq (u'_1, \ldots, u'_n)$ when $u_i \trianglelefteq_i u'_i$ for each $i = 1, \ldots, n$. It follows immediately from the definition that $\trianglelefteq$ is transitive if and only if each $\trianglelefteq_i$ is transitive. A map $\alpha : U_1 \to U_2$ is *order-respecting* when $u \trianglelefteq_1 u'$ implies $\alpha(u) \trianglelefteq_2 \alpha(u')$, and it is *perfectly order-respecting* when the converse also holds. Observe that, when α is a perfectly order-respecting bijection, the relations $\trianglelefteq_1, \trianglelefteq_2$ are simultaneously transitive. The symbol $\approx$ indicates a perfectly order-respecting bijection.

We pose one final question in this section, motivated by the following well-known equality of Hilbert-Samuel multiplicities: If R is a local Cohen-Macaulay ring with dualizing module ω, then $e(\omega) = e(R)$.

QUESTION 2.7. Let $(R, \mathfrak{m})$ be a local ring and I an $\mathfrak{m}$-primary ideal. If C is a semidualizing R-module, must there be an equality $e(I, C) = e(I, R)$?

Consult Matsumura [34, §14] for the basics of the Hilbert-Samuel multiplicity. We answer Question 2.7 in the affirmative for several classes of rings in Corollaries 3.14 and 4.11. To do so, we need the following lemma, which addresses this question when R is generically Gorenstein.

LEMMA 2.8. *Let $(R, \mathfrak{m})$ be a local ring and C a semidualizing R-module.*

(a) *If $R_\mathfrak{p}$ is Gorenstein for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim(R/\mathfrak{p}) = \dim(R)$, then $e(J, C) = e(J, R)$ for each $\mathfrak{m}$-primary ideal J.*

(b) *Assume that R is equidimensional and, for each $\mathfrak{p} \in \operatorname{Min}(R)$, the rings $R_\mathfrak{p}$ and $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R \widehat{R}$ are Gorenstein. If $\mathbf{y} \in \mathfrak{m}$ is an R-sequence and $R' = R/\mathbf{y}$, then $e(J, C \otimes R') = e(J, R')$ for each $\mathfrak{m}R'$-primary ideal J.*

Proof. (a) Set $\operatorname{Minh}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim(R/\mathfrak{p}) = \dim(R)\}$. For each $\mathfrak{p} \in \operatorname{Minh}(R)$, there is an isomorphism $C_\mathfrak{p} \cong R_\mathfrak{p}$ by [12, (8.6)] since $R_\mathfrak{p}$ is Gorenstein. This provides the second equality in the following sequence

$$e(J, C) = \sum_{\mathfrak{p} \in \operatorname{Minh}(R)} \operatorname{length}(C_\mathfrak{p}) e(J, R/\mathfrak{p}) = \sum_{\mathfrak{p} \in \operatorname{Minh}(R)} \operatorname{length}(R_\mathfrak{p}) e(J, R/\mathfrak{p}) = e(J, R),$$

while the others are the additivity formulas for multiplicies [10, (4.7.t)].

(b) Recall the following fact: If $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is a flat local homomorphism such that $\mathfrak{m}_B = \mathfrak{m}_A B$, and if M is a finite A module, then for each $\mathfrak{m}_A$-primary ideal I there is an equality of Hilbert functions

$$\operatorname{length}_A(M/I^n M) = \operatorname{length}_B((M \otimes_A B)/(IB)^n (M \otimes_A B)),$$

which yields an equality of multiplicities; see [32, (2.3)]:

$$e(I, M) = e(IB, M \otimes_A B). \tag{2.8.1}$$

Next, from [29, (0.10.3.1)] one has a flat local homomorphism $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ such that S has infinite residue field and $\mathfrak{n} = \mathfrak{m}S$. Since S is flat over R, the sequence $\mathbf{y}$ is S-regular. Set $S' = S/(\mathbf{y})S$ and let $\tau\colon S \to S'$ denote the natural surjection. Note that the induced map $R' \to S'$ is flat and local and that the maximal ideal of R' extends to that of S'. It follows that the ideal JS' is $\mathfrak{m}S'$-primary.

For every $\mathfrak{q} \in \operatorname{Minh}(S)$, the local ring $S_{\mathfrak{q}}$ is Gorenstein. To see this, fix $\mathfrak{q} \in \operatorname{Minh}(S)$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. Since φ is faithfully flat, the going-down theorem implies $\operatorname{ht}(\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{q}) = 0$, and so $\mathfrak{p} \in \operatorname{Min}(R)$. By assumption, the rings $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_R \widehat{R}$ and $S/\mathfrak{m}S$ are Gorenstein, and it follows from [6, Main Theorem] that the fibre $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_R S$ is Gorenstein. Thus, the induced map $\varphi_{\mathfrak{q}} \colon R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is flat and local with Gorenstein source and Gorenstein closed fibre. This implies that $S_{\mathfrak{q}}$ is Gorenstein.

The ring S' has a parameter ideal $\mathbf{x} = (x_1, \ldots, x_r)S' \subseteq JS'$ such that

$$(2.8.2) \quad e(JS', C \otimes_R S') = e(\mathbf{x}, C \otimes_R S') \qquad \text{and} \qquad e(JS', S') = e(\mathbf{x}, S');$$

see [10, (4.6.10)]. Fix $\widetilde{x}_1, \ldots, \widetilde{x}_r \in S$ such that $\tau(\widetilde{x}_i) = x_i$ for each $i = 1, \ldots, r$, and set $\widetilde{J} = (\widetilde{\mathbf{x}}, \mathbf{y})S$. The third equality in the next sequence is from part (a), the first one comes from the isomorphism $(C \otimes_R S) \otimes_S S' \cong C \otimes_R S'$, and the second and fourth hold by [34, (14.11)].

$$e(\mathbf{x}, C \otimes_R S') = e(\mathbf{x}, (C \otimes_R S) \otimes_S S') = e(\widetilde{J}, C \otimes_R S) = e(\widetilde{J}, S) = e(\mathbf{x}, S').$$

This yields the third equality below, while the remaining ones are from equations (2.8.1) and (2.8.2).

$$\begin{aligned} e(J, C \otimes_R R') &= e(JS', C \otimes_R S') = e(\mathbf{x}, C \otimes_R S') \\ &= e(\mathbf{x}, S') = e(JS', S') = e(J, R'). \end{aligned} \qquad \square$$

2.9. Let $\varphi \colon R \to S$ be a ring homomorphism. The *flat dimension* of φ is $\operatorname{fd}(\varphi) = \operatorname{fd}_R(S)$. Assume that φ is surjective and $\operatorname{fd}(\varphi) < \infty$. The map φ is *Cohen-Macaulay* of grade d if S is a perfect R-module of grade d. The map is *Gorenstein* of grade d if it is Cohen-Macaulay of grade d and, for each prime ideal $\mathfrak{q} \subset S$, the $S_{\mathfrak{q}}$-module $\operatorname{Ext}^d_R(S, R)_{\mathfrak{q}}$ is cyclic.

2.10. This section concludes with the definition of the divisor class group of a normal domain R with field of fractions Q. Let $(-)^{\star}$ denote the functor $\operatorname{Hom}_R(-, R)$. An R-module M is *reflexive*[1] if it is finitely generated and the natural biduality map $b^S_M \colon M \to M^{\star\star}$ is bijective. The *divisor class group* of R, denoted $\operatorname{Cl}(R)$, is the set of isomorphism classes of reflexive R-modules of rank 1. The isomorphism class of a reflexive module M is denoted $[M]_R$ or $[M]$ when there is no risk of confusion. The set $\operatorname{Cl}(R)$ admits an Abelian group structure: when M, N are rank 1 reflexive modules, then

$$[M] + [N] = [(M \otimes_R N)^{\star\star}], \qquad [M] - [N] = [\operatorname{Hom}_R(N, M)].$$

If $\mathfrak{a}, \mathfrak{b}$ are ideals with $\mathfrak{a} \cong M$ and $\mathfrak{b} \cong N$, then $[M] + [N] = [\mathfrak{a}] + [\mathfrak{b}] = [(\mathfrak{a}\mathfrak{b})^{\star\star}]$.

The fact that the operations described above make $\operatorname{Cl}(R)$ into an Abelian group seems to be part of the folklore of this subject; see [33, Sec. 0]. We sketch a proof of this fact below, which also indicates why this definition is equivalent to other formulations that may be more familiar to some readers.

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[1]Not to be confused with "K-reflexive" or "totally C-reflexive".

A *fractionary* ideal of R is a nonzero finitely generated R-submodule of Q. From the proof of [19, (2.2.iv)], one has $\operatorname{Hom}_R(\mathfrak{a},\mathfrak{b}) \cong \mathfrak{a} :_Q \mathfrak{b}$ for every pair of fractionary ideals $\mathfrak{a},\mathfrak{b}$. Let $D(R)$ denote the set of reflexive fractionary ideals of R, and let $P(R)$ denote the set of principal fractionary ideals of R. As R is a normal domain, we learn from [19, (3.4)] that $D(R)$ is an Abelian group via the operations

$$\mathfrak{a}+\mathfrak{b} = R :_Q (R :_Q (\mathfrak{a}\mathfrak{b})), \qquad \mathfrak{a}-\mathfrak{b} = \mathfrak{a} :_Q \mathfrak{b}$$

with identity R. One checks that $P(R)$ is a subgroup of $D(R)$ and that $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ in $D(R)/P(R)$ if and only if $\mathfrak{a} = a\mathfrak{b}$ for some $a \in Q^\times$ if and only if $\mathfrak{a} \cong \mathfrak{b}$.

Let $Z(R)$ denote the set of all height 1 prime ideals $\mathfrak{p} \subset R$. For each $\mathfrak{p} \in Z(R)$, the localization $R_\mathfrak{p}$ is a discrete valuation ring because R is a normal domain, and we let $v_\mathfrak{p} \colon Q^\times \to \mathbb{Z}$ denote the associated valuation. For each $\mathfrak{a} \in D(R)$ and $\mathfrak{p} \in Z(R)$, set $v_\mathfrak{p}(\mathfrak{a}) = \inf\{v_\mathfrak{p}(a) \mid a \in \mathfrak{a}\}$. Set $\operatorname{Div}(R) = \oplus_{\mathfrak{p}\in Z(R)}\mathbb{Z}\cdot[R/\mathfrak{p}]$ and consider the function $\operatorname{div}\colon D(R) \to \operatorname{Div}(R)$ given by $\operatorname{div}(\mathfrak{a}) = (v_\mathfrak{p}(\mathfrak{a})[R/\mathfrak{p}])_{\mathfrak{p}\in Z(R)}$. This is an Abelian group isomorphism by [19, (5.9)] and we set $\operatorname{Prin}(R) = \operatorname{div}(P(R)) \subseteq \operatorname{Div}(R)$. It is routine to verify that div induces a group isomorphism $D(R)/P(R) \cong \operatorname{Div}(R)/\operatorname{Prin}(R)$.

Many readers will undoubtedly recognize $\operatorname{Div}(R)/\operatorname{Prin}(R)$ as the definition of the divisor class group from [8]. To see that this is equivalent to the definition formulated above (and that our formulation yields an Abelian group) it suffices to construct a bijection $f\colon D(R)/P(R) \to \operatorname{Cl}(R)$ such that

$$f(\overline{\mathfrak{a}}+\overline{\mathfrak{b}}) = f(\overline{\mathfrak{a}})+f(\overline{\mathfrak{b}}), \qquad f(\overline{\mathfrak{a}}-\overline{\mathfrak{b}}) = f(\overline{\mathfrak{a}})-f(\overline{\mathfrak{b}}), \qquad f(\overline{R}) = [R]$$

for each $\mathfrak{a},\mathfrak{b} \in D(R)$. Since $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $\mathfrak{a} \cong \mathfrak{b}$, one sees that the assignment $\overline{\mathfrak{a}} \mapsto [\mathfrak{b}]$ describes a well-defined injection $f\colon D(R)/P(R) \to \operatorname{Cl}(R)$. That this map is surjective follows from a standard exercise; see, for instance, [10, (1.4.18)]. For the displayed relations it suffices to show, for each $\mathfrak{a},\mathfrak{b} \in D(R)$,

$$R :_Q (R :_Q \mathfrak{a}\mathfrak{b}) \cong (\mathfrak{a}\otimes_R \mathfrak{b})^{\star\star}, \qquad \mathfrak{a} :_Q \mathfrak{b} \cong \operatorname{Hom}_R(\mathfrak{b},\mathfrak{a}).$$

(The third condition is obvious from the definition of f.) The second of these has already been discussed. For the first, it suffices to show

$$(\mathfrak{a}\mathfrak{b})^{\star\star} \cong (\mathfrak{a}\otimes_R \mathfrak{b})^{\star\star}.$$

Let K be the kernel of the multiplication map $\mu\colon \mathfrak{a}\otimes_R \mathfrak{b} \to \mathfrak{a}\mathfrak{b}$. One checks that the map $\mu\otimes_R K$ is an isomorphism, and so K is torsion. It follows that $K^\star = 0$ and so $(\mathfrak{a}\mathfrak{b})^\star \cong (\mathfrak{a}\otimes_R \mathfrak{b})^\star$ and $(\mathfrak{a}\mathfrak{b})^{\star\star} \cong (\mathfrak{a}\otimes_R \mathfrak{b})^{\star\star}$.

It follows readily from the isomorphisms described above that, if $\mathfrak{p} \in Z(R)$ and $\ell > 0$, then $\ell[\mathfrak{p}] = [\mathfrak{p}^{(\ell)}]$ in $\operatorname{Cl}(R)$.

3. Semidualizing modules as divisor classes

The following proposition compares directly to the "classical" result for the dualizing module which is the prime motivation for our techniques; see, e.g., [10, (3.3.18)]. Recall that a finite R-module M has *rank* (respectively, *rank* r) if $M_{\mathfrak{p}}$ is free (respectively, free of rank r) over $R_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Ass}(R)$. Of course, condition (i) is satisfied if R is a domain. Also, using $C = R$ one sees that the implication (ii) $\Longrightarrow$ (i) fails in general; see Proposition 3.2(c).

PROPOSITION 3.1. *Let C be a semidualizing R-module, and consider the following conditions.*

(i) *For each $\mathfrak{p} \in \operatorname{Ass}(R)$, the localization $R_{\mathfrak{p}}$ is Gorenstein.*
(ii) *C has rank 1.*
(iii) *C has rank.*
(iv) *C is isomorphic to an ideal of R.*
(v) *C is isomorphic to an ideal $\mathfrak{a}$ of R with torsion quotient $R/\mathfrak{a}$.*

The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) *hold.*

Proof. (i) $\Longrightarrow$ (ii). For each associated prime $\mathfrak{p}$, the ring $R_{\mathfrak{p}}$ is Gorenstein and therefore the semidualizing $R_{\mathfrak{p}}$-module $C_{\mathfrak{p}}$ is isomorphic to $R_{\mathfrak{p}}$ by [12, (8.6)].

(ii) $\Longrightarrow$ (iii) is trivial. For the converse, since $C_{\mathfrak{p}}$ is semidualizing for $R_{\mathfrak{p}}$, it is routine to check that, if $C_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$, then it is free of rank 1.

(ii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v). It is straightforward to show that the semidualizing module C is torsion-free; in fact, $\operatorname{Ass}(R) = \operatorname{Ass}_R(C)$. The desired biimplications now follow from a standard exercise; see, for instance, [10, (1.4.18)]. $\square$

A *semidualizing ideal* is an ideal that is semidualizing as an R-module. One consequence of Proposition 3.1 is that, when R is a domain, every semidualizing module is isomorphic to a semidualizing ideal. The next result provides basic properties of such ideals; it compares directly to [10, (3.3.18)]. We restrict our attention to proper ideals as the case $\mathfrak{a} = R$ is tedious. Since a principal ideal generated by a non-zerodivisor is semidualizing, but is dualizing if and only if R is Gorenstein, the implication (iii) $\Longrightarrow$ (ii) fails in general.

PROPOSITION 3.2. *Let R be a Cohen-Macaulay ring of dimension d and $\mathfrak{a}$ a proper semidualizing ideal of R.*

(a) $\operatorname{ht}(\mathfrak{a}) = 1$ *and $R/\mathfrak{a}$ is Cohen-Macaulay of dimension $d-1$.*
(b) *The quotient $R/\mathfrak{a}$ has $G_{\mathfrak{a}}$-dimension 1 and there are isomorphisms*

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathfrak{a}) \cong \begin{cases} R/\mathfrak{a} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) *Consider the following conditions:*
  (i) *The quotient $R/\mathfrak{a}$ is a Gorenstein ring.*
  (ii) *The ideal $\mathfrak{a}$ is dualizing for R.*
  (iii) *R is generically Gorenstein.*

  The implications (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) hold.

Proof. The proof of (a) is nearly identical to that of [10, (3.3.18.b)], so we omit it here. For part (b), use the exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0 \tag{3.2.1}$$

with the fact that $\mathfrak{a}$ and R are both totally $\mathfrak{a}$-reflexive to conclude that $R/\mathfrak{a}$ has $\mathrm{G}_{\mathfrak{a}}$-dimension at most 1. In particular, $\operatorname{Ext}^i_R(R/\mathfrak{a},\mathfrak{a}) = 0$ for $i > 1$. Furthermore, since $\mathfrak{a}$ has rank, it contains an element that is both R-regular and $\mathfrak{a}$-regular, and thus $\operatorname{Hom}_R(R/\mathfrak{a},\mathfrak{a}) = 0$. Applying $\operatorname{Hom}_R(-,\mathfrak{a})$ to (3.2.1) supplies the exact sequence

$$0 \to \underbrace{\operatorname{Hom}_R(R,\mathfrak{a})}_{\mathfrak{a}} \to \underbrace{\operatorname{Hom}_R(\mathfrak{a},\mathfrak{a})}_{R} \to \operatorname{Ext}^1_R(R/\mathfrak{a},\mathfrak{a}) \to 0,$$

which yields an isomorphism $\operatorname{Ext}^1_R(R/\mathfrak{a},\mathfrak{a}) \cong R/\mathfrak{a}$ and the desired conclusions.

For part (c) we may assume that R is local. In the following sequence of formal equalities of series, the first is by [12, (1.6.7)] and the third is standard, while the second is a consequence of part (b).

$$I^{\mathfrak{a}}_R(t) = I^{\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a},\mathfrak{a})}_{R/\mathfrak{a}}(t) = I^{\Sigma^{-1}R/\mathfrak{a}}_{R/\mathfrak{a}}(t) = t \cdot I^{R/\mathfrak{a}}_{R/\mathfrak{a}}(t).$$

It follows that $\operatorname{id}_R(\mathfrak{a})$ and $\operatorname{id}_{R/\mathfrak{a}}(R/\mathfrak{a})$ are simultaneously finite. This gives the equivalence of (i) and (ii), and the implication (ii) $\Longrightarrow$ (iii) is part of [10, (3.3.18)]. $\square$

The next result simplifies the computation of $[\mathfrak{a}]+[\mathfrak{b}]$ in $\mathrm{Cl}(R)$ for certain semidualizing modules $\mathfrak{a},\mathfrak{b}$ and is a key tool for the proof of Theorem 4.2.

Proposition 3.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be semidualizing ideals such that $\mathfrak{a}\otimes_R\mathfrak{b}$ is semidualizing. The natural multiplication map $\mathfrak{a}\otimes_R\mathfrak{b} \to \mathfrak{a}\mathfrak{b}$ is an isomorphism.*

Proof. The map $\mathfrak{a}\otimes_R\mathfrak{b} \to \mathfrak{a}\mathfrak{b}$ is always surjective, so it remains to verify injectivity. Let U denote the complement in R of the union of the associated primes of R. Since $\mathfrak{a}$ and $\mathfrak{b}$ have rank, the same is true of $\mathfrak{a}\otimes_R\mathfrak{b}$. Furthermore, the fact that $\mathfrak{a}\otimes_R\mathfrak{b}$ is semidualizing implies that $\mathfrak{a}\otimes_R\mathfrak{b}$ is torsion-free. This yields the injectivity of the localization map $\mathfrak{a}\otimes_R\mathfrak{b} \to U^{-1}(\mathfrak{a}\otimes_R\mathfrak{b})$ in the following commuting diagram, where the maps (1) and (2) are given by the appropriate multiplication and the others are the natural ones.

$$\begin{tikzcd}
\mathfrak{a}\otimes_R\mathfrak{b} \arrow[r, "(1)"] \arrow[d, hook] & \mathfrak{a}\mathfrak{b} \arrow[rr, hook] & & R \arrow[r, hook] & U^{-1}R \arrow[d, "="] \\
U^{-1}(\mathfrak{a}\otimes_R\mathfrak{b}) \arrow[r, "\cong"] & U^{-1}\mathfrak{a}\otimes_{U^{-1}R}U^{-1}\mathfrak{b} \arrow[r, hook, "(2)"] & (U^{-1}\mathfrak{a})(U^{-1}\mathfrak{b}) \arrow[rr, hook] & & U^{-1}R
\end{tikzcd}$$

The map (2) is injective, since $U^{-1}\mathfrak{a}$ and $U^{-1}\mathfrak{b}$ are $U^{-1}R$-free of rank 1. It follows that the map (1) must be injective, as desired. $\square$

The next result supplies the main tool for this investigation. Note that Theorem 4.2 shows that $\mathfrak{S}_0(R)$ cannot be given a group structure making the inclusion into a group homomorphism.

PROPOSITION 3.4. *Let R be a normal domain. Each semidualizing R-module C is a rank 1 reflexive module, so there is a natural inclusion $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$.*

Proof. It suffices to verify the first statement. Proposition 3.1 shows that C has rank 1. For each prime ideal $\mathfrak{p}$ of height 1, the ring $R_\mathfrak{p}$ is regular as R is (R_1), and so $C_\mathfrak{p} \cong R_\mathfrak{p}$. Since R is (S_2) and $\operatorname{depth}_{R_\mathfrak{p}}(C_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p})$ for each prime ideal $\mathfrak{p}$, the reflexivity of C follows from [10, (1.4.1)]. □

We record an immediate corollary.

COROLLARY 3.5. *Every normal domain with finite divisor class group is $\mathfrak{S}_0$-finite, and every Cohen-Macaulay normal domain with finite divisor class group is $\mathfrak{S}$-finite.* □

Since a Cohen-Macaulay normal domain R with $\mathrm{Cl}(R) = 0$ is Gorenstein, we note that there are non-Gorenstein rings that satisfy the hypotheses of the corollary. For instance, if k is a field and $\mathbf{X}$ a symmetric $n \times n$ matrix of variables and r an integer such that $0 < r < n$, then the ring $R = k[\mathbf{X}]/I_{r+1}(\mathbf{X})$ is a Cohen-Macaulay normal domain with $\mathrm{Cl}(R) \cong \mathbb{Z}/(2)$ and is non-Gorenstein if and only if $r \equiv n \pmod 2$. Here $I_{r+1}(X)$ is the ideal generated by the minors of $\mathbf{X}$ of size $r+1$; see [10, (7.3.7.c)]. Determinantal rings will be of particular interest in Section 4.

Proposition 3.4 points toward a plethora of examples of nonlocal rings that are neither $\mathfrak{S}$-finite nor $\mathfrak{S}_0$-finite. Hence the local hypothesis in Question 1.1.

3.6. The *Picard group* of a normal domain R, denoted $\mathrm{Pic}(R)$, is the set of isomorphism classes of finitely generated locally free (i.e., projective) R-modules of rank 1 with operation given by tensor product. The inverse of an element $[P] \in \mathrm{Pic}(R)$ is the class $[P]^{-1} = [\mathrm{Hom}_R(P, R)]$; see [19, p. 105]. It is straightforward to show that there are natural inclusions $\mathrm{Pic}(R) \subseteq \mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$ for any normal domain R. Using [23, (3.2)] one sees that the first inclusion is an equality when R is Gorenstein. Each inclusion is an equality when R is a Dedekind domain by Fossum [19, (18.5)].

A result of Claborn [19, (14.10)] states that any Abelian group G can be realized as the divisor class group of a Dedekind domain. In particular, for any Abelian group G, regardless of the cardinality, there is a Dedekind domain R such that the sets $\mathfrak{S}(R)$ and $\mathfrak{S}_0(R)$ are in bijection with G.

We observe that [23, (3.1.b)] implies that the hypothesis of the next result is satisfied when C is C''-reflexive and $C' = \mathrm{Hom}_R(C, C'')$. Compare to Proposition 3.3.

Proposition 3.7. *Let R be a normal domain and C, C' semidualizing R-modules. If $C' \otimes_R C$ is R-semidualizing, then $[C' \otimes_R C] = [C'] + [C]$ in* $\mathrm{Cl}(R)$.

Proof. Since $C' \otimes_R C$ is semidualizing, it is reflexive, so $(C' \otimes_R C)^{\star\star} \cong C' \otimes_R C$, and so $[C' \otimes_R C] = (C' \otimes_R C)^{\star\star} = [C'] + [C]$; see 2.10. □

The inclusion $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$ is well-behaved with respect to certain operations that are defined on both sets. The remainder of this section is devoted to describing some of this behavior. We begin by describing base-change maps for Picard groups and sets of semidualizing objects.

3.8. Let $\varphi\colon R \to S$ be a ring homomorphism.

(a) The assignment $L \mapsto L \otimes^{\mathbf{L}}_R S$ yields a well-defined group homomorphism $\mathrm{Pic}(\varphi)\colon \mathrm{Pic}(R) \to \mathrm{Pic}(S)$; see [19, discussion after (18.3)].
(b) When $\mathrm{fd}(\varphi)$ is finite and K is a semidualizing complex, the S-complex $K \otimes^{\mathbf{L}}_R S$ is semidualizing by [23, (4.5)], and the assignment $K \mapsto K \otimes^{\mathbf{L}}_R S$ gives rise to a well-defined order-respecting map $\mathfrak{S}(\varphi)\colon \mathfrak{S}(R) \to \mathfrak{S}(S)$ by [23, (4.7)].
(c) When $\mathrm{fd}(\varphi)$ is finite and C is a semidualizing R-module, the S-module $C \otimes_R S$ is semidualizing and $\mathrm{Tor}^R_{\geqslant 1}(C, S) = 0$ by [23, (4.5)]; the assignment $C \mapsto C \otimes_R S$ induces a well-defined order-respecting map $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ by [23, (4.7)].

Next we consider the divisor class group. Part (d) in the following lemma is well-known, but we include it here for completeness; see [19, Sec. 6].

Lemma 3.9. *Let $\varphi\colon R \to S$ be a homomorphism of finite flat dimension between normal domains.*

(a) *Fix $\mathfrak{q} \in \mathrm{Spec}(S)$ and set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. If $\mathrm{ht}(\mathfrak{q}) \leq 1$, then $R_\mathfrak{p}$ is regular.*
(b) *If M is a reflexive R-module of rank 1, then $\mathrm{rank}_S(M \otimes_R S) = 1$.*
(c) *If φ is module-finite, the assignment $M \mapsto \mathrm{Hom}_S(\mathrm{Hom}_S(M \otimes_R S, S), S)$ yields a well-defined group homomorphism $\mathrm{Cl}(\varphi)\colon \mathrm{Cl}(R) \to \mathrm{Cl}(S)$.*
(d) *If φ is flat, then the assignment $M \mapsto M \otimes_R S$ yields a well-defined group homomorphism $\mathrm{Cl}(\varphi)\colon \mathrm{Cl}(R) \to \mathrm{Cl}(S)$.*

Proof. (a) The induced map $\varphi_\mathfrak{q}\colon R_\mathfrak{p} \to S_\mathfrak{q}$ has finite flat dimension. Since S is normal, it satisfies Serre's condition (R_1) and so the local ring $S_\mathfrak{q}$ is regular. It follows from [1, Theorem R] that $R_\mathfrak{p}$ is also regular.

(b) To show that $M \otimes_R S$ has rank 1, it suffices to set $\mathfrak{q} = (0)S$ and exhibit an isomorphism $(M \otimes_R S)_\mathfrak{q} \cong S_\mathfrak{q}$. With $\mathfrak{p} = \mathrm{Ker}(\varphi)$, part (a) implies $\mathrm{Cl}(R_\mathfrak{p}) = 0$ and so $M_\mathfrak{p} \cong R_\mathfrak{p}$. The next isomorphisms now follow readily

$$(M \otimes_R S)_\mathfrak{q} \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q} \cong R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q} \cong S_\mathfrak{q}$$

and provide the desired conclusion.

(c) Using part (a), this follows from [35, (1.2.1)].

(d) For a finitely generated R-module U, one has a natural S-linear map

$$\begin{aligned} f_U \colon \operatorname{Hom}_R(U,R)\otimes_R S &\to \operatorname{Hom}_S(U\otimes_R S, S)\\ \psi\otimes s &\mapsto [u\otimes s' \mapsto \varphi(\psi(u))ss'], \end{aligned}$$

which is readily seen to be an isomorphism because φ is flat.

Let M and N be rank 1 reflexive R-modules. The S-module $M\otimes_R S$ has rank 1 by part (b). The flatness of φ provides the following commutative diagram from which one concludes that $M\otimes_R S$ is a reflexive S-module.

$$\begin{tikzcd} M\otimes_R S \arrow[r, "b^R_M\otimes_R S", "\cong"'] \arrow[d, "b^S_{M\otimes_R S}"'] & \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)\otimes_R S \arrow[d, "f_{\operatorname{Hom}_R(M,R)}"', "\cong"] \\ \operatorname{Hom}_S(\operatorname{Hom}_S(M\otimes_R S,S),S) \arrow[r, "\operatorname{Hom}_S(f_M,S)", "\cong"'] & \operatorname{Hom}_S(\operatorname{Hom}_R(M,R)\otimes_R S,S) \end{tikzcd}$$

Hence, the map $\operatorname{Cl}(\varphi)$ is well-defined. The fact that $\operatorname{Cl}(\varphi)$ is a group homomorphism follows from the next sequence of isomorphisms

$$\begin{aligned} \operatorname{Hom}_R(\operatorname{Hom}_R(M\otimes_R N,R),R)\otimes_R S &\overset{(1)}{\cong} \operatorname{Hom}_S(\operatorname{Hom}_R(M\otimes_R N,R)\otimes_R S,S)\\ &\overset{(2)}{\cong} \operatorname{Hom}_S(\operatorname{Hom}_S((M\otimes_R N)\otimes_R S,S),S)\\ &\overset{(3)}{\cong} \operatorname{Hom}_S(\operatorname{Hom}_S((M\otimes_R S)\otimes_S (N\otimes_R S),S),S), \end{aligned}$$

where (1) is $f_{\operatorname{Hom}_R(M\otimes_R N,R)}$, (2) is $\operatorname{Hom}_S(f_{M\otimes_R N},S)$, and (3) is standard. □

LEMMA 3.10. *Let $\varphi\colon R\to S$ be a homomorphism of finite flat dimension.*

(a) *If R and S are normal domains and φ is module-finite or flat, then the following diagram commutes.*

$$\begin{tikzcd} \mathfrak{S}_0(R) \arrow[r, hook] \arrow[d, "\mathfrak{S}_0(\varphi)"'] & \operatorname{Cl}(R) \arrow[d, "\operatorname{Cl}(\varphi)"'] \\ \mathfrak{S}_0(S) \arrow[r, hook] & \operatorname{Cl}(S) \end{tikzcd}$$

In particular, if $\operatorname{Cl}(\varphi)$ is injective, then so is $\mathfrak{S}_0(\varphi)$.

(b) *Assume that the image of the map $\operatorname{Spec}(\varphi)\colon \operatorname{Spec}(S)\to\operatorname{Spec}(R)$ contains $\operatorname{m-Spec}(R)$. If the map $\operatorname{Pic}(\varphi)\colon \operatorname{Pic}(R)\to\operatorname{Pic}(S)$ is injective, e.g., if φ is surjective or local, then $\mathfrak{S}_0(\varphi)$ and $\mathfrak{S}(\varphi)$ are also injective.*

(c) *Assume that R,S are normal domains and φ is faithfully flat. If $\operatorname{Cl}(\varphi)$ is surjective, then so is $\mathfrak{S}_0(\varphi)$.*

(d) *Assume that R,S are normal domains and φ is faithfully flat. If $\operatorname{Cl}(\varphi)$ is bijective, then $\mathfrak{S}_0(\varphi)$ is a perfectly order-respecting bijection.*

Proof. When φ is flat, the commutativity of the diagram in (a) follows readily from the definitions. When φ is module finite and C is a semidualizing R-module, the fact that $C \otimes_R S$ is semidualizing for S implies that it is reflexive, and the commutativity of the diagram follows easily. Part (b) is contained in [23, (4.9),(4.11)], and (c) is in [23, (4.5)]. When $\mathrm{Cl}(\varphi)$ is bijective, the map $\mathrm{Pic}(\varphi)$ is injective, so (d) follows from parts (b) and (c) with [23, (4.8)]. $\square$

When $\varphi\colon R \to S$ is a local homomorphism of finite flat dimension, it is a straightforward exercise to show that, if $\mathfrak{S}(\varphi)$ is a perfectly order-respecting bijection, then it is also an isometry with respect to the metric structure defined in [24]. For instance, this holds under the hypotheses of Lemma 3.10(d) when S is Cohen-Macaulay. Some particular instances of this are provided in the next corollary. Others are given in Corollary 3.13 and Proposition 3.15.

COROLLARY 3.11. *Let R be a normal domain and $\mathbf{X} = X_1, \ldots, X_n$ a sequence of variables. For the following flat R-algebras S, the map $\mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ is a perfectly order-respecting bijection:*

(a) $S = R[\mathbf{X}]$;
(b) $S = R[\mathbf{X}][f_1^{-1}, \ldots, f_i^{-1}]$, *where $f_1, \ldots, f_i$ are prime elements of $R[\mathbf{X}]$ and the ring homomorphism $R \to R[\mathbf{X}][f_1^{-1}, \ldots, f_i^{-1}]$ is faithfully flat;*
(c) $S = R[\mathbf{X}]_{\mathfrak{m}R[\mathbf{X}]}$ *when R is local with maximal ideal $\mathfrak{m}$;*
(d) $S = R[\![\mathbf{X}]\!]_{\mathfrak{m}R[\![\mathbf{X}]\!]}$ *when R is local with maximal ideal $\mathfrak{m}$ and the $\mathfrak{m}$-adic completion of R is normal.*

Proof. By Lemma 3.10(d), it suffices to note that the maps $\mathrm{Cl}(R) \to \mathrm{Cl}(S)$ are bijective; see [19, (7.3),(8.1),(8.9),(19.15)]. $\square$

The following is an important case when localization induces a bijection on the set of semidualizing modules.

PROPOSITION 3.12. *Let $R = \coprod_{i\geq 0} R_i$ be a graded normal domain such that $(R_0, \mathfrak{m}_0)$ is local. Setting $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geq 1} R_i$, the natural map $\mathfrak{S}_0(R) \to \mathfrak{S}_0(R_\mathfrak{m})$ is a perfectly order-respecting bijection.*

Proof. Let $\varphi\colon R \to R_\mathfrak{m}$ be the localization map. Using [23, (2.14)], the argument of [19, (10.3)] shows that $\mathrm{Cl}(\varphi)$ is bijective. From Lemma 3.10(a) it follows that $\mathfrak{S}_0(\varphi)$ is injective. To show surjectivity, fix a semidualizing $R_\mathfrak{m}$-module L. Use the surjectivity of $\mathrm{Cl}(\varphi)$ and [19, (10.2)] to obtain a homogeneous reflexive ideal $\mathfrak{a}$ of R such that $\mathfrak{a}_\mathfrak{m} \cong L$. Since L is $R_\mathfrak{m}$-semidualizing, the R-module $\mathfrak{a}$ is R-semidualizing by [23, (2.15.a)], and it follows that $\mathfrak{S}_0(\varphi)$ is bijective. The fact that $\mathfrak{S}_0(\varphi)$ is perfectly order-respecting then follows from [23, (2.15.b)]. $\square$

If R is a local ring with completion map $\varphi\colon R \to \widehat{R}$, then the map $\mathfrak{S}_0(\varphi)$ is not usually surjective. Indeed, there exists a Cohen-Macaulay local ring R that does not admit a dualizing module; the complete local ring $\widehat{R}$ does admit a dualizing module ω, and it is straightforward to show that $[\omega] \in \mathfrak{S}_0(\widehat{R})$ cannot be in the image of $\mathfrak{S}_0(\varphi)$. However, a result of Flenner [18, (1.4)] can be applied in certain cases to provide bijectivity. See Corollary 3.14(a) for a generalization, and also [14, Thm. D]. Recall that a ring is *super-normal* if it satisfies Serre's conditions (S_3) and (R_2). Further, note that a ring R satisfying the hypotheses of the next result is excellent because it is finitely generated over the complete local ring R_0. In particular, the complete ring $\widehat{R}$ is also a super-normal domain.

Corollary 3.13. *Let $R = \coprod_{i\geq 0} R_i$ be a graded super-normal domain with $(R_0, \mathfrak{m}_0)$ local and complete, and set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geq 1} R_i$ and $\widehat{R} = \prod_{i\geq 0} R_i$. The induced maps $\mathfrak{S}_0(R) \to \mathfrak{S}_0(\widehat{R})$ and $\mathfrak{S}_0(R_\mathfrak{m}) \to \mathfrak{S}_0(\widehat{R})$ are perfectly order-respecting bijections.*

Proof. The ring $\widehat{R}$ is the $\mathfrak{m}$-adic completion of $R_\mathfrak{m}$, and since R is excellent and super-normal, the same is true of $R_\mathfrak{m}$ and $\widehat{R}$. Let $\varphi\colon R \to R_\mathfrak{m}$ be the localization map and $\psi\colon R_\mathfrak{m} \to \widehat{R}$ the completion map. By Proposition 3.12, the map $\mathfrak{S}_0(\varphi)$ is a perfectly order-respecting bijection, so the equality $\mathfrak{S}_0(\psi\varphi) = \mathfrak{S}_0(\psi)\mathfrak{S}_0(\varphi)$ shows that we need only verify the same for $\mathfrak{S}_0(\psi)$. Since ψ is flat and local, Lemma 3.10(b) supplies the injectivity of $\mathfrak{S}_0(\psi)$. The surjectivity is a consequence of Lemma 3.10 (c), as [18, (1.4)] guarantees that $\mathrm{Cl}(\psi\varphi) = \mathrm{Cl}(\psi)\,\mathrm{Cl}(\varphi)$ is surjective, and therefore that $\mathrm{Cl}(\psi)$ is surjective. □

Here is the first indication that our methods have applications outside the normal domain arena. See Corollary 4.11 for a more general statement.

Corollary 3.14. *With $R, \mathfrak{m}, \widehat{R}$ as in Corollary 3.13, fix an $R_\mathfrak{m}$-regular sequence $\mathbf{y} = y_1, \ldots, y_q \in \mathfrak{m}R_\mathfrak{m}$.*

(a) *The natural homomorphisms $R_\mathfrak{m} \to R_\mathfrak{m}/(\mathbf{y}) \to \widehat{R}/(\mathbf{y})$ induce perfectly order-respecting bijections*

$$\mathfrak{S}_0(R_\mathfrak{m}) \xrightarrow{\approx} \mathfrak{S}_0(R_\mathfrak{m}/(\mathbf{y})) \xrightarrow{\approx} \mathfrak{S}_0(\widehat{R}/(\mathbf{y})).$$

(b) *Let R' denote either $R_\mathfrak{m}/(\mathbf{y})$ or $\widehat{R}/(\mathbf{y})$ and fix an $\mathfrak{m}R'$-primary ideal $J \subset R'$. If C' is a semidualizing R'-module, then $e(J, C) = e(J, R')$.*

(c) *If $\mathbf{y}$ is an R-regular sequence in $\mathfrak{m}$, then the composition of induced maps*

$$\mathfrak{S}_0(R) \to \mathfrak{S}_0(R/(\mathbf{y})) \to \mathfrak{S}_0(R_\mathfrak{m}/(\mathbf{y}))$$

is a perfectly order-respecting bijection; thus, the first map is a perfectly order-respecting injection and the second is surjective.

Proof. (a) The rings under consideration fit into the commutative diagram of local ring homomorphisms on the left

$$\begin{tikzcd} R_{\mathfrak{m}} \arrow[r, "\beta"] \arrow[d, "\alpha_{\mathfrak{m}}"'] & \widehat{R} \arrow[d, "\widehat{\alpha}"] \\ R_{\mathfrak{m}}/(\mathbf{y}) \arrow[r, "\beta'"] & \widehat{R}/(\mathbf{y}) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{S}_0(R_{\mathfrak{m}}) \arrow[r, "\mathfrak{S}_0(\beta)", "\approx"'] \arrow[d, "\mathfrak{S}_0(\alpha_{\mathfrak{m}})"'] & \mathfrak{S}_0(\widehat{R}) \arrow[d, "\mathfrak{S}_0(\widehat{\alpha})"', "\approx"] \\ \mathfrak{S}_0(R_{\mathfrak{m}}/(\mathbf{y})) \arrow[r, "\mathfrak{S}_0(\beta')"] & \mathfrak{S}_0(\widehat{R}/(\mathbf{y})) \end{tikzcd}$$

and the second commutative diagram arises by applying $\mathfrak{S}_0(-)$ to the first. The maps $\mathfrak{S}_0(\beta)$ and $\mathfrak{S}_0(\widehat{\alpha})$ are perfectly order-respecting bijections by Corollary 3.13 and [24, (4.2)], respectively; see also Gerko [27, (3)]. From the diagram, it follows that $\mathfrak{S}_0(\beta')$ is surjective, and so is bijective by Lemma 3.10(b). That it is perfectly order-respecting is then a consequence of [23, (4.8)]. Thus, $\mathfrak{S}_0(\alpha_{\mathfrak{m}})$ is a perfectly order-respecting bijection, as well.

(b) Set $\widetilde{R} = R_{\mathfrak{m}}$ or $\widehat{R}$, according to whether $R' = R_{\mathfrak{m}}/(\mathbf{y})$ or $\widehat{R}/(\mathbf{y})$. Fix a semidualizing $\widetilde{R}$-module C such that $C' \cong C \otimes_{\widetilde{R}} R'$. Since $\widetilde{R}$ is an excellent local domain, one applies Lemma 2.8(b) to obtain the desired conclusion.

(c) When $\mathbf{y}$ is an R-regular sequence in $\mathfrak{m}$, there is a commutative diagram

$$\begin{tikzcd} \mathfrak{S}_0(R) \arrow[r, "\mathfrak{S}_0(\gamma)", "\approx"'] \arrow[d, "\mathfrak{S}_0(\alpha)"'] & \mathfrak{S}_0(R_{\mathfrak{m}}) \arrow[d, "\mathfrak{S}_0(\alpha_{\mathfrak{m}})"', "\approx"] \\ \mathfrak{S}_0(R/(\mathbf{y})) \arrow[r, "\mathfrak{S}_0(\gamma')"] & \mathfrak{S}_0(R_{\mathfrak{m}}/(\mathbf{y})) \end{tikzcd}$$

where $\mathfrak{S}_0(\gamma)$ and $\mathfrak{S}_0(\alpha_{\mathfrak{m}})$ are perfectly order-respecting bijections by Proposition 3.12 and part (a). The injectivity of $\mathfrak{S}_0(\alpha)$ and surjectivity of $\mathfrak{S}_0(\gamma')$ follow immediately. To see that $\mathfrak{S}_0(\alpha)$ is perfectly order-respecting, fix $[C], [C'] \in \mathfrak{S}_0(R)$ such that $\mathfrak{S}_0(\alpha)([C]) \trianglelefteq \mathfrak{S}_0(\alpha)([C'])$. It follows that

$$\begin{aligned} \mathfrak{S}_0(\alpha_{\mathfrak{m}})(\mathfrak{S}_0(\gamma)([C])) &= \mathfrak{S}_0(\gamma)(\mathfrak{S}_0(\alpha)([C])) \\ &\trianglelefteq \mathfrak{S}_0(\gamma)(\mathfrak{S}_0(\alpha)([C'])) \\ &= \mathfrak{S}_0(\alpha_{\mathfrak{m}})(\mathfrak{S}_0(\gamma)([C'])), \end{aligned}$$

and so $[C] \trianglelefteq [C']$ since $\mathfrak{S}_0(\alpha_{\mathfrak{m}})$ and $\mathfrak{S}_0(\gamma)$ are perfectly order-respecting. $\square$

The surjectivity of the natural map $\mathfrak{S}_0(R_{\mathfrak{m}}) \to \mathfrak{S}_0(R_{\mathfrak{m}}/(\mathbf{y}))$ in the corollary does not hold for more general local rings. Indeed, let $(A, \mathfrak{n})$ be a local Cohen-Macaulay ring that does not admit a dualizing module. If $\mathbf{y} \in \mathfrak{n}$ is a system of parameters of A, then the map $\mathfrak{S}_0(A) \to \mathfrak{S}_0(A/(\mathbf{y}))$ is not surjective, as $A/(\mathbf{y})$ is Artinian and therefore admits a dualizing module. See [14, (5.5),(5.6)] for further discussion.

When the homomorphism $\varphi\colon R \to S$ is part of a retract pair, the next result sometimes allows one to conclude that $\mathfrak{S}_0(\varphi)$ is bijective. Examples of such retract pairs can be found in power series and localized polynomial extensions:

(a) The natural maps $R \to R[\![\mathbf{X}]\!]$ and $R[\![\mathbf{X}]\!] \to R$.
(b) The natural maps $R \to R[\mathbf{X}]_{\mathfrak{n}}$ and $R[\mathbf{X}]_{\mathfrak{n}} \to R$, when $(R, \mathfrak{m})$ is local and $\mathfrak{n} = (\mathfrak{m}, X_1 - a_1, \ldots, X_n - a_n)R[\mathbf{X}]$ for a sequence $a_1, \ldots, a_n \in R$.

Note that the rings involved are not assumed to be normal domains, so one cannot use the divisor class group directly. However, the method of proof is taken directly from the corresponding results for divisor class groups.

Proposition 3.15. *Let $\varphi\colon R \to S$ and $\psi\colon S \to R$ be homomorphisms of finite flat dimension such that the composition $\psi\varphi$ is the identity on R. If $\ker(\psi)$ is contained in the Jacobson radical of S, then the induced maps $\mathfrak{S}_0(\varphi), \mathfrak{S}_0(\psi), \mathfrak{S}(\varphi), \mathfrak{S}(\psi)$ are perfectly order-respecting bijections.*

Proof. (a) Since the composition $\psi\varphi\colon R \to R$ is the identity, the same is true of the composition $\mathfrak{S}_0(\psi)\mathfrak{S}_0(\varphi)$. In particular, $\mathfrak{S}_0(\psi)$ is surjective. Since $\ker(\psi)$ is in the Jacobson radical of S, Lemma 3.10(b) guarantees that this map is bijective, and therefore so is $\mathfrak{S}_0(\varphi)$. The same argument works for $\mathfrak{S}(\psi)$ and $\mathfrak{S}(\varphi)$. □

This is a surprising departure from the parallels we have seen between the behavior of $\mathrm{Cl}(-)$ and $\mathfrak{S}_0(-)$, as it is known that, when R is a normal domain, the map $\mathrm{Cl}(R) \to \mathrm{Cl}(R[\![\mathbf{X}]\!])$ need not be bijective; see Danilov [15], [16], [17].

The proof of Proposition 3.15 can be translated easily to show that the natural maps $R \to R[\mathbf{X}] \to R$ induce injections and surjections respectively on sets of semidualizing objects. However, we can only say more about these maps when we assume that R is a normal domain, by using Corollary 3.11.

Question 3.16. Must the induced maps $\mathfrak{S}_0(R) \to \mathfrak{S}_0(R[X]) \to \mathfrak{S}_0(R)$ and $\mathfrak{S}(R) \to \mathfrak{S}(R[X]) \to \mathfrak{S}(R)$ all be bijective?

4. Analysis of special cases

We begin with some notation and facts on determinantal rings from [11].

4.1. Let A be a Noetherian ring and m, n, r nonnegative integers satisfying $r < \min\{m, n\}$. If $\mathbf{X} = \{X_{ij}\}$ is an $m \times n$ matrix of variables, then set

$$R = R_{r+1}(A; m, n) = A[\mathbf{X}]/I_{r+1}(\mathbf{X}),$$

where $I_{r+1}(\mathbf{X})$ denotes the ideal generated by the minors of $\mathbf{X}$ of size $r+1$. If A is a normal domain (respectively, is Cohen-Macaulay or is (S_3) or is (R_2)), then so is R; see [11, (6.3),(5.17),(5.16),(6.12)]. The ring R is Gorenstein if and only if A is Gorenstein and either $m = n$ or $r = 0$ by [11, (8.9)].

Assume that A is a normal domain and $r > 0$. Let $\mathfrak{p}$ be the ideal of R generated by the r-minors of the first r rows of the residue matrix $\mathbf{x}$. The ideal $\mathfrak{p}$ is prime, and there is an isomorphism $\mathrm{Cl}(R) \cong \mathrm{Cl}(B) \oplus \mathbb{Z}$, where the summand $\mathbb{Z}$ is generated by $[\mathfrak{p}]$; see [11, (8.4)]. For $\ell > 0$ one has $-[\mathfrak{p}] = [\mathrm{Hom}_R(\mathfrak{p}, R)] = [\mathfrak{q}]$, where $\mathfrak{q}$ is the prime ideal of R generated by the r-minors of the first r columns of $\mathbf{x}$. For $\ell \geq 0$ one has $\ell[\mathfrak{p}] = [\mathfrak{p}^{(\ell)}] = [\mathfrak{p}^\ell]$ and $-\ell[\mathfrak{p}] = [\mathfrak{q}^{(\ell)}] = [\mathfrak{q}^\ell]$ by [11, (7.10)], and we write $\mathfrak{p}^{-\ell}$ in place of $\mathfrak{q}^\ell$. If A is also Gorenstein local and $m \geq n$, then R admits a unique (up to isomorphism) dualizing module $\omega \cong \mathfrak{p}^{(m-n)} = \mathfrak{p}^{m-n}$; see [11, (7.10),(8.8)].

Assume that A is a field and $m \geq n \geq r > 1$. Let $\mathbf{Y} = \{Y_{pq}\}$ be an $(m-1) \times (n-1)$ matrix of variables and set $R' = R_r(A; m-1, n-1)$. Let $\mathfrak{p}'$ be the ideal of R' generated by the $(r-1)$-minors of the first $r-1$ rows of the residue matrix $\mathbf{y}$. The discussion above implies that R' has a unique dualizing module, namely, the ideal $(\mathfrak{p}')^{(m-1)-(n-1)} = (\mathfrak{p}')^{m-n}$. We consider three homomorphisms of normal domains

$$R \xrightarrow{\varphi} R_{x_{11}} \xleftarrow[\cong]{\rho} R'[X_{11}, \ldots, X_{m1}, X_{12}, \ldots, X_{1n}]_{X_{11}} \xleftarrow{\psi} R',$$

where ρ is given by $y_{ij} \mapsto x_{i+1j+1} - x_{1j+1}x_{i+1,1}x_{11}^{-1}$ and φ and ψ are the natural flat maps. By [10, (7.3.3)] the map ρ is an isomorphism. Further, the induced maps are all isomorphisms between groups isomorphic to $\mathbb{Z}$,

$$\mathrm{Cl}(R) \xrightarrow{\cong} \mathrm{Cl}(R_{x_{11}}) \xleftarrow{\cong} \mathrm{Cl}(R'[X_{11}, \ldots, X_{m1}, X_{12}, \ldots, X_{1n}]_{X_{11}}) \xleftarrow{\cong} \mathrm{Cl}(R'),$$

and $\mathrm{Cl}(\varphi)([\mathfrak{p}]) = \mathrm{Cl}(\rho\psi)([\mathfrak{p}'])$; see Lemma 3.9(d) and [10, proof of (7.3.6)]. For each integer $\ell \geq 0$, the additivity of $\mathrm{Cl}(\varphi)$ and $\mathrm{Cl}(\rho\psi)$ provides the second equality in the next sequence while the others are from the previous paragraph.

$$\mathrm{Cl}(\varphi)([\mathfrak{p}^\ell]) = \mathrm{Cl}(\varphi)(\ell[\mathfrak{p}]) = \mathrm{Cl}(\rho\psi)(\ell[\mathfrak{p}']) = \mathrm{Cl}(\rho\psi)([(\mathfrak{p}')^\ell]).$$

Lemma 3.10(d) implies that $\mathfrak{S}_0(\psi)$ is bijective. Let C be a semidualizing R-module and let c be the unique integer with $[C] = c[\mathfrak{p}] = [\mathfrak{p}^c]$ in $\mathrm{Cl}(R)$. The $R_{x_{11}}$-module $C \otimes_R R_{x_{11}}$ is semidualizing by 3.8(c), and we compute its class in $\mathfrak{S}_0(R_{x_{11}})$ in the next sequence, where the equalities follow from Lemma 3.10(a), the choice of c, and the previous displayed sequence.

$$\mathfrak{S}_0(\varphi)([C]) = \mathrm{Cl}(\varphi)([C]) = \mathrm{Cl}(\varphi)([\mathfrak{p}^c]) = \mathrm{Cl}(\rho\psi)([(\mathfrak{p}')^c]).$$

As $\rho\psi$ is flat, this provides an isomorphism of $R_{x_{11}}$-modules

$$(\mathfrak{p}')^c \otimes_{R'} R_{x_{11}} \cong C \otimes_R R_{x_{11}}.$$

Since C is R-semidualizing, the module $C \otimes_R R_{x_{11}}$ is $R_{x_{11}}$-semidualizing, and the last isomorphism implies that $(\mathfrak{p}')^c$ is R'-semidualizing by [23, (4.5)].

Theorem 4.2. *Let k be a field and m, n, r nonnegative integers such that $r < \min\{m, n\}$. The ring $R = R_{r+1}(k; m, n)$ satisfies $\mathfrak{S}_0(R) = \{[R], [\omega]\}$, where ω is a dualizing module for R. In particular, the cardinality of $\mathfrak{S}_0(R)$*

is

$$\operatorname{card}\mathfrak{S}_0(R) = \begin{cases} 1 & \textit{when } m = n \textit{ or } r = 0 \\ 2 & \textit{when } m \neq n \textit{ and } r \neq 0. \end{cases}$$

Proof. If $r = 0$ or $m = n$, then R is Gorenstein and the result follows from 3.6. Assume for the remainder of the proof that $r > 0$ and $m \neq n$. We may also assume that $n \leq m$, as replacing $\mathbf{X}$ with its transpose yields an isomorphism $R_{r+1}(k; m, n) \cong R_{r+1}(k; n, m)$. Let x_{ij} denote the residue of X_{ij} in R.

We have the containment $\mathfrak{S}_0(R) \supseteq \{[R], [\mathfrak{p}^{m-n}]\}$ from 4.1, so it remains to verify the containment $\mathfrak{S}_0(R) \subseteq \{[R], [\mathfrak{p}^{m-n}]\}$. Let C be a semidualizing R-module and let c be the unique integer with $[C] = c[\mathfrak{p}]$ in $\mathrm{Cl}(R) \cong \mathbb{Z}[\mathfrak{p}]$.

Following the proof of [10, (7.3.6)], we use induction on r to reduce to the case $r = 1$. Suppose that $r > 1$ and employ the notation of the last paragraph of 4.1. The final conclusion of 4.1 says that $(\mathfrak{p}')^c$ is R'-semidualizing, and the induction hypothesis states that $\mathfrak{S}_0(R') = \{[R'], [(\mathfrak{p}')^{m-n}]\}$. Hence, either $c = 0$ or $c = m - n$. By our choice of c, this implies either $[C] = [\mathfrak{p}^0] = [R]$ or $[C] = [\mathfrak{p}^{m-n}]$, as desired.

Assume $r = 1$. As above, it suffices to show that $c = 0$ or $c = m - n$. The ring R is a standard graded ring over a field and $\mathfrak{p}$ is a homogeneous prime ideal. For each $v \geq 0$ the power $\mathfrak{p}^v$ is homogeneous, and so we may speak of its minimal number of generators, denoted $\beta_0(\mathfrak{p}^v)$. As is noted in [11, (9.20)], the homogeneous minimal generators of $\mathfrak{p}^v$ are in bijection with the monomials of degree v in the ring $k[Z_1, \ldots, Z_n]$. Since $n \geq 2$, a routine argument shows

$$\beta_0(\mathfrak{p}^u)\beta_0(\mathfrak{p}^v) > \beta_0(\mathfrak{p}^{u+v}) > \beta_0(\mathfrak{p}^v). \tag{4.2.1}$$

Also, when $u \leq v$, the following sequence implies $\mathfrak{p}^{v-u} \cong \mathrm{Hom}_R(\mathfrak{p}^u, \mathfrak{p}^v)$:

$$[\mathfrak{p}^{v-u}] = (v-u)[\mathfrak{p}] = v[\mathfrak{p}] - u[\mathfrak{p}] = [\mathfrak{p}^v] - [\mathfrak{p}^u] = [\mathrm{Hom}_R(\mathfrak{p}^u, \mathfrak{p}^v)].$$

Suppose that $0 < c < m - n$. Then $C \cong \mathfrak{p}^c$, and $\mathfrak{p}^{m-n}$ is a dualizing module for R. It follows from [23, (2.14.a),(3.1.a)] and [12, (3.4.a)] that $\mathrm{Hom}_R(\mathfrak{p}^c, \mathfrak{p}^{m-n}) \cong \mathfrak{p}^{m-n-c}$ is semidualizing. Furthermore, Proposition 3.3 yields an isomorphism

$$\mathfrak{p}^c \otimes_R \mathfrak{p}^{m-n-c} \xrightarrow{\cong} \mathfrak{p}^{m-n}$$

and thus the equality $\beta_0(\mathfrak{p}^{m-n}) = \beta_0(\mathfrak{p}^c)\beta_0(\mathfrak{p}^{m-n-c})$, contradicting (4.2.1).

Next, suppose that $c > m - n$. As above, we have

$$\beta_0(\mathfrak{p}^c) > \beta_0(\mathfrak{p}^{m-n}) = \beta_0(\mathfrak{p}^c)\beta_0(\mathrm{Hom}_R(\mathfrak{p}^c, \mathfrak{p}^{m-n})) > \beta_0(\mathfrak{p}^c),$$

again yielding a contradiction.

Finally, suppose that $c < 0$. Then $\mathrm{Hom}_R(C, \mathfrak{p}^{m-n}) \cong \mathfrak{p}^{m-n-c}$ is semidualizing. However, $c < 0$ implies that $m - n - c > m - n$, contradicting the previous case. $\square$

Next, we present the local analogue of Theorem 4.2.

COROLLARY 4.3. *With notation as in Theorem 4.2, let* $\mathfrak{m}$ *denote the maximal ideal of* R *generated by the residues of the variables* X_{ij}. *If* R' *is either the localization* $R_{\mathfrak{m}}$ *or its* $\mathfrak{m}$*-adic completion* $\widehat{R}$, *then* $\mathfrak{S}_0(R') = \{[R'], [\omega']\}$, *where* ω' *is a dualizing module for* R'. *In particular, the cardinality of* $\mathfrak{S}_0(R')$ *is*

$$\operatorname{card}\mathfrak{S}_0(R') = \begin{cases} 1 & \text{when } m = n \text{ or } r = 0 \\ 2 & \text{when } m \neq n \text{ and } r \neq 0. \end{cases}$$

Proof. The ring R satisfies the hypotheses of Corollary 3.13 as it is Cohen-Macaulay and (R_2) by [11, (6.12)]. □

The next result is a considerable generalization of Theorem 4.2 that encompasses Theorem 1.2 from the introduction. Its proof requires more notation.

4.4. Let A be a normal domain and m, n, r nonnegative integers such that $r < \min\{m, n\}$. Set $R = R_{r+1}(A; m, n)$ and consider the commutative diagram of natural ring homomorphisms

(4.4.1)
$$\begin{tikzcd} & A[\mathbf{X}] \arrow[dr, "\varphi'"] & \\ A \arrow[ur, "\dot\varphi"] \arrow[rr, "\varphi"] & & R \end{tikzcd}$$

wherein φ and $\dot\varphi$ are faithfully flat, and φ' is surjective and Cohen-Macaulay of grade $d = mn - r(m + n - r)$ by [11, (5.18)]; see 2.9 for terminology. If C is a semidualizing A-module, then the following R-module is semidualizing by [23, (6.1)]:

$$C(\varphi) = \operatorname{Ext}^d_{A[\mathbf{X}]}(R, C \otimes_A A[\mathbf{X}]).$$

THEOREM 4.5. *Let* A *be a normal domain and* m, n, r *nonnegative integers such that* $r < \min\{m, n\}$. *The ring* $R = R_{r+1}(A; m, n)$ *is* $\mathfrak{S}_0$*-finite if and only if* A *is so, and the ordering on* $\mathfrak{S}_0(R)$ *is transitive if and only if the ordering on* $\mathfrak{S}_0(A)$ *is so. More specifically, one has the following cases.*

(a) *If* $r = 0$ *or* $m = n$, *then* $\mathfrak{S}_0(\varphi)$ *is a perfectly order-respecting bijection*

$$\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(A) \xrightarrow{\approx} \mathfrak{S}_0(R).$$

(b) *If* $r > 0$ *and* $m \neq n$, *then the assignment*

$$([C]_A, 0) \mapsto [C(\varphi)]_R, \qquad ([C]_A, 1) \mapsto [C \otimes_A R]_R$$

describes a perfectly order-respecting bijection

$$h\colon \mathfrak{S}_0(A) \times \{0, 1\} \xrightarrow{\approx} \mathfrak{S}_0(R).$$

The proof of this result is rather long, so it is presented at the end of the section in 4.14. For now we focus on some consequences of the theorem.

4.6. Continue with the notation of 4.12. Let $\mathfrak{n}$ be a prime ideal of A and consider the prime ideals $\mathfrak{N} = (\mathfrak{n}, \mathbf{X})A[\mathbf{X}]$ and $\mathfrak{m} = (\mathfrak{n}, \mathbf{x})R$. Localizing and completing the diagram (4.4.1) yield similar commutative diagrams

$$\begin{tikzcd} & A[\mathbf{X}]_{\mathfrak{N}} \arrow[dr, two heads, "\varphi'_{\mathfrak{m}}"] & \\ A_{\mathfrak{n}} \arrow[ur, "\dot{\varphi}_{\mathfrak{N}}"] \arrow[rr, "\varphi_{\mathfrak{m}}"] & & R_{\mathfrak{m}} \end{tikzcd} \qquad \begin{tikzcd} & \widehat{A[\mathbf{X}]_{\mathfrak{N}}} \arrow[dr, two heads, "\widehat{\varphi'_{\mathfrak{m}}}"] & \\ \widehat{A} = \widehat{A_{\mathfrak{n}}} \arrow[ur, "\widehat{\dot{\varphi}_{\mathfrak{N}}}"] \arrow[rr, "\widehat{\varphi}=\widehat{\varphi_{\mathfrak{m}}}"] & & \widehat{R_{\mathfrak{m}}} = \widehat{R} \end{tikzcd}$$

For semidualizing $A_{\mathfrak{n}}$- and $\widehat{A}$-modules C_0 and C_1, respectively, we set

$$C_0(\varphi_{\mathfrak{m}}) = \operatorname{Ext}^d_{A[\mathbf{X}]_{\mathfrak{N}}}(R_{\mathfrak{m}}, C_0 \otimes_{A_{\mathfrak{n}}} A[\mathbf{X}]_{\mathfrak{N}}) \qquad \text{(semidualizing for } R_{\mathfrak{m}}\text{)},$$
$$C_1(\widehat{\varphi}) = \operatorname{Ext}^d_{\widehat{A[\mathbf{X}]_{\mathfrak{N}}}}(\widehat{R}, C_1 \otimes_{\widehat{A}} \widehat{A[\mathbf{X}]_{\mathfrak{N}}}) \qquad \text{(semidualizing for } \widehat{R}\text{)}.$$

These local constructions are discussed extensively in [23, Section 6].

What follows is the localized version of Theorem 4.5.

Corollary 4.7. *Let $A = \coprod_{i \geq 0} A_i$ be a graded normal domain with $(A_0, \mathfrak{n}_0)$ local, and set $\mathfrak{n} = \mathfrak{n}_0 + \coprod_{i \geq 1} A_i$.*

(a) *If $r = 0$ or $m = n$, then $\mathfrak{S}_0(\varphi_{\mathfrak{m}})$ is a perfectly order-respecting bijection*

$$\mathfrak{S}_0(\varphi_{\mathfrak{m}}) \colon \mathfrak{S}_0(A_{\mathfrak{n}}) \xrightarrow{\approx} \mathfrak{S}_0(R_{\mathfrak{m}}).$$

(b) *If $r > 0$ and $m \neq n$, then the assignment*

$$([C]_{A_{\mathfrak{n}}}, 0) \mapsto [C(\varphi_{\mathfrak{n}})]_{R_{\mathfrak{m}}}, \qquad ([C]_{A_{\mathfrak{n}}}, 1) \mapsto [C \otimes_{A_{\mathfrak{n}}} R_{\mathfrak{m}}]_{R_{\mathfrak{m}}}$$

describes a perfectly order-respecting bijection

$$\mathfrak{S}_0(A_{\mathfrak{n}}) \times \{0, 1\} \xrightarrow{\approx} \mathfrak{S}_0(R_{\mathfrak{m}}).$$

Proof. The following diagrams (one for each of our cases) commute.

$$\begin{tikzcd} \mathfrak{S}_0(A) \arrow[r, "\approx"] \arrow[d, "\approx"', "\mathfrak{S}_0(\varphi)"] & \mathfrak{S}_0(A_{\mathfrak{n}}) \arrow[d, "\mathfrak{S}_0(\varphi_{\mathfrak{m}})"] \\ \mathfrak{S}_0(R) \arrow[r, "\approx"] & \mathfrak{S}_0(R_{\mathfrak{m}}) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{S}_0(A) \times \{0,1\} \arrow[r, "\approx"] \arrow[d, "\approx"', "h"] & \mathfrak{S}_0(A_{\mathfrak{n}}) \times \{0,1\} \arrow[d, "h_{\mathfrak{m}}"] \\ \mathfrak{S}_0(R) \arrow[r, "\approx"] & \mathfrak{S}_0(R_{\mathfrak{m}}) \end{tikzcd}$$

The four horizontal maps are perfectly order-respecting bijections by Proposition 3.12, as are two of the vertical ones by Theorem 4.5. Thus, the two remaining maps are so as well. $\square$

Corollary 4.8. *Let $A = \coprod_{i \geq 0} A_i$ be a graded super-normal domain with $(A_0, \mathfrak{n}_0)$ local and complete. Set $\mathfrak{n} = \mathfrak{n}_0 + \coprod_{i \geq 1} A_i$ and let $\mathfrak{m}, \widehat{A}, \widehat{R}$ be as in 4.6.*

(a) *If $r = 0$ or $m = n$, then $\mathfrak{S}_0(\widehat{\varphi})$ is a perfectly order-respecting bijection*

$$\mathfrak{S}_0(\widehat{\varphi})\colon \mathfrak{S}_0(\widehat{A}) \xrightarrow{\approx} \mathfrak{S}_0(\widehat{R}).$$

(b) *If $r > 0$ and $m \neq n$, then the assignment*

$$([C]_{\widehat{A}}, 0) \mapsto [C(\widehat{\varphi})]_{\widehat{R}}, \qquad ([C]_{\widehat{A}}, 1) \mapsto [C \otimes_{\widehat{A}} \widehat{R}]_{\widehat{R}}$$

describes a perfectly order-respecting bijection

$$\mathfrak{S}_0(\widehat{A}) \times \{0,1\} \xrightarrow{\approx} \mathfrak{S}_0(\widehat{R}).$$

Proof. The proof is almost identical to the previous one, using Corollary 3.13 in place of Proposition 3.12. One needs only note that, since A is super-normal, the same is true of R by [11, (5.17),(6.12)]. □

The next step is to iterate the previous three results.

Corollary 4.9. *Let A be a normal domain and t a positive integer. For $l = 1, \ldots, t$ fix integers r_l, m_l, n_l such that $0 \leq r_l < \min\{m_l, n_l\}$ and let $\mathbf{X}_{l**} = \{X_{lij}\}$ be an $m_l \times n_l$ matrix of variables. Let $\mathbf{X}$ denote the entire list of variables $X_{111}, \ldots, X_{tm_tn_t}$ and set*

$$R = A[\mathbf{X}]/\textstyle\sum_{l=1}^{t} I_{r_l+1}(\mathbf{X}_{l**})$$

with $\mathbf{x}$ the image in R of the sequence $\mathbf{X}$. Let s be the number of indices l such that $r_l > 0$ and $m_l \neq n_l$.

(a) *There is a perfectly order-respecting bijection*

$$\mathfrak{S}_0(A) \times \{0,1\}^s \xrightarrow{\approx} \mathfrak{S}_0(R).$$

(b) *With $A, \mathfrak{n}$ as in Corollary 4.7 and $\mathfrak{m} = (\mathfrak{n}, \mathbf{x})R$, there is a perfectly order-respecting bijection*

$$\mathfrak{S}_0(A_{\mathfrak{n}}) \times \{0,1\}^s \xrightarrow{\approx} \mathfrak{S}_0(R_{\mathfrak{m}}).$$

(c) *With $A, \mathfrak{n}$ as in Corollary 4.8 and $\mathfrak{m} = (\mathfrak{n}, x)R$, let $\widehat{A}$ and $\widehat{R}$ denote the $\mathfrak{n}$-adic and $\mathfrak{m}$-adic completions of A and R, respectively. There is a perfectly order-respecting bijection*

$$\mathfrak{S}_0(\widehat{A}) \times \{0,1\}^s \xrightarrow{\approx} \mathfrak{S}_0(\widehat{R}).$$

Proof. Write $R_0 = A$ and for $l = 1, \ldots, t$ set $R_l = R_{r_l+1}(R_{l-1}; m_l, n_l)$. Then $R_t \cong R$ and part (a) is proved by induction on t using Theorem 4.5. Parts (b) and (c) now follow from Proposition 3.12 and Corollary 3.13, respectively. □

Before continuing, we present some notation.

4.10. Let A be a ring and fix an integer $m \geq 1$ and an A-regular sequence $\mathbf{y} = y_1, \ldots, y_q \in A$. Set $n = m + q - 1$ and let $\mathbf{X}$ be an $m \times n$ matrix of variables. The discussion before and after [11, (2.14)] exhibits a surjection

$$A[\mathbf{X}]/I_m(\mathbf{X}) \twoheadrightarrow A/(\mathbf{y})^m$$

whose kernel is generated by an $A[\mathbf{X}]/I_m(\mathbf{X})$-regular sequence.

For $l = 1, \ldots, t$ fix integers $m_l, q_l \geq 1$ and a sequence $\mathbf{y}_{l*} = y_{l1}, \ldots, y_{lq_l} \in A$. Assume that the sequence $\mathbf{y}_{**}$ is A-regular and set

$$B(A, \mathbf{y}, \mathbf{m}, \mathbf{q}) = A/\textstyle\sum_{l=1}^{t}(\mathbf{y}_{l*})^{m_l}.$$

With R as in Corollary 4.9, tensoring copies of the surjection from the previous paragraph provides a surjection

$$\tau\colon R \twoheadrightarrow B(A, \mathbf{y}, \mathbf{m}, \mathbf{q}) \tag{4.10.1}$$

whose kernel is generated by an R-regular sequence.

Let B be a ring and u a positive integer. For $l = 1, \ldots, u$ fix a positive integer p_l and variables $\mathbf{Z}_{l*} = Z_{l1}, \ldots, Z_{lp_l}$. We consider the ring

$$S = S(B, \mathbf{p}) = B[\mathbf{Z}_{1*}]/(\mathbf{Z}_{1*})^2 \otimes_B \cdots \otimes_B B[\mathbf{Z}_{u*}]/(\mathbf{Z}_{u*})^2,$$

which can be thought of in several different ways. Each ring $B[\mathbf{Z}_{l*}]/(\mathbf{Z}_{l*})^2$ is isomorphic to the trivial extension $B \ltimes B^{q_l}$, so there is an isomorphism

$$S \cong (B \ltimes B^{q_1}) \otimes_B \cdots \otimes_B (B \ltimes B^{q_u}).$$

Next, set $S_0 = B$ and take successive trivial extensions $S_l = S_{l-1} \ltimes (S_{l-1})^{q_l}$. From the previous description, there is an isomorphism $S \cong S_u$. Finally, let $\mathbf{Z}$ denote the full list of variables $\mathbf{Z} = Z_{11}, \ldots, Z_{up_u}$ and let $\mathbf{z}$ denote the image in S of the sequence $\mathbf{Z}$. From the definition of S, one obtains the isomorphism

$$S \cong B[\mathbf{Z}]/\textstyle\sum_{l=1}^{u}(\mathbf{Z}_{l*})^2.$$

If B is (complete) local with maximal ideal $\mathfrak{r}$, then S is (complete) local with maximal ideal $(\mathfrak{r}, \mathbf{z})S$.

The final result of this paper generalizes Corollary 3.14 and contains Theorems 1.3 and 1.4 from the introduction.

Corollary 4.11. *With $A, \mathfrak{n}$ as in Corollary 4.8, let t, u be nonnegative integers. For $l = 1, \ldots, t$ fix positive integers m_l, q_l and a sequence $\mathbf{y}_{l*} = y_{l1}, \ldots, y_{lq_l} \in \mathfrak{n}\widehat{A}$, and let s be the number of indices l such that $m_l, q_l > 1$. For $l = 1, \ldots, u$ fix a positive integer p_l, and let r denote the number of indices l such that $p_l > 1$.*

(a) *Set $\widehat{B} = B(\widehat{A}, \mathbf{y}, \mathbf{m}, \mathbf{q})$ and $\widehat{S} = S(\widehat{B}, \mathbf{p})$. If $\mathbf{y}$ is $\widehat{A}$-regular, then there is a perfectly order-respecting bijection*

$$\mathfrak{S}_0(\widehat{A}) \times \{0,1\}^{r+s} \xrightarrow{\approx} \mathfrak{S}_0(\widehat{S}).$$

Furthermore, if $J \subset \widehat{S}$ is an ideal primary to the maximal ideal of $\widehat{S}$ and C is a semidualizing $\widehat{S}$-module, then $e(J,C) = e(J,\widehat{S})$.

(b) *Assume that* $\mathbf{y}$ *is an* $A_{\mathfrak{n}}$*-sequence in* $\mathfrak{n}A_{\mathfrak{n}}$*, and set* $B' = B(A_{\mathfrak{n}}, \mathbf{y}, \mathbf{m}, \mathbf{q})$ *and* $S' = S(B', \mathbf{p})$*. There are perfectly order-respecting bijections*

$$\mathfrak{S}_0(A_{\mathfrak{n}}) \times \{0,1\}^{r+s} \xrightarrow{\approx} \mathfrak{S}_0(S') \xrightarrow{\approx} \mathfrak{S}_0(\widehat{S}).$$

Furthermore, if $J \subset S'$ is an ideal primary to the maximal ideal of S' and C is a semidualizing S'-module, then $e(J,C) = e(J,S')$.

(c) *Assume that* $\mathbf{y}$ *is an* A*-regular sequence in* $\mathfrak{n}$*, and set* $B = B(A, \mathbf{y}, \mathbf{m}, \mathbf{q})$ *and* $S = S(B, \mathbf{p})$*. There is a perfectly order-respecting injection*

$$\mathfrak{S}_0(A) \times \{0,1\}^{r+s} \hookrightarrow \mathfrak{S}_0(S).$$

Proof. We prove part (a); argue similarly for the other parts. Let $\mathbf{Z}$ be as in 4.10. Then there are isomorphisms

$$\widehat{S} \cong \widehat{B}[\mathbf{Z}]/\textstyle\sum_{l=1}^{u}(\mathbf{Z}_{l*})^2 \cong \widehat{A}[\mathbf{Z}]/(\sum_{l=1}^{t}(\mathbf{y}_{l*})^{k_l} + \sum_{l=1}^{u}(\mathbf{Z}_{l*})^2).$$

By Corollary 3.11(a), the natural map $\mathfrak{S}_0(\widehat{A}) \to \mathfrak{S}_0(\widehat{A}[\mathbf{Z}])$ is a perfectly order-respecting bijection. Pass to the ring $\widehat{A}[\mathbf{Z}]$ and use the fact that $\mathbf{Z}$ is $\widehat{A}[\mathbf{Z}]/(\mathbf{y})$-regular, to reduce to the case $u = 0 = r$, that is, $\widehat{S} = \widehat{B}$.

For $l = 1, \ldots, t$ set $r_l = m_l - 1$ and $n_l = m_l + q_l - 1$, and let $R, \mathfrak{m}, \widehat{R}$ be as in Corollary 4.9. The surjection (4.10.1) completes to a surjection $\widehat{\tau}\colon \widehat{R} \to \widehat{S}$ whose kernel is generated by a $\widehat{R}$-regular sequence. The perfectly order-respecting bijections in the next sequence are in Corollary 4.9(c) and [24, (4.2)], respectively:

$$\mathfrak{S}_0(\widehat{A}) \times \{0,1\}^{r+s} \xrightarrow{\approx} \mathfrak{S}_0(\widehat{R}) \xrightarrow{\approx} \mathfrak{S}_0(\widehat{S}).$$

The equality of multiplicities follows from Corollary 3.14(b). □

4.12. To keep things tangible, we give an explicit description of the injection

$$\mathfrak{S}_0(A) \times \{0,1\}^{r+s} \hookrightarrow \mathfrak{S}_0(S)$$

from the previous corollary. (The two bijections are described analogously.) For $l = t+1, \ldots, t+u$ set $m_l = 2$ and $q_l = p_{l-t}$. Set $S_0 = A$. For $l = 1, \ldots, t$ take quotients $S_l = S_{l-1}/(\mathbf{y}_{l*})^{m_l}$, and for $l = t+1, \ldots, t+u$ take trivial extensions $S_l = S_{l-1} \ltimes (S_{l-1})^{p_l}$, so that $S \cong S_{t+u}$. Each homomorphism $\varphi_{l-1}\colon S_{l-1} \to S_l$ induces an injective map:

(a) If $q_l = 1$ or $m_l = 1$, then set $f_{l-1} = \mathfrak{S}_0(\varphi_{l-1})\colon \mathfrak{S}_0(S_{l-1}) \to \mathfrak{S}_0(S_l)$.

(b) If $m_l, q_l > 1$, then let $f_{l-1}\colon \mathfrak{S}_0(S_{l-1}) \times \{0,1\} \to \mathfrak{S}_0(S_l)$ be given by

$$([C]_{S_{l-1}}, 0) \mapsto \begin{cases} [\operatorname{Ext}^{q_l}_{S_{l-1}}(S_l, C)]_{S_l} & \text{if } l \leq t \\ [\operatorname{Hom}_{S_{l-1}}(S_l, C)]_{S_l} & \text{if } l > t \end{cases}$$

$$([C]_{S_{l-1}}, 1) \mapsto [C \otimes_{S_{l-1}} S_l]_{S_l}.$$

The desired inclusion is exactly the composition $f_{t+u-1}\cdots f_0$.

The calculations of this section motivate a refinement of Question 1.1.

QUESTION 4.13. If R is a local ring, must the cardinalities of the sets $\mathfrak{S}_0(R)$ and $\mathfrak{S}(R)$ be powers of 2?

Paragraph 3.6 explains the need for the "local" hypothesis. Beyond the results of this section, evidence justifying this question can be found in [24, (3.4)]: If R is a non-Gorenstein ring admitting a dualizing complex and $\mathfrak{S}(R)$ is a finite set, then $\mathfrak{S}(R)$ has even cardinality.

We conclude this section with the proof of Theorem 4.5.

4.14. *(Proof of Theorem 4.5.)* Let x_{ij} denote the residue of X_{ij} in R and set

$$\Delta = \det\begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & & \vdots \\ X_{r1} & \dots & X_{rr}\end{pmatrix} \in A[\mathbf{X}], \qquad \delta = \det\begin{pmatrix} x_{11} & \dots & x_{1r} \\ \vdots & & \vdots \\ x_{r1} & \dots & x_{rr}\end{pmatrix} \in R,$$

noting $\delta = \varphi'(\Delta)$. Also, set $e = \dim R - \dim A$ and note that [11, (5.18)] implies $e = (m+n-r)r$. By [11, (6.4)] there is a prime element $\zeta \in A[T_1,\dots,T_e]$ and an isomorphism ϵ as in the next display; the isomorphism τ is clear.

$$R\otimes_{A[\mathbf{X}]} A[\mathbf{X}]_\Delta \xrightarrow[\tau]{\cong} R_\delta \xleftarrow[\epsilon]{\cong} A[T_1,\dots,T_e]_\zeta\ .$$

Furthermore, the natural map $\alpha\colon A \to A[T_1,\dots,T_e]_\zeta$ is faithfully flat, so $\mathfrak{S}_0(\alpha)$ is bijective by Corollary 3.11(b).

Set $U = A \smallsetminus (0)$ and $F = U^{-1}A$. Using Lemma 3.10(a), the natural maps $\beta\colon R \to R_\delta$ and $\gamma\colon R \to U^{-1}R$ along with $\epsilon\alpha$ yield a commutative diagram

$$\begin{tikzcd} \mathfrak{S}_0(R) \arrow[r, "f"] \arrow[d, hook] & \mathfrak{S}_0(R_\delta)\times\mathfrak{S}_0(U^{-1}R) \arrow[d, hook] & \mathfrak{S}_0(A)\times\mathfrak{S}_0(U^{-1}R) \arrow[l, "g"'] \arrow[d, hook] \\ \mathrm{Cl}(R) \arrow[r, "\cong", "f'"'] & \mathrm{Cl}(R_\delta)\times\mathrm{Cl}(U^{-1}R) & \mathrm{Cl}(A)\times\mathrm{Cl}(U^{-1}R) \arrow[l, "\cong"', "-g'"] \end{tikzcd} \tag{4.14.1}$$

where the horizontal maps are given by

$$\begin{gathered} [C] \xmapsto{f} ([C\otimes_R R_\delta\,], [C\otimes_R U^{-1}R]) \\ ([C'\otimes_A R_\delta], [C'']) \xleftarrow{g} ([C'], [C'']) \end{gathered} \tag{4.14.2}$$

and the vertical arrows are induced by the respective inclusions. The maps f' and g' are bijective by [11, (8.3)] and [19, (7.3),(8.1)], respectively. In particular, the maps f, g are injective, and g is bijective by Corollary 3.11(b).

(a) Assuming that $r = 0$ or $m = n$, Theorem 4.2 implies that $\mathfrak{S}_0(U^{-1}R)$ is trivial since $U^{-1}R \cong R_{r+1}(F; m, n)$. Thus, the top row of (4.14.1) reduces to

$$\mathfrak{S}_0(R) \xhookrightarrow{\mathfrak{S}_0(\beta)} \mathfrak{S}_0(R_\delta) \xleftarrow[\approx]{\mathfrak{S}_0(\epsilon\alpha)} \mathfrak{S}_0(A). \tag{4.14.3}$$

The functoriality of $\mathfrak{S}_0(-)$ and the following commutative diagram of ring homomorphisms

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varphi"] & A[T_1, \ldots, T_e]_\zeta \arrow[d, "\epsilon"] \\ R \arrow[r, "\beta"] & R_\delta \end{tikzcd}$$

yield the equality $\mathfrak{S}_0(\beta)\mathfrak{S}_0(\varphi) = \mathfrak{S}_0(\epsilon\alpha)$. Since (4.14.3) shows that $\mathfrak{S}_0(\epsilon\alpha)$ is bijective, it follows that $\mathfrak{S}_0(\beta)$ is surjective. As noted above, $\mathfrak{S}_0(\beta)$ is also injective, so it follows that $\mathfrak{S}_0(\varphi)$ is bijective. That it is a perfectly order-respecting bijection follows from [23, (4.8)] since φ is faithfully flat.

(b) Assume now that $r > 0$ and $m \neq n$. The isomorphism $U^{-1}R \cong R_{r+1}(F; m, n)$ in conjunction with Theorem 4.2 yields a bijection $i\colon \{0, 1\} \xrightarrow{\approx} \mathfrak{S}_0(U^{-1}R)$ given by $i(0) = [\omega_{U^{-1}R}]$ and $i(1) = [U^{-1}R]$, where $\omega_{U^{-1}R}$ is a dualizing module for $U^{-1}R$. Let $i'\colon \mathfrak{S}_0(A) \times \{0, 1\} \to \mathfrak{S}_0(A) \times \mathfrak{S}_0(U^{-1}R)$ be the induced bijection.

Recall that $h\colon \mathfrak{S}_0(A) \times \{0, 1\} \to \mathfrak{S}_0(R)$ is defined as $h([C]_A, 0) = [C(\varphi)]_R$ and $h([C]_A, 1) = [C \otimes_A R]_R$. Below we construct a bijection

$$j\colon \mathfrak{S}_0(R_\delta) \times \mathfrak{S}_0(U^{-1}R) \to \mathfrak{S}_0(R_\delta) \times \mathfrak{S}_0(U^{-1}R)$$

such that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{S}_0(A) \times \{0, 1\} \arrow[r, "h"] \arrow[d, "i'"', "\approx"] & \mathfrak{S}_0(R) \arrow[r, hook, "f"] & \mathfrak{S}_0(R_\delta) \times \mathfrak{S}_0(U^{-1}R) \arrow[d, "j"', "\approx"] \\ \mathfrak{S}_0(A) \times \mathfrak{S}_0(U^{-1}R) \arrow[rr, "g", "\approx"'] & & \mathfrak{S}_0(R_\delta) \times \mathfrak{S}_0(U^{-1}R) \end{tikzcd}$$

Once this is done, a simple diagram chase provides the bijectivity of h.

Localize the surjection $\varphi'\colon A[\mathbf{X}] \to R$ by inverting Δ to obtain a surjection $\rho\colon A[\mathbf{X}]_\Delta \to R_\delta$. We claim that ρ is Gorenstein; see 2.9 for terminology. In 4.4 it is observed that φ' is Cohen-Macaulay of grade d. Hence, the same is true of ρ. The diagram (4.4.1) fits in the next commutative diagram of ring homomorphisms.

$$
\begin{tikzcd}
& A[\mathbf{X}] \arrow[r, "\psi"] \arrow[d, "\varphi'"] & A[\mathbf{X}]_\Delta \arrow[d, "\rho"] \\
A \arrow[ur, "\dot{\varphi}"] \arrow[r, "\varphi"] \arrow[dr, "\alpha"'] & R \arrow[r, "\beta"] & R_\delta \\
& A[T_1, \dots, T_e]_\zeta \arrow[ur, "\epsilon", "\cong"'] &
\end{tikzcd}
$$

The map α is faithfully flat. Furthermore, for each prime ideal $\mathfrak{p}$ of A, the fibre $\kappa(\mathfrak{p}) \otimes_A A[T_1, \dots, T_e]_\zeta \cong \kappa(\mathfrak{p})[T_1, \dots, T_e]_\zeta$ is Gorenstein. That ρ is Gorenstein now follows from Avramov and Foxby [5, (6.2),(6.3)].

Set $\omega_\rho = \operatorname{Ext}^d_{A[\mathbf{X}]_\Delta}(R_\delta, A[\mathbf{X}]_\Delta)$, which is R_δ-semidualizing by [23, (5.6.a)]. Moreover, it is locally free of rank 1 by [23, (5.6.b)]. Setting

$$\omega_\rho^{-1} = \operatorname{Hom}_{R_\delta}(\omega_\rho, R_\delta),$$

the discussion in 3.6 yields an isomorphism

$$\omega_\rho \otimes_{R_\delta} \omega_\rho^{-1} \cong R_\delta. \tag{4.14.4}$$

We now define the aforementioned map j and demonstrate that it has the desired properties. For each semidualizing R_δ-module C, set

$$j([C], [U^{-1}R]) = ([C], [U^{-1}R]), \qquad j([C], [\omega_{U^{-1}R}]) = ([\omega_\rho^{-1} \otimes_{R_\delta} C], [\omega_{U^{-1}R}]).$$

It follows from the isomorphism (4.14.4) that the assignment

$$([C], [U^{-1}R]) \mapsto ([C], [U^{-1}R]), \qquad ([C], [\omega_{U^{-1}R}]) \mapsto ([\omega_\rho \otimes_{R_\delta} C], [\omega_{U^{-1}R}])$$

describes an inverse of j, so that j is bijective. It remains only to show that $gi' = jfh$, so fix a semidualizing A-module C. First, there are isomorphisms

$$C \otimes_A U^{-1}R \cong (C \otimes_A U^{-1}A) \otimes_{U^{-1}A} U^{-1}R \cong U^{-1}A \otimes_{U^{-1}A} U^{-1}R \cong U^{-1}R,$$

the first and third of which are standard, and the second of which is due to the fact that $U^{-1}A$ is a field. This yields equality (1) in the following sequence

$$
\begin{aligned}
jfh([C], 1) &= jf([C \otimes_A R]) \\
&= j([C \otimes_A R \otimes_R R_\delta)], [C \otimes_A R \otimes_R U^{-1}R]) \\
&\overset{(1)}{=} j([C \otimes_A R_\delta], [U^{-1}R]) \\
&= ([C \otimes_A R_\delta], [U^{-1}R]) \\
&= g([C], [U^{-1}R]) \\
&= gi'([C], 1),
\end{aligned}
$$

where each of the unmarked equalities follows either from a definition (e.g., (4.14.2)) or by a standard isomorphism.

To compute $jfh([C],0)$, we first describe some isomorphisms:

$$
\begin{aligned}
C(\varphi)\otimes_R R_\delta &= \operatorname{Ext}^d_{A[\mathbf{X}]}(R, C\otimes_A A[\mathbf{X}])\otimes_R R_\delta \\
&\overset{(2)}{\cong} \operatorname{Ext}^d_{A[\mathbf{X}]}(R, C\otimes_A A[\mathbf{X}])\otimes_R (R\otimes_{A[\mathbf{X}]} A[\mathbf{X}]_\Delta) \\
&\cong \operatorname{Ext}^d_{A[\mathbf{X}]}(R, C\otimes_A A[\mathbf{X}])\otimes_{A[\mathbf{X}]} A[\mathbf{X}]_\Delta \\
&\overset{(3)}{\cong} \operatorname{Ext}^d_{A[\mathbf{X}]_\Delta}(R\otimes_{A[\mathbf{X}]} A[\mathbf{X}]_\Delta, C\otimes_A A[\mathbf{X}]\otimes_{A[\mathbf{X}]} A[\mathbf{X}]_\Delta) \\
&\overset{(4)}{\cong} \operatorname{Ext}^d_{A[\mathbf{X}]_\Delta}(R_\delta, C\otimes_A A[\mathbf{X}]_\Delta) \\
&\overset{(5)}{\cong} \omega_\rho\otimes_{R_\delta}(C\otimes_A A[\mathbf{X}]_\Delta\otimes_{A[\mathbf{X}]_\Delta} R_\delta) \\
&\cong \omega_\rho\otimes_{R_\delta}(C\otimes_A R_\delta).
\end{aligned}
$$

Each of the unmarked isomorphisms is either by definition or standard. Isomorphisms (2) and (4) are via the isomorphism τ, whereas (3) is from the flatness of ψ. For (5) use the equality $\operatorname{pd}_{A[\mathbf{X}]_\Delta}(R_\delta)=d$ to apply [23, (1.7.b)] and the definition of ω_ρ. Similar explanations yield all but one of the following isomorphisms.

$$
\begin{aligned}
C(\varphi)\otimes_R U^{-1}R &= \operatorname{Ext}^d_{A[\mathbf{X}]}(R, C\otimes_A A[\mathbf{X}])\otimes_R U^{-1}R \\
&\cong \operatorname{Ext}^d_{A[\mathbf{X}]}(R, C\otimes_A A[\mathbf{X}])\otimes_R (R\otimes_{A[\mathbf{X}]} U^{-1}A[\mathbf{X}]) \\
&\cong \operatorname{Ext}^d_{A[\mathbf{X}]}(R, C\otimes_A A[\mathbf{X}])\otimes_{A[\mathbf{X}]} U^{-1}A[\mathbf{X}] \\
&\cong \operatorname{Ext}^d_{U^{-1}A[\mathbf{X}]}(R\otimes_{A[\mathbf{X}]} U^{-1}A[\mathbf{X}], C\otimes_A A[\mathbf{X}]\otimes_{A[\mathbf{X}]} U^{-1}A[\mathbf{X}]) \\
&\cong \operatorname{Ext}^d_{U^{-1}A[\mathbf{X}]}(U^{-1}R, C\otimes_A U^{-1}A[\mathbf{X}]) \\
&\cong \operatorname{Ext}^d_{U^{-1}A[\mathbf{X}]}(U^{-1}R, (C\otimes_A U^{-1}A)\otimes_{U^{-1}A} U^{-1}A[\mathbf{X}]) \\
&\cong \operatorname{Ext}^d_{U^{-1}A[\mathbf{X}]}(U^{-1}R, U^{-1}A\otimes_{U^{-1}A} U^{-1}A[\mathbf{X}]) \\
&\cong \operatorname{Ext}^d_{U^{-1}A[\mathbf{X}]}(U^{-1}R, U^{-1}A[\mathbf{X}]) \\
&\overset{(6)}{\cong} \omega_{U^{-1}R}.
\end{aligned}
$$

For isomorphism (6), the ring $U^{-1}A[\mathbf{X}]$ is regular and surjects onto $U^{-1}R$ so that $\operatorname{Ext}^d_{U^{-1}A[\mathbf{X}]}(U^{-1}R, U^{-1}A[\mathbf{X}])$ is a dualizing module for $U^{-1}R$, and is therefore isomorphic to $\omega_{U^{-1}R}$ since the dualizing module of $U^{-1}R$ is unique up to isomorphism.

The preceding isomorphisms yield equality (7) in the next computation, while (8) is by (4.14.4), and the others are by definition; see, e.g., (4.14.2).

$$
\begin{aligned}
jfh([C],0) &= jf([C(\varphi)]) \\
&= j([C(\varphi)\otimes_R R_\delta],[C(\varphi)\otimes_R U^{-1}R]) \\
&\overset{(7)}{=} j([\omega_\rho\otimes_{R_\delta}(C\otimes_A R_\delta)],[\omega_{U^{-1}R}])
\end{aligned}
$$

$$
\begin{aligned}
&= ([\omega_\rho^{-1} \otimes_{R_\delta} \omega_\rho \otimes_{R_\delta} (C \otimes_A R_\delta)], [\omega_{U^{-1}R}]) \\
&\overset{(8)}{=} ([C \otimes_A R_\delta], [\omega_{U^{-1}R}]) \\
&= g([C], [\omega_{U^{-1}R}]) \\
&= gi'([C], 0).
\end{aligned}
$$

To complete the proof, we verify the behavior of the orderings. One implication follows from [23, (4.7),(5.7),(5.12)]: If $[C]_A \trianglelefteq [C']_A$ and $i \leq i'$, then $h([C]_A, i) \trianglelefteq h([C']_A, i')$. For the converse, assume that $h([C]_A, i) \trianglelefteq h([C']_A, i')$. By way of contradiction, suppose that $i > i'$, that is, $i = 1$ and $i' = 0$. Then our assumption is $[C \otimes_A R]_R \trianglelefteq [C'(\varphi)]_R$. The computations above provide isomorphisms

$$C \otimes_A R \otimes_R U^{-1}R \cong U^{-1}R \qquad \text{and} \qquad C'(\varphi) \otimes_R U^{-1}R \cong \omega_{U^{-1}R},$$

so that the order-respecting map $\mathfrak{S}_0(\gamma)$ yields $[U^{-1}R]_{U^{-1}R} \trianglelefteq [\omega_{U^{-1}R}]_{U^{-1}R}$, a contradiction since $U^{-1}R$ is not Gorenstein. Thus, we have $i \leq i'$. The final conclusion $[C]_A \trianglelefteq [C']_A$ follows from [23, (4.8),(5.8),(5.13)]. $\square$

Acknowledgments. I am grateful to Luchezar Avramov, Lars Christensen, Neil Epstein, Anders Frankild, Srikanth Iyengar, Graham Leuschke, Paul Roberts, and Roger Wiegand for stimulating conversations about this research. Many thanks also to the referee for his/her very thorough comments. I am especially grateful to Phillip Griffith for his tireless support and motivation. His work on the divisor class group and my many conversations with him inspired the research contained in this paper.

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