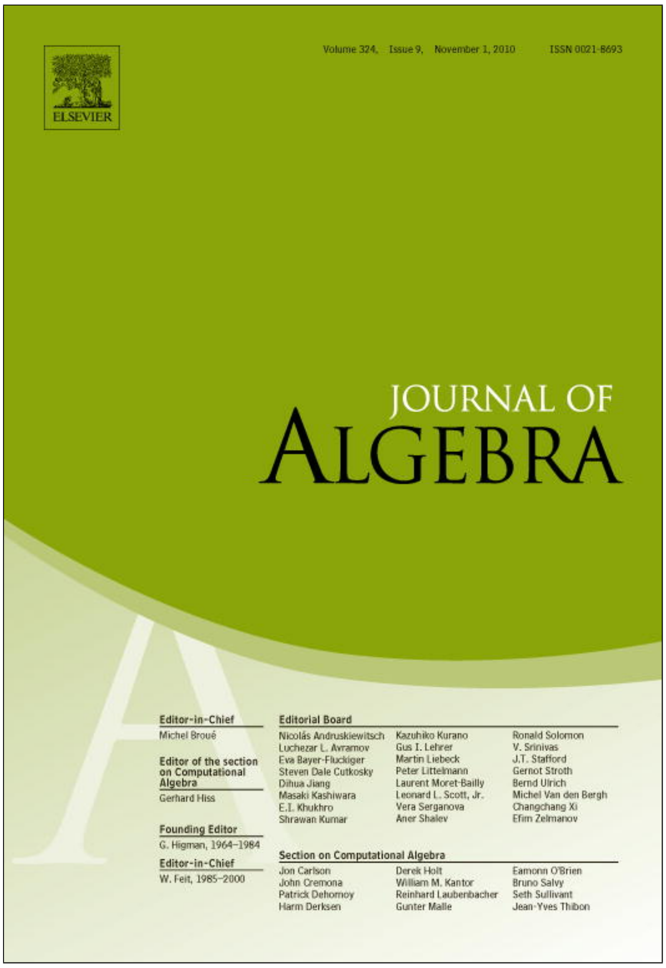
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Tate cohomology with respect to semidualizing modules

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ABSTRACT

We investigate Tate cohomology of modules over a commutative noetherian ring with respect to semidualizing modules. We identify classes of modules admitting Tate resolutions and analyze the interaction between the corresponding relative and Tate cohomology modules. As an application of our approach, we prove a general balance result for Tate cohomology. Our results are based on an analysis of Tate cohomology in abelian categories.

Introduction

This paper investigates Tate cohomology of objects in abelian categories, inspired by the work of Avramov and Martsinkovsky [3] and building from our own works [19,17,18]. Much of our motivation comes from certain categories of modules over a commutative ring R. For this introduction, we focus

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on this specific situation. (All rings in this paper are commutative with identity, and all modules are unital.)

An R-module C is *semidualizing* if $R \cong \mathrm{Hom}_R(C, C)$ and $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$. (See Section 2 for background information about these modules.) For example, the free module R is semidualizing, as is a dualizing module.

Each semidualizing R-module C comes equipped with a certain number of classes of R-modules that have good homological properties with respect to C. One example is the class of *C-projective* R-modules $\mathcal{P}_C(R)$, consisting of the modules of the form $P \otimes_R C$ for some projective R-module P. Another example is the class $\mathcal{G}(\mathcal{P}_C(R))$, containing the modules that are built by taking *complete resolutions* by modules in $\mathcal{P}_C(R)$. Other examples are the categories of modules M that admit a bounded resolution by modules from $\mathcal{P}_C(R)$ or from $\mathcal{G}(\mathcal{P}_C(R))$; these are the modules M with $\mathcal{P}_C\text{-pd}_R(M) < \infty$ or $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) < \infty$. For example, when $C = R$, the modules in $\mathcal{G}(\mathcal{P}_C(R))$ are the *Gorenstein projective R-modules*, and $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M)$ is the *Gorenstein projective dimension* of M.

The first step in constructing a theory of Tate cohomology with respect to C is to identify the modules M that admit appropriate resolutions: A *Tate $\mathcal{P}_C$-resolution* of M is a diagram of chain maps $T \to W \to M$ where T and W are certain chain complexes of modules from $\mathcal{P}_C(R)$. The complexes T and W contain slightly different homological information about M. For instance, W is a resolution of M which measures $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M)$ and $\mathcal{P}_C\text{-pd}_R(M)$. The following result characterizes the modules which admit Tate $\mathcal{P}_C$-resolutions. It is contained in Theorem 3.7.

Theorem A. *Let R be a commutative ring, and let C be a semidualizing R-module. An R-module M admits a Tate $\mathcal{P}_C$-resolution if and only if $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M)$ is finite.*

Given an R-module M with a Tate $\mathcal{P}_C$-resolution $T \to W \to M$, one uses the complex W to define the *relative cohomology* functors $\mathrm{Ext}^n_{\mathcal{G}(\mathcal{P}_C)}(M, -)$ and $\mathrm{Ext}^n_{\mathcal{P}_C}(M, -)$. The complex T is used to define the *Tate cohomology* functors $\widehat{\mathrm{Ext}}^n_{\mathcal{P}_C}(M, -)$. These cohomology functors are connected by the following result; it is proved in (4.11), and the dual result is Corollary 4.13. The special case where $C = R$ and M is finitely generated is in [3, (7.1)].

Theorem B. *Let R be a commutative ring, and let C be a semidualizing R-module. Let M and N be R-modules, and assume that $d = \mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) < \infty$. There is a long exact sequence that is natural in M and N*

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathcal{G}(\mathcal{P}_C)}(M,N) \to \mathrm{Ext}^1_{\mathcal{P}_C}(M,N) \to \widehat{\mathrm{Ext}}^1_{\mathcal{P}_C}(M,N) \\ \to \mathrm{Ext}^2_{\mathcal{G}(\mathcal{P}_C)}(M,N) \to \mathrm{Ext}^2_{\mathcal{P}_C}(M,N) \to \widehat{\mathrm{Ext}}^2_{\mathcal{P}_C}(M,N) \\ \cdots \to \mathrm{Ext}^d_{\mathcal{G}(\mathcal{P}_C)}(M,N) \to \mathrm{Ext}^d_{\mathcal{P}_C}(M,N) \to \widehat{\mathrm{Ext}}^d_{\mathcal{P}_C}(M,N) \to 0 \end{aligned}$$

and there are isomorphisms $\mathrm{Ext}^n_{\mathcal{P}_C}(M,N) \xrightarrow{\cong} \widehat{\mathrm{Ext}}^n_{\mathcal{P}_C}(M,N)$ for each $n > d$.

The next result shows how Tate cohomology detects the finiteness of $\mathcal{P}_C$-projective dimension. The proof is in (5.3); see also Corollary 5.5.

Theorem C. *Let R be a commutative ring, and let C be a semidualizing R-module. For an R-module M with $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) < \infty$, the next conditions are equivalent*:

(i) $\mathcal{P}_C\text{-pd}_R(M) < \infty$;
(ii) $\widehat{\mathrm{Ext}}^n_{\mathcal{P}_C}(-, M) = 0$ *for each* (*equivalently, for some*) $n \in \mathbb{Z}$;
(iii) $\widehat{\mathrm{Ext}}^n_{\mathcal{P}_C}(M, -) = 0$ *for each* (*equivalently, for some*) $n \in \mathbb{Z}$; *and*
(iv) $\widehat{\mathrm{Ext}}^0_{\mathcal{P}_C}(M, M) = 0$.

The following balance result is another one of our main theorems; it is proved in (6.2). Corollary 6.3 shows how it improves upon a result of Asadollahi and Salarian [1, (4.8)]. It also compliments work of Iacob [16, Theorem 2] and implies some of the main results of [20]; see Corollary 6.5.

Theorem D. *Let R be a commutative ring, and let B and C be semidualizing R-modules such that B is in $\mathcal{GP}_C(R)$. Set $B^\dagger = \operatorname{Hom}_R(B, C)$. Let M and N be R-modules such that $\mathcal{G}(\mathcal{P}_B)\text{-}\mathrm{pd}_R(M) < \infty$ and $\mathcal{G}(\mathcal{I}_{B^\dagger})\text{-}\mathrm{id}_R(N) < \infty$. Then there are isomorphisms for each $n \geqslant 1$*

$$\widehat{\operatorname{Ext}}^n_{\mathcal{P}_B}(M, N) \cong \widehat{\operatorname{Ext}}^n_{\mathcal{I}_{B^\dagger}}(M, N).$$

If R is noetherian and C is dualizing for R, this isomorphism holds for all $n \in \mathbb{Z}$.

We conclude this section by summarizing the contents of this paper. Section 1 contains notation and background information on the relevant subcategories of abelian categories. Section 2 specifies the examples arising from semidualizing modules. Section 3 focuses on the main properties of Tate resolutions; it contains the proof of Theorem A. In Section 4, we investigate the fundamental properties of Tate cohomology and prove Theorem B. Section 5 analyzes the vanishing behavior of these functors and contains the proof of Theorem C. Finally, Section 6 deals with balance for Tate cohomolgy including the proof of Theorem D.

1. Categories, resolutions, and relative cohomology

We begin with some notation and terminology for use throughout this paper.

Definition 1.1. Throughout this work $\mathcal{A}$ is an abelian category, and $\mathcal{A}b$ is the category of abelian groups. Write $\mathcal{P} = \mathcal{P}(\mathcal{A})$ and $\mathcal{I} = \mathcal{I}(\mathcal{A})$ for the subcategories of projective and injective objects in $\mathcal{A}$, respectively. We use the term "subcategory" to mean a "full and additive subcategory that is closed under isomorphisms". A subcategory $\mathcal{X}$ of $\mathcal{A}$ is *exact* if it is closed under direct summands and extensions; it satisfies the *two-of-three property* when it is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms.

Definition 1.2. We fix subcategories $\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{V} \subseteq \mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$. Write $\mathcal{X} \perp \mathcal{Y}$ if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(X, Y) = 0$ for each object $X \in \mathcal{X}$ and each object $Y \in \mathcal{Y}$. For an object $M \in \mathcal{A}$, write $M \perp \mathcal{Y}$ (resp., $\mathcal{X} \perp M$) if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(M, Y) = 0$ for each object $Y \in \mathcal{Y}$ (resp., if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(X, M) = 0$ for each object $X \in \mathcal{X}$). We say that $\mathcal{W}$ is a *cogenerator* for $\mathcal{X}$ if, for each object $X \in \mathcal{X}$, there is an exact sequence

$$0 \to X \to W \to X' \to 0$$

with $W \in \mathcal{W}$ and $X' \in \mathcal{X}$; and $\mathcal{W}$ is an *injective cogenerator* for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ such that $\mathcal{X} \perp \mathcal{W}$. The terms *generator* and *projective generator* are defined dually.

Definition 1.3. An *$\mathcal{A}$-complex* is a sequence of homomorphisms in $\mathcal{A}$

$$M = \cdots \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots$$

such that $\partial^M_{n-1}\partial^M_n = 0$ for each integer n. We frequently (and without warning) identify objects in $\mathcal{A}$ with complexes concentrated in degree 0.

Fix an integer i and an $\mathcal{A}$-complex M. The ith *homology object* of M is $\mathrm{H}_i(M) = \operatorname{Ker}(\partial^M_i)/\operatorname{Im}(\partial^M_{i+1})$. The ith *suspension* (or *shift*) of M, denoted by $\Sigma^i M$, is the complex with $(\Sigma^i M)_n = M_{n-i}$ and $\partial^{\Sigma^i M}_n = (-1)^i \partial^M_{n-i}$. We set $\Sigma M = \Sigma^1 M$. The *hard truncation* $M_{\geqslant i}$ is the complex

$$M_{\geqslant i} = \cdots \xrightarrow{\partial^M_{i+2}} M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \to 0$$

and the hard truncations $M_{>i}$, $M_{\leqslant i}$, and $M_{<i}$ are defined similarly.

Definition 1.4. Let M and N be $\mathcal{A}$-complexes. The *Hom-complex* $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is the complex of abelian groups defined as $\mathrm{Hom}_{\mathcal{A}}(M,N)_n = \prod_p \mathrm{Hom}_{\mathcal{A}}(M_p, N_{p+n})$ with nth differential $\partial_n^{\mathrm{Hom}_{\mathcal{A}}(M,N)}$ given by $\{f_p\} \mapsto \{\partial^N_{p+n} f_p - (-1)^n f_{p-1}\partial^M_p\}$. A *morphism* from M to N is an element of $\mathrm{Ker}(\partial_0^{\mathrm{Hom}_{\mathcal{A}}(M,N)})$; it is *null-homotopic* if it is in $\mathrm{Im}(\partial_1^{\mathrm{Hom}_{\mathcal{A}}(M,N)})$. The identity morphism $M \to M$ is denoted by id_M. The complex M is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact* if $\mathrm{Hom}_{\mathcal{A}}(X,M)$ is exact for each object $X \in \mathcal{X}$. The term $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$*-exact* is defined dually.

Fix morphisms of $\mathcal{A}$-complexes $\alpha, \alpha' : M \to N$. We say that α and α' are *homotopic* if the difference $\alpha - \alpha'$ is null-homotopic. The morphism α is a *homotopy equivalence* if there is a morphism $\beta : N \to M$ such that $\beta\alpha$ is homotopic to id_M and $\alpha\beta$ is homotopic to id_N. The complex M is *contractible* if id_M is null-homotopic.

For each integer i, the morphism α induces a morphism on homology objects $\mathrm{H}_i(\alpha) : \mathrm{H}_i(M) \to \mathrm{H}_i(N)$, and α is a *quasiisomorphism* when each $\mathrm{H}_i(\alpha)$ is an isomorphism. The *mapping cone* of α is the complex $\mathrm{Cone}(\alpha)$ defined as $\mathrm{Cone}(\alpha)_n = N_n \oplus M_{n-1}$ with nth differential $\partial_n^{\mathrm{Cone}(\alpha)} = \left(\begin{smallmatrix} \partial^N_n & \alpha_{n-1} \\ 0 & -\partial^M_{n-1} \end{smallmatrix}\right)$.

Fact 1.5. Let $\alpha : M \to N$ be a morphism of $\mathcal{A}$-complexes. There is a degreewise split exact sequence $0 \to \Sigma^{-1}N \to \Sigma^{-1}\mathrm{Cone}(\alpha) \to M \to 0$ of $\mathcal{A}$-complexes. The complex $\mathrm{Cone}(\mathrm{id}_M)$ is contractible.

If M is contractible, then it is exact and for every $\mathcal{A}$-complex L, the complexes $\mathrm{Hom}_{\mathcal{A}}(M,L)$ and $\mathrm{Hom}_{\mathcal{A}}(L,M)$ are exact.

Definition 1.6. Let X be an $\mathcal{A}$-complex. It is *bounded* if $X_n = 0$ for $|n| \gg 0$.

Assume that $X_{-n} = 0 = \mathrm{H}_n(X)$ for all $n > 0$ and that $M \cong \mathrm{H}_0(X)$. The natural morphism $X \to M$ is a quasiisomorphism. If each X_n is in $\mathcal{X}$, then X is an $\mathcal{X}$*-resolution* of M, and the associated exact sequence

$$X^+ = \cdots \xrightarrow{\partial^X_2} X_1 \xrightarrow{\partial^X_1} X_0 \to M \to 0$$

is the *augmented* $\mathcal{X}$*-resolution* of M associated to X. Sometimes we call the quasiisomorphism $X \xrightarrow{\simeq} M$ a resolution of M.

An $\mathcal{X}$-resolution X is *proper* if X^+ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact. We set

$$\operatorname{res}\widetilde{\mathcal{X}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a proper } \mathcal{X}\text{-resolution.}$$

The $\mathcal{X}$*-projective dimension* of M is the quantity

$$\mathcal{X}\text{-pd}(M) = \inf\bigl\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \bigm| X \text{ is an } \mathcal{X}\text{-resolution of } M\bigr\}.$$

The objects of $\mathcal{X}$-projective dimension 0 are exactly the objects of $\mathcal{X}$. We set

$$\operatorname{res}\widehat{\mathcal{X}} = \text{the subcategory of objects } M \in \mathcal{A} \text{ with } \mathcal{X}\text{-pd}(M) < \infty.$$

One checks readily that $\operatorname{res}\widetilde{\mathcal{X}}$ and $\operatorname{res}\widehat{\mathcal{X}}$ are subcategories of $\mathcal{A}$ that contain $\mathcal{X}$.

We define (*proper*) $\mathcal{Y}$*-coresolutions* and $\mathcal{Y}$*-injective dimension* dually. The *augmented* $\mathcal{Y}$*-coresolution* associated to a $\mathcal{Y}$-coresolution Y is denoted by ${}^+Y$, and the $\mathcal{Y}$-injective dimension of M is $\mathcal{Y}$-id(M). We set

$$\operatorname{cores}\widetilde{\mathcal{Y}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a proper } \mathcal{Y}\text{-coresolution},$$
$$\operatorname{cores}\widehat{\mathcal{Y}} = \text{the subcategory of objects } N \in \mathcal{A} \text{ with } \mathcal{Y}\text{-id}(N) < \infty$$

which are subcategories of $\mathcal{A}$ that contain $\mathcal{Y}$.

Auslander and Buchweitz [2, (1.1)] provide the next important constructions.

Definition 1.7. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are exact and that $\mathcal{W}$ and $\mathcal{V}$ are closed under direct summands. Assume that $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and that $\mathcal{V}$ is a generator for $\mathcal{Y}$, and fix an object $M \in \operatorname{res}\widehat{\mathcal{X}}$. There exist exact sequences in $\mathcal{A}$

$$0 \to K \to X_0 \to M \to 0, \qquad 0 \to M \to K' \to X' \to 0$$

such that $K, K' \in \operatorname{res}\widehat{\mathcal{W}}$ and $X_0, X' \in \mathcal{X}$. The first sequence is a *$\mathcal{WX}$-approximation* of M, and the second sequence is a *$\mathcal{WX}$-hull* of M. It follows that M admits a *bounded strict $\mathcal{WX}$-resolution*, that is, a bounded $\mathcal{X}$-resolution $X \xrightarrow{\simeq} M$ such that $X_i \in \mathcal{W}$ for each $i \geqslant 1$. This resolution is obtained by splicing a bounded $\mathcal{W}$-resolution of K with the $\mathcal{WX}$-approximation.

Similarly, an object N in $\operatorname{cores}\widehat{\mathcal{Y}}$ admits a *bounded strict $\mathcal{YV}$-coresolution*, that is, a bounded $\mathcal{Y}$-coresolution $N \xrightarrow{\simeq} Y$ such that $Y_i \in \mathcal{V}$ for each $i \leqslant -1$.

Definition 1.8. Let $f : M \to M'$ and $g : N \to N'$ be morphisms in $\mathcal{A}$. If M admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\gamma} M$, then for each integer n the *nth relative cohomology group* $\operatorname{Ext}^n_{\mathcal{WA}}(M, N)$ is

$$\operatorname{Ext}^n_{\mathcal{WA}}(M, N) = \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(W, N)\big).$$

If M' also admits a proper $\mathcal{W}$-resolution $W' \xrightarrow{\gamma} M'$, then [17, (1.8.a)] yields a lifting $\overline{f} : W \to W'$ of f that is unique up to homotopy, and we define

$$\operatorname{Ext}^n_{\mathcal{WA}}(f, N) = \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(\overline{f}, N)\big) : \operatorname{Ext}^n_{\mathcal{WA}}\big(M', N\big) \to \operatorname{Ext}^n_{\mathcal{WA}}(M, N),$$
$$\operatorname{Ext}^n_{\mathcal{WA}}(M, g) = \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(W, g)\big) : \operatorname{Ext}^n_{\mathcal{WA}}(M, N) \to \operatorname{Ext}^n_{\mathcal{WA}}\big(M, N'\big).$$

We write $\operatorname{Ext}^{\geqslant 1}_{\mathcal{WA}}(M, \mathcal{Y}) = 0$ if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{WA}}(M, Y) = 0$ for each object $Y \in \mathcal{Y}$. When $\mathcal{X} \subseteq \operatorname{res}\widetilde{\mathcal{W}}$, we write $\operatorname{Ext}^{\geqslant 1}_{\mathcal{WA}}(\mathcal{X}, \mathcal{Y}) = 0$ if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{WA}}(X, Y) = 0$ for each object $X \in \mathcal{X}$ and each object $Y \in \mathcal{Y}$.

When N and N' admit proper $\mathcal{V}$-coresolutions, the *nth relative cohomology group* $\operatorname{Ext}^n_{\mathcal{AV}}(M, N)$ is defined dually, as are the maps

$$\operatorname{Ext}^n_{\mathcal{AV}}(f, N) : \operatorname{Ext}^n_{\mathcal{AV}}\big(M', N\big) \to \operatorname{Ext}^n_{\mathcal{AV}}(M, N),$$
$$\operatorname{Ext}^n_{\mathcal{AV}}(M, g) : \operatorname{Ext}^n_{\mathcal{AV}}(M, N) \to \operatorname{Ext}^n_{\mathcal{AV}}\big(M, N'\big)$$

and similarly for the conditions $\operatorname{Ext}^{\geqslant 1}_{\mathcal{AV}}(\mathcal{X}, N) = 0$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{AV}}(\mathcal{X}, \mathcal{Y}) = 0$.

Definition 1.9. Let M, N be objects in $\mathcal{A}$. If M admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\gamma} M$ and a proper $\mathcal{X}$-resolution $X \xrightarrow{\gamma'} M$, let $\overline{\mathrm{id}_M} : W \to X$ be a lifting of the identity $\mathrm{id}_M : M \to M$, cf. [17, (1.8.a)]. This is a quasiisomorphism such that $\gamma = \gamma' \overline{\mathrm{id}_M}$. We set

$$\vartheta^n_{\mathcal{XWA}}(M, N) = \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(\overline{\mathrm{id}_M}, N)\big) : \operatorname{Ext}^n_{\mathcal{XA}}(M, N) \to \operatorname{Ext}^n_{\mathcal{WA}}(M, N).$$

When N admits a proper $\mathcal{Y}$-coresolution and a proper $\mathcal{V}$-coresolution, the map

$$\vartheta^n_{\mathcal{A}\mathcal{Y}\mathcal{V}}(M,N):\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,N)\to\operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}(M,N)$$

is defined similarly.

Fact 1.10. Let R be a commutative ring, and assume that $\mathcal{W}$ is a subcategory of $\mathcal{A}=\mathcal{M}(R)$. Let M, M', N, N' be R-modules equipped with R-module homomorphisms $f:M\to M'$ and $g:N\to N'$. If M admits a proper $\mathcal{W}$-resolution, then each group $\operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M,N)$ is an R-module. If M' also admits a proper $\mathcal{W}$-resolution, then the maps $\operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(f,N)$ and $\operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M,g)$ are R-module homomorphisms. Similar comments hold for $\operatorname{Ext}_{\mathcal{A}\mathcal{V}}$ and the maps from Definition 1.9.

Fact 1.11. The uniqueness of the liftings in [17, (1.8)] shows that

$$\operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}:\operatorname{res}\widetilde{\mathcal{W}}\times\mathcal{A}\to\mathcal{A}b\quad\text{and}\quad\operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}:\mathcal{A}\times\operatorname{cores}\widetilde{\mathcal{V}}\to\mathcal{A}b$$

are well-defined bifunctors, and

$$\vartheta_{\mathcal{X}\mathcal{W}\mathcal{A}}:\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}\big|_{(\operatorname{res}\widetilde{\mathcal{W}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{A}}\to\operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}\big|_{(\operatorname{res}\widetilde{\mathcal{W}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{A}},$$
$$\vartheta^n_{\mathcal{A}\mathcal{Y}\mathcal{V}}:\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}\big|_{\mathcal{A}\times(\operatorname{cores}\widetilde{\mathcal{V}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})}\to\operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}\big|_{\mathcal{A}\times(\operatorname{cores}\widetilde{\mathcal{V}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})}$$

are the well-defined natural transformations, independent of resolutions and liftings.

One has $\operatorname{Ext}^{\geqslant 1}_{\mathcal{X}\mathcal{A}}(\mathcal{X},-)=0$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{Y}}(-,\mathcal{Y})=0$. There is a natural equivalence $\operatorname{Ext}^0_{\mathcal{X}\mathcal{A}}\cong\operatorname{Hom}_{\mathcal{A}}$ on $\operatorname{res}\widetilde{\mathcal{X}}\times\mathcal{A}$, and similarly $\operatorname{Ext}^0_{\mathcal{A}\mathcal{Y}}\cong\operatorname{Hom}_{\mathcal{A}}$ on $\mathcal{A}\times\operatorname{cores}\widetilde{\mathcal{Y}}$.

We conclude this section by summarizing some aspects of [18].

Definition 1.12. An exact complex in $\mathcal{W}$ is *totally $\mathcal{W}$-acyclic* if it is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Let $\mathcal{G}(\mathcal{W})$ denote the subcategory of $\mathcal{A}$ whose modules are of the form $M\cong\operatorname{Coker}(\partial^W_1)$ for some totally $\mathcal{W}$-acyclic complex W in $\mathcal{W}$; we say that W is a *complete $\mathcal{W}$-resolution of M*.

Fact 1.13. A contractible $\mathcal{W}$-complex is totally $\mathcal{W}$-acyclic; see Fact 1.5.

It is straightforward to show that $\mathcal{W}$ is a subcategory of $\mathcal{G}(\mathcal{W})$: if $N\in\mathcal{W}$, then the complex $0\to N\xrightarrow{\operatorname{id}_N}N\to 0$ is a complete $\mathcal{W}$-resolution of N.

Let M be an object in $\mathcal{G}(\mathcal{W})$ with complete $\mathcal{W}$-resolution W. The hard truncation $W_{\geqslant 0}$ is a proper $\mathcal{W}$-resolution of M such that $W^+_{\geqslant 0}$ is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, and $W_{<0}$ is a proper $\mathcal{W}$-coresolution of M such that ${}^+W_{<0}$ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact. So, one has $M\in\operatorname{res}\widetilde{\mathcal{W}}\cap\operatorname{cores}\widetilde{\mathcal{W}}$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(M,\mathcal{W})=0=\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{W}}(\mathcal{W},M)$.

Using standard arguments, one sees readily that any complete $\mathcal{W}$-resolution is $\operatorname{Hom}_{\mathcal{A}}(\operatorname{cores}\widehat{\mathcal{W}},-)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-,\operatorname{res}\widehat{\mathcal{W}})$-exact.

Fact 1.14. Assume that $\mathcal{W}\perp\mathcal{W}$. We have $\mathcal{G}(\mathcal{W})\perp\operatorname{res}\widehat{\mathcal{W}}$ and $\operatorname{cores}\widehat{\mathcal{W}}\perp\mathcal{G}(\mathcal{W})$. The category $\mathcal{G}(\mathcal{W})$ is exact, and $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{W})$. If $\mathcal{W}$ is closed under kernels of epimorphisms or under cokernels of monomorphisms, then so is $\mathcal{G}(\mathcal{W})$. See [18, (4.3), (4.5), (4.7), (4.11), (4.12)].

2. Semidualizing modules and associated categories

Much of the motivation for this work comes from the module categories discussed in this section, wherein R is a commutative ring.

Definition 2.1. Let $\mathcal{M}(R)$ denote the category of R-modules. We write $\mathcal{P}(R)$ and $\mathcal{I}(R)$ for the subcategories of projective R-modules and injective R-modules.

The study of semidualizing modules was initiated independently (with different names) by Foxby [8], Golod [12], and Vasconcelos [22].

Definition 2.2. An R-module C is *semidualizing* if it satisfies the following:

(1) C admits a (possibly unbounded) resolution by finite rank free R-modules;
(2) The natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism; and
(3) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

A finitely generated projective R-module of rank 1 is semidualizing. If R is Cohen–Macaulay, then C is *dualizing* if it is semidualizing and $\mathrm{id}_R(C)$ is finite.

Over a noetherian ring, the next categories were introduced by Foxby [9] when C is dualizing, and by Vasconcelos [22, Section 4.4] for arbitrary C, with different notation. In the non-noetherian setting, see Holm and White [15] and White [23].

Definition 2.3. Let C be a semidualizing R-module.
The *Auslander class* of C is the subcategory $\mathcal{A}_C(R)$ of R-modules M such that

(1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0 = \mathrm{Ext}_R^{\geqslant 1}(C, C \otimes_R M)$, and
(2) the natural map $M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* of C is the subcategory $\mathcal{B}_C(R)$ of R-modules N such that

(1) $\mathrm{Ext}_R^{\geqslant 1}(C, M) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, M))$, and
(2) the natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

Based on the work of Enochs and Jenda [7], the following notions were introduced and studied in this generality by Holm and Jørgensen [14] and White [23].

Definition 2.4. Let C be a semidualizing R-module, and set

$$\mathcal{P}_C(R) = \text{the subcategory of modules } M \cong P \otimes_R C \text{ where } P \text{ is } R\text{-projective},$$

$$\mathcal{I}_C(R) = \text{the subcategory of modules } N \cong \mathrm{Hom}_R(C, I) \text{ where } I \text{ is } R\text{-injective}.$$

Modules in $\mathcal{P}_C(R)$ and $\mathcal{I}_C(R)$ are called C-*projective* and C-*injective*, respectively.
A *complete* $\mathcal{P}\mathcal{P}_C$*-resolution* is a complex X of R-modules satisfying the following:

(1) X is exact and $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact; and
(2) X_i is projective for $i \geqslant 0$ and X_i is C-projective for $i < 0$.

An R-module M is G_C*-projective* if there exists a complete $\mathcal{P}\mathcal{P}_C$-resolution X such that $M \cong \mathrm{Coker}(\partial_1^X)$, and X is a *complete* $\mathcal{P}\mathcal{P}_C$*-resolution of* M. Set

$$\mathcal{GP}_C(R) = \text{the subcategory of } \mathrm{G}_C\text{-projective } R\text{-modules}.$$

In the case $C = R$ we use the more common terminology "complete projective resolution" and "Gorenstein projective module" and the notation $\mathcal{GP}(R)$.

A *complete* $\mathcal{I}_C\mathcal{I}$*-coresolution* is a complex Y of R-modules such that:

(1) Y is exact and $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$-exact; and
(2) Y_i is injective for $i \leqslant 0$ and Y_i is C-injective for $i > 0$.

An R-module N is G_C*-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-coresolution Y such that $N \cong \mathrm{Ker}(\partial_0^Y)$, and Y is a *complete* $\mathcal{I}_C\mathcal{I}$*-coresolution of* N. Set

$$\mathcal{GI}_C(R) = \text{the subcategory of } \mathrm{G}_C\text{-injective } R\text{-modules}.$$

In the case $C = R$ we use the more common terminology "complete injective resolution" and "Gorenstein injective module" and the notation $\mathcal{GI}(R)$.

Notation 2.5. Let C be a semidualizing R-module. We abbreviate as follows:

$$\begin{aligned}
\mathrm{pd}_R(-) &= \mathcal{P}(R)\text{-pd}(-), & \mathrm{id}_R(-) &= \mathcal{I}(R)\text{-id}(-),\\
\mathcal{P}_C\text{-pd}_R(-) &= \mathcal{P}_C(R)\text{-pd}(-), & \mathcal{I}_C\text{-id}_R(-) &= \mathcal{I}_C(R)\text{-id}(-),\\
\mathcal{GP}\text{-pd}_R(-) &= \mathcal{GP}(R)\text{-pd}(-), & \mathcal{GI}\text{-id}_R(-) &= \mathcal{GI}(R)\text{-id}(-),\\
\mathcal{GP}_C\text{-pd}_R(-) &= \mathcal{GP}_C(R)\text{-pd}(-), & \mathcal{GI}_C\text{-id}_R(-) &= \mathcal{GI}_C(R)\text{-id}(-),\\
\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(-) &= \mathcal{G}\big(\mathcal{P}_C(R)\big)\text{-pd}(-), & \mathcal{G}(\mathcal{I}_C)\text{-id}_R(-) &= \mathcal{G}\big(\mathcal{I}_C(R)\big)\text{-id}(-).
\end{aligned}$$

Fact 2.6. Let B and C be semidualizing R-modules. The Auslander class $\mathcal{A}_C(R)$ contains every projective R-module and every C-injective R-module, and the Bass class $\mathcal{B}_C(R)$ contains every injective R-module and every C-projective R-module; see [15, Lemmas 4.1, 5.1]. These classes also satisfy the two-of-three property by [15, Corollary 6.3]. Hence, $\mathcal{A}_C(R)$ contains the R-modules of finite projective dimension and the R-modules of finite $\mathcal{I}_C$-injective dimension, and $\mathcal{B}_C(R)$ contains the R-modules of finite injective dimension and the R-modules of finite $\mathcal{P}_C$-projective dimension. From [21, (2.8)] we know that an R-module M is in $\mathcal{B}_C(R)$ if and only if $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C(R)$, and that $M \in \mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(R)$.

The category $\mathcal{P}_C(R)$ is exact and closed under kernels of epimorphisms by [15, Proposition 5.1(b)] and [23, (2.8)]. Also, $\mathcal{P}_C(R)$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{P}_C(R)) = \mathcal{GP}_C(R) \cap \mathcal{B}_C(R)$, and $\mathcal{G}(\mathcal{P}_C(R))$ is exact and closed under kernels of epimorphisms; see [18, Sections 4–5]. In particular, $\mathcal{P}_C(R) \perp \mathcal{P}_C(R)$.

The category $\mathcal{I}_C(R)$ is exact and closed under cokernels of monomorphisms. Also, $\mathcal{I}_C(R)$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{I}_C(R)) = \mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$, and $\mathcal{G}(\mathcal{I}_C(R))$ is exact and closed under cokernels of monomorphisms; see [18, Sections 4–5]. In particular, we have $\mathcal{I}_C(R) \perp \mathcal{I}_C(R)$.

If $B \in \mathcal{GP}_C(R)$, then $\mathrm{Hom}_R(B, C)$ is also semidualizing; see, e.g. [6, (2.11)]. If C is dualizing, then $B \in \mathcal{GP}_C(R)$ and $C \cong B \otimes_R \mathrm{Hom}_R(B, C)$; see [5, (3.3.10)], [11, (3.3)] and [23, (4.4)].

The next lemma is from an early version of [21]. We are grateful to Takahashi and White for allowing us to include it here.

Lemma 2.7. *Let R be a commutative ring, and let C be a semidualizing R-module. Assume that R is Cohen–Macaulay with a dualizing module D, and set $C^\dagger = \mathrm{Hom}_R(C, D)$. For each R-module M, one has $\mathcal{P}_C\text{-pd}_R(M) < \infty$ if and only if $\mathcal{I}_{C^\dagger}\text{-id}_R(M) < \infty$.*

Proof. We prove the forward implication; the proof of the converse is similar. Assume that $n = \mathcal{P}_C\text{-pd}_R(M) < \infty$. The category of modules of finite $\mathcal{I}_{C^\dagger}$-id satisfies the two-of-three property by [21, (3.4)]. Hence, using a routine induction argument on n, it suffices to assume that $n = 0$ and prove that $\mathcal{I}_{C^\dagger}\text{-id}_R(M) < \infty$.

So, we assume that there is a projective R-module P such that $M \cong C \otimes_R P$. This yields the second equality in the next sequence

$$\mathcal{I}_{C^\dagger}\text{-id}_R(M) = \text{id}_R\big(C^\dagger \otimes_R M\big) = \text{id}_R\big(C^\dagger \otimes_R C \otimes_R P\big) = \text{id}_R(D \otimes_R P) < \infty.$$

The first equality is from [21, (2.11.b)], and the third equality is from Fact 2.6. The finiteness follows from the fact that $\text{id}_R(D)$ is finite and P is projective. □

The next three lemmas are for use in Corollary 6.5.

Lemma 2.8. *Let R be a commutative ring, and let C be a semidualizing R-module. Let M be an R-module.*

(a) *M is in $\mathcal{G}(\mathcal{P}_C(R))$ if and only if $\text{Hom}_R(C, M)$ is in $\mathcal{GP}(R) \cap \mathcal{A}_C(R)$.*
(b) *M is in $\mathcal{G}(\mathcal{I}_C(R))$ if and only if $C \otimes_R M$ is in $\mathcal{GI}(R) \cap \mathcal{B}_C(R)$.*

Proof. We prove part (b); the proof of (a) is dual.

Assume first that $M \in \mathcal{G}(\mathcal{I}_C(R))$, and fix a complete $\mathcal{I}_C$-resolution Y of M. Fact 2.6 implies that $M \in \mathcal{A}_C(R)$, and that $C \otimes_R M \in \mathcal{B}_C(R)$. Since each module Y_i is in $\mathcal{I}_C(R)$, it is straightforward to show that the complex $C \otimes_R Y$ is a complex of injective R-modules; see, e.g., [15, Theorem 1]. Also, since the modules $M \cong \text{Ker}(\partial_0^Y)$ and Y_i are all in $\mathcal{A}_C(R)$, it is straightforward to show that $C \otimes_R Y$ is exact and that $C \otimes_R M \cong \text{Ker}(\partial_0^{C \otimes_R Y})$.

By assumption, the complex Y is $\text{Hom}_R(\mathcal{I}_C(R), -)$-exact. Hence, for each injective R-module I, the following complex is exact

$$\begin{aligned}\text{Hom}_R\big(\text{Hom}_R(C, I), Y\big) &\cong \text{Hom}_R\big(\text{Hom}_R(C, I), \text{Hom}_R(C, C \otimes_R Y)\big)\\ &\cong \text{Hom}_R\big(C \otimes_R \text{Hom}_R(C, I), C \otimes_R Y\big)\\ &\cong \text{Hom}_R(I, C \otimes_R Y).\end{aligned}$$

In this sequence, the first isomorphism comes from the fact that each Y_i is in $\mathcal{A}_C(R)$. The second isomorphism is Hom-tensor adjointness, and the third isomorphism is due to the condition $I \in \mathcal{B}_C(R)$. It follows that $C \otimes_R Y$ is a complete injective resolution of $C \otimes_R M$, so we have $C \otimes_R M \in \mathcal{GI}(R)$.

For the converse, assume that $C \otimes_R M \in \mathcal{GI}(R) \cap \mathcal{B}_C(R)$. Fact 2.6 implies that $M \in \mathcal{A}_C(R)$. Let Z be a complete injective resolution of $C \otimes_R M$. Since the modules $C \otimes_R M$ and Z_i are in $\mathcal{B}_C(R)$, we conclude that the complex $\text{Hom}_R(C, Z)$ is exact with $M \cong \text{Ker}(\partial_0^{\text{Hom}_R(C,Z)})$. Thus, to conclude the proof, we need to show that $\text{Hom}_R(C, Z)$ is $\text{Hom}_R(\mathcal{I}_C(R), -)$-exact and $\text{Hom}_R(-, \mathcal{I}_C(R))$-exact. Let J be an injective R-module. Since Z is $\text{Hom}_R(\mathcal{I}(R), -)$-exact, the next complex is exact

$$\begin{aligned}\text{Hom}_R(J, Z) &\cong \text{Hom}_R\big(C \otimes_R \text{Hom}_R(C, J), Z\big)\\ &\cong \text{Hom}_R\big(\text{Hom}_R(C, J), \text{Hom}_R(C, Z)\big).\end{aligned}$$

The isomorphisms are from the condition $J \in \mathcal{B}_C(R)$ and from Hom-tensor adjointness, respectively. Thus, $\text{Hom}_R(C, Z)$ is $\text{Hom}_R(\mathcal{I}_C(R), -)$-exact.

The complex $\text{Hom}_R(C, Z)$ consists of modules in $\mathcal{I}_C(R) \subseteq \mathcal{A}_C(R)$ and has $\text{Ker}(\partial_0^{\text{Hom}_R(C,Z)}) \cong \text{Hom}_R(C, M) \in \mathcal{A}_C(R)$. It follows that $C \otimes_R \text{Hom}_R(C, Z)$ is exact. The fact that J is injective implies that the next complex is exact

$$\mathrm{Hom}_R\big(C \otimes_R \mathrm{Hom}_R(C,Z),\ J\big) \cong \mathrm{Hom}_R\big(\mathrm{Hom}_R(C,Z), \mathrm{Hom}_R(C,\ J)\big)$$

where the isomorphism is from Hom-tensor adjointness. It follows that $\mathrm{Hom}_R(C,Z)$ is $\mathrm{Hom}_R(-,\mathcal{I}_C(R))$-exact, as desired. □

The next two results improve upon Lemma 2.8 and compliment [14, (4.2), (4.3)]. The proof of Lemma 2.10 is dual to that of Lemma 2.9.

Lemma 2.9. *Let R be a commutative ring, and let C be a semidualizing R-module. For an R-module M, the following conditions are equivalent*:

(i) $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) < \infty$;
(ii) $\mathcal{GP}_C\text{-pd}_R(M) < \infty$ *and* $M \in \mathcal{B}_C(R)$; *and*
(iii) $\mathcal{GP}\text{-pd}_R(\mathrm{Hom}_R(C,M)) < \infty$ *and* $M \in \mathcal{B}_C(R)$.

When these conditions are satisfied, we have

$$\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) = \mathcal{GP}_C\text{-pd}_R(M) = \mathcal{GP}\text{-pd}_R\big(\mathrm{Hom}_R(C,M)\big). \tag{2.9.1}$$

Proof. (i) ⇒ (ii). Assume that $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) < \infty$. Since $\mathcal{G}(\mathcal{P}_C(R))$ is a subcategory of $\mathcal{GP}_C(R)$, we have $\mathcal{GP}_C\text{-pd}_R(M) \leqslant \mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) < \infty$. As $\mathcal{B}_C(R)$ satisfies the two-of-three property and contains $\mathcal{G}(\mathcal{P}_C(R))$, we have $M \in \mathcal{B}_C(R)$.

(ii) ⇒ (iii). Assume that $g = \mathcal{GP}_C\text{-pd}_R(M) < \infty$ and $M \in \mathcal{B}_C(R)$. The condition $M \in \mathcal{B}_C(R)$ implies that M has a proper $\mathcal{P}_C(R)$-resolution T by [21, (2.3)]. In particular, the complex $\mathrm{Hom}_R(C,T^+)$ is exact, and it follows that $\mathrm{Hom}_R(C,T)$ is a projective resolution of $\mathrm{Hom}_R(C,M)$ with

$$\mathrm{Coker}\big(\partial_{g+1}^{\mathrm{Hom}_R(C,T)}\big) \cong \mathrm{Hom}_R\big(C, \mathrm{Coker}\big(\partial_{g+1}^T\big)\big).$$

Since each T_i is in $\mathcal{P}_C(R) \subseteq \mathcal{GP}_C(R)$, the condition $g = \mathcal{GP}_C\text{-pd}_R(M) < \infty$ implies that $K_g = \mathrm{Coker}(\partial_{g+1}^T)$ is G_C-projective; see [14, (2.16)] and [13, (2.20)]. Since M is in $\mathcal{B}_C(R)$ and each T_i is in $\mathcal{B}_C(R)$, Fact 2.6 implies that $K_g \in \mathcal{B}_C(R)$. It follows that K_g is in $\mathcal{GP}_C(R) \cap \mathcal{B}_C(R) = \mathcal{G}(\mathcal{P}_C(R))$, so Lemma 2.8(a) implies that $\mathrm{Hom}_R(C,K_g) \in \mathcal{GP}(R)$. Hence, the exact sequence

$$0 \to \mathrm{Hom}_R(C,K_g) \to \mathrm{Hom}_R(C,T_{g-1}) \to \cdots \to \mathrm{Hom}_R(C,T_0) \to \mathrm{Hom}_R(C,M) \to 0$$

shows that $\mathcal{GP}\text{-pd}_R(\mathrm{Hom}_R(C,M)) \leqslant g = \mathcal{GP}_C\text{-pd}_R(M) < \infty$.

(iii) ⇒ (i). Assume that $d = \mathcal{GP}\text{-pd}_R(\mathrm{Hom}_R(C,M)) < \infty$ and $M \in \mathcal{B}_C(R)$. Let T be a proper $\mathcal{P}_C(R)$-resolution of M, and set $K_d = \mathrm{Coker}(\partial_{d+1}^T)$. As in the previous paragraph, $\mathrm{Hom}_R(C,T)$ is a projective resolution of $\mathrm{Hom}_R(C,M)$ with

$$\mathrm{Hom}_R(C,K_d) \cong \mathrm{Coker}\big(\partial_{d+1}^{\mathrm{Hom}_R(C,T)}\big) \in \mathcal{A}_C(R).$$

The fact that $d = \mathcal{GP}\text{-pd}_R(\mathrm{Hom}_R(C,M)) < \infty$ implies that $\mathrm{Coker}(\partial_{d+1}^{\mathrm{Hom}_R(C,T)})$ is Gorenstein projective, and we conclude from Lemma 2.8(a) that $K_d \in \mathcal{G}(\mathcal{P}_C(R))$. Hence the exact sequence $0 \to K_d \to T_{d-1} \to \cdots \to T_0 \to 0$ shows that we have $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) \leqslant d = \mathcal{GP}\text{-pd}_R(\mathrm{Hom}_R(C,M)) < \infty$.

Finally, assume that conditions (i)–(iii) are satisfied. The proofs of the three implications yield the inequalities in the next sequence:

$$\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) \leqslant \mathcal{GP}_C\text{-pd}_R(M) \leqslant \mathcal{GP}\text{-pd}_R\big(\mathrm{Hom}_R(C,M)\big) \leqslant \mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M).$$

This verifies the equalities in (2.9.1). □

Lemma 2.10. *Let R be a commutative ring, and let C be a semidualizing R-module. For an R-module M, the following conditions are equivalent:*

(i) $\mathcal{G}(\mathcal{I}_C)\text{-id}_R(M) < \infty$;
(ii) $\mathcal{GI}_C\text{-id}_R(M) < \infty$ *and* $M \in \mathcal{A}_C(R)$; *and*
(iii) $\mathcal{GI}\text{-id}_R(C \otimes_R M) < \infty$ *and* $M \in \mathcal{A}_C(R)$.

When these conditions are satisfied, we have

$$\mathcal{G}(\mathcal{I}_C)\text{-id}_R(M) = \mathcal{GI}_C\text{-id}_R(M) = \mathcal{GI}\text{-id}_R(C \otimes_R M).$$

Remark 2.11. Lemmas 2.9–2.10 have the following interpretations in terms of "Foxby equivalence". Fact 2.6 shows that the functors $C \otimes_R -$ and $\operatorname{Hom}_R(C, -)$ provide natural equivalences between the Auslander and Bass classes, as we indicate in the middle row of the following diagram:

$$\begin{array}{ccc}
\mathcal{A}_C(R) \cap \text{res-}\widehat{\mathcal{GP}}_{\leqslant n} & \underset{\sim}{\rightleftarrows} & \text{res-}\widehat{\mathcal{G}(\mathcal{P}_C)}_{\leqslant n} \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}_C(R) & \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{B}_C(R) \\
\big\uparrow & & \big\uparrow \\
\text{cores-}\widehat{\mathcal{G}(\mathcal{I}_C)}_{\leqslant n} & \underset{\sim}{\rightleftarrows} & \mathcal{B}_C(R) \cap \text{cores-}\widehat{\mathcal{GI}}_{\leqslant n}.
\end{array}$$

The equivalences in the top and bottom rows of the diagram follow from Lemmas 2.9 and 2.10, using the equivalence in the middle row. Here, the notation $\text{res-}\widehat{\mathcal{GP}}_{\leqslant n}$ stands for the category of R-modules M with $\mathcal{GP}\text{-pd}_R(M) \leqslant n$, et cetera.

The final three results of this section are proved like Lemmas 2.8–2.10.

Lemma 2.12. *Let R be a commutative ring, and let C be a semidualizing R-module. Let M be an R-module.*

(a) *M is in $\mathcal{GP}(R) \cap \mathcal{A}_C(R)$ if and only if $C \otimes_R M$ is in $\mathcal{G}(\mathcal{P}_C(R))$.*
(b) *M is in $\mathcal{GI}(R) \cap \mathcal{B}_C(R)$ if and only if $\operatorname{Hom}_R(C, M)$ is in $\mathcal{G}(\mathcal{I}_C(R))$.*

Lemma 2.13. *Let R be a commutative ring, and let C be a semidualizing R-module. For an R-module M, the following conditions are equivalent*:

(i) $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(C \otimes_R M) < \infty$;
(ii) $\mathcal{GP}_C\text{-pd}_R(C \otimes_R M) < \infty$ *and* $M \in \mathcal{A}_C(R)$; *and*
(iii) $\mathcal{GP}\text{-pd}_R(M) < \infty$ *and* $M \in \mathcal{A}_C(R)$.

When these conditions are satisfied, we have

$$\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(C \otimes_R M) = \mathcal{GP}_C\text{-pd}_R(C \otimes_R M) = \mathcal{GP}\text{-pd}_R(M).$$

Lemma 2.14. *Let R be a commutative ring, and let C be a semidualizing R-module. For an R-module M, the following conditions are equivalent*:

(i) $\mathcal{G}(\mathcal{I}_C)\text{-id}_R(\operatorname{Hom}_R(C, M)) < \infty$;

(ii) $\mathcal{GI}_C\text{-id}_R(\mathrm{Hom}_R(C,M)) < \infty$ *and* $M \in \mathcal{B}_C(R)$; *and*
(iii) $\mathcal{GI}\text{-id}_R(M) < \infty$ *and* $M \in \mathcal{B}_C(R)$.

When these conditions are satisfied, we have

$$\mathcal{G}(\mathcal{I}_C)\text{-id}_R\big(\mathrm{Hom}_R(C,M)\big) = \mathcal{GI}_C\text{-id}_R\big(\mathrm{Hom}_R(C,M)\big) = \mathcal{GI}\text{-id}_R(M).$$

3. Tate resolutions

In this section, we study the resolutions used to define our Tate cohomology functors. In many cases, the objects admitting such resolutions are precisely the objects of finite $\mathcal{G}(\mathcal{W})$-projective/injective dimension; see Theorems 3.6 and 3.7.

Definition 3.1. Let M and N be objects in $\mathcal{A}$.

A *Tate $\mathcal{W}$-resolution* of M is a diagram $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ of morphisms of $\mathcal{A}$-complexes wherein T is an exact $\mathcal{W}$-complex that is totally $\mathcal{W}$-acyclic, γ is a proper $\mathcal{W}$-resolution of M, and α_n is an isomorphism for $n \gg 0$. We set

$$\mathrm{res}\,\overline{\mathcal{W}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a Tate } \mathcal{W}\text{-resolution.}$$

A *Tate $\mathcal{V}$-coresolution* of N is a diagram $N \xrightarrow{\delta} V \xrightarrow{\beta} S$ of morphisms of $\mathcal{A}$-complexes wherein S is an exact $\mathcal{V}$-complex that is totally $\mathcal{V}$-acyclic, δ is a proper $\mathcal{V}$-coresolution of N, and β_n is an isomorphism for $n \ll 0$. We set

$$\mathrm{cores}\,\overline{\mathcal{V}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a Tate } \mathcal{V}\text{-coresolution.}$$

Fact 3.2. Given $M', M'' \in \mathrm{res}\,\overline{\mathcal{W}}$ with Tate $\mathcal{W}$-resolutions $T' \xrightarrow{\alpha'} W' \xrightarrow{\gamma'} M'$ and $T'' \xrightarrow{\alpha''} W'' \xrightarrow{\gamma''} M''$, one readily shows that the direct sum

$$T' \oplus T'' \xrightarrow{\left(\begin{smallmatrix}\alpha' & 0\\ 0 & \alpha''\end{smallmatrix}\right)} W \oplus W'' \xrightarrow{\left(\begin{smallmatrix}\gamma' & 0\\ 0 & \gamma''\end{smallmatrix}\right)} M \oplus M''$$

is a Tate $\mathcal{W}$-resolution of $M \oplus M''$. It follows that $\mathrm{res}\,\overline{\mathcal{W}}$ is a subcategory of $\mathcal{A}$. Similarly, we see that $\mathrm{cores}\,\overline{\mathcal{V}}$ is a subcategory of $\mathcal{A}$.

If M admits a Tate $\mathcal{W}$-resolution $T \to W \to M$, then W is a proper $\mathcal{W}$-resolution of M. Hence, $\mathrm{res}\,\overline{\mathcal{W}} \subseteq \mathrm{res}\,\widetilde{\mathcal{W}}$, and similarly, $\mathrm{cores}\,\overline{\mathcal{V}} \subseteq \mathrm{cores}\,\widetilde{\mathcal{V}}$.

If M is in $\mathcal{G}(\mathcal{W})$ with complete $\mathcal{W}$-resolution T, then M admits a Tate $\mathcal{W}$-resolution $T \to T_{\geqslant 0} \to M$ and a Tate $\mathcal{W}$-coresolution $M \to \Sigma T_{<0} \to \Sigma T$. Hence, $\mathcal{G}(\mathcal{W})$ is a subcategory of $\mathrm{res}\,\overline{\mathcal{W}} \cap \mathrm{cores}\,\overline{\mathcal{W}}$.

Assume that $\mathcal{W} \perp \mathcal{W}$. If $M \in \mathrm{res}\,\widehat{\mathcal{W}}$, then any bounded $\mathcal{W}$-resolution $W \xrightarrow{\gamma} M$ is proper by [17, (3.2.a)], and this yields a Tate $\mathcal{W}$-resolution $0 \to W \xrightarrow{\gamma} M$. In particular, we have $\mathrm{res}\,\widehat{\mathcal{W}} \subseteq \mathrm{res}\,\overline{\mathcal{W}}$. Similarly, if $\mathcal{V} \perp \mathcal{V}$, then $\mathrm{cores}\,\widehat{\mathcal{V}} \subseteq \mathrm{cores}\,\overline{\mathcal{V}}$.

The next two results are tools for the proof of Theorem 3.6.

Lemma 3.3. *One has* $\mathrm{res}\,\overline{\mathcal{W}} \subseteq \mathrm{res}\,\widehat{\mathcal{G}(\mathcal{W})}$ *and* $\mathrm{cores}\,\overline{\mathcal{V}} \subseteq \mathrm{cores}\,\widehat{\mathcal{G}(\mathcal{V})}$.

Proof. We prove the first containment; the proof of the second containment is dual.

Let M be an object in $\mathrm{res}\,\overline{\mathcal{W}}$, and fix a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$. Since T is an exact totally $\mathcal{W}$-acyclic complex in $\mathcal{W}$, the object $\mathrm{Coker}(\partial_n^T)$ is in $\mathcal{G}(\mathcal{W})$ for each integer n. By assumption, the homomorphism α_n is an isomorphism for $n \gg 0$, and it follows that $\operatorname{Coker}(\partial_n^W) \cong \operatorname{Coker}(\partial_n^T) \in \mathcal{G}(\mathcal{W})$ for $n \gg 0$. Also, each object W_i is in $\mathcal{G}(\mathcal{W})$, so the exact sequence

$$0 \to \operatorname{Coker}\bigl(\partial_n^W\bigr) \to W_{n-1} \to \cdots \to W_0 \to M \to 0$$

is a bounded augmented $\mathcal{G}(\mathcal{W})$-resolution of M. $\square$

Lemma 3.4. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are exact, that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$, and that $\mathcal{V}$ is both an injective cogenerator and a projective generator for $\mathcal{Y}$.*

(a) *Let M be an object in $\operatorname{res}\widehat{\mathcal{X}}$. If $\mathcal{X}$ is closed under kernels of epimorphisms, then M admits a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ such that α_n is an isomorphism for each $n \geqslant \mathcal{X}\text{-}\operatorname{pd}(M)$ and each object $\operatorname{Ker}(\partial_i^T)$ is in $\mathcal{X}$. Moreover, this resolution can be built so that α_n is a split surjection for all n.*

(b) *Let N be an object in $\operatorname{cores}\widehat{\mathcal{Y}}$. If $\mathcal{Y}$ is closed under cokernels of monomorphisms, then N admits a Tate $\mathcal{V}$-coresolution $N \xrightarrow{\delta} V \xrightarrow{\beta} S$ such that β_n is an isomorphism for each $n \leqslant -\mathcal{Y}\text{-}\operatorname{id}(N)$ and each object $\operatorname{Ker}(\partial_i^S)$ is in $\mathcal{Y}$. Moreover, this resolution can be built so that β_n is a split injection for all n.*

Proof. We prove part (a); the proof of (b) is dual.

Since $\mathcal{W}$ is a projective generator and an injective generator for $\mathcal{X}$, we have $\mathcal{X} \subseteq \operatorname{res}\widetilde{\mathcal{W}} \cap \operatorname{cores}\widetilde{\mathcal{W}}$ and $\operatorname{res}\widehat{\mathcal{X}} \subseteq \operatorname{res}\widetilde{\mathcal{W}}$ by [17, (3.3)]. In particular, the object M admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\gamma} M$. Set $d = \mathcal{X}\text{-}\operatorname{pd}(M)$. Since $\mathcal{X}$ is closed under kernels of epimorphisms, it follows from [2, (3.3)] that $X = \operatorname{Ker}(\partial_{d-1}^W)$ is in $\mathcal{X}$, and hence X admits a proper $\mathcal{W}$-coresolution $X \xrightarrow{\simeq} \widetilde{W}$ such that each $\operatorname{Ker}(\partial_i^{\widetilde{W}})$ is in $\mathcal{X}$; see [18, (1.8)]. A standard argument using the condition $\mathcal{W} \perp \mathcal{X}$ shows that ${}^{+}\widetilde{W}$ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact.

Set $\widehat{W} = \Sigma^{d-1}\widetilde{W}$. The properness of $\widetilde{W}$ yields a morphism $\gamma : \widehat{W} \to W_{<d}$ making the following diagram commute.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X & \longrightarrow & \widehat{W}_{d-1} & \longrightarrow & \cdots & \longrightarrow & \widehat{W}_0 & \longrightarrow & \widehat{W}_{-1} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \operatorname{id}_X} & & \big\downarrow{\scriptstyle \gamma_{d-1}} & & & & \big\downarrow{\scriptstyle \gamma_0} & & \big\downarrow{\scriptstyle \gamma_{-1}} & & \\
0 & \longrightarrow & X & \longrightarrow & W_{d-1} & \longrightarrow & \cdots & \longrightarrow & W_0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \tag{3.4.1}$$

The top row of this diagram is both $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. The truncation $W_{\geqslant d}$ is a proper $\mathcal{W}$-resolution of X, hence the complex $W_{\geqslant d}^{+}$ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact; a standard argument using the condition $\mathcal{X} \perp \mathcal{W}$ shows that it is also $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Let $T^{(1)}$ be the complex obtained by splicing $W_{\geqslant d}$ and $\widehat{W}$ along X. It follows that each $T_n^{(1)}$ is in $\mathcal{W}$ and that $T^{(1)}$ is both $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Set

$$\alpha_n^{(1)} = \begin{cases} \gamma_n & \text{for } n < d, \\ \operatorname{id}_{W_n} & \text{for } n \geqslant d. \end{cases}$$

The diagram (3.4.1) shows that $\alpha^{(1)} : T^{(1)} \to W$ is a morphism, and it follows that the diagram $T \xrightarrow{\alpha^{(1)}} W \xrightarrow{\gamma} M$ is a Tate $\mathcal{W}$-resolution.

Next we show how to modify the Tate $\mathcal{W}$-resolution $T^{(1)} \xrightarrow{\alpha^{(1)}} W \xrightarrow{\gamma} M$ to build a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ such that each α_n is a split surjection and such that $\alpha_n = \alpha_n'$ for all $n \geqslant d$. To this end, it suffices to construct a contractible $\mathcal{W}$-complex $T^{(2)}$ and a morphism $\alpha : T^{(1)} \oplus T^{(2)} \to W$ such that α_n is a split surjection for each $n < d$, and such that $T_n^{(2)} = 0$ for each $n \geqslant d$.

Consider the truncation $W_{<d}$. The complex $T^{(2)} = \Sigma^{-1}\operatorname{Cone}(\mathrm{id}_{W_{<d}})$ is contractible, and $T^{(2)}_n = 0$ for each $n \geqslant d$; see Fact 1.5. Let $f : T^{(2)} \to W$ denote the composition of the natural morphisms $T^{(2)} = \Sigma^{-1}\operatorname{Cone}(\mathrm{id}_{W_{<d}}) \to W_{<d} \to W$. Note that f_n is a split epimorphism for each $n < d$, and $f_n = 0$ for each $n \geqslant d$. One checks readily that the morphisms $\alpha_n = (\alpha^{(1)}_n\ \ f_n) : T^{(1)}_n \oplus T^{(2)}_n \to W_n$ describe a morphism of complexes satisfying the desired properties. $\square$

The next result is a version of Lemma 3.4 for objects in $\mathcal{X}$ with fewer hypotheses on the categories.

Proposition 3.5. *Let M be an object in $\mathcal{A}$. Assume that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, and that $\mathcal{V}$ is a projective generator for $\mathcal{Y}$.*

(a) *If $M \in \mathcal{X}$, then $M \in \operatorname{res}\overline{\mathcal{W}}$ if and only if $M \in \mathcal{G}(\mathcal{W})$.*
(b) *If $M \in \mathcal{Y}$, then $M \in \operatorname{cores}\overline{\mathcal{V}}$ if and only if $M \in \mathcal{G}(\mathcal{V})$.*

Proof. We prove part (a); the proof of (b) is dual. One implication is covered by the containment $\mathcal{G}(\mathcal{W}) \subseteq \operatorname{cores}\overline{\mathcal{W}}$ from Remark 3.2.

For the converse, assume that M is in $\operatorname{cores}\overline{\mathcal{W}}$ and fix a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$. By assumption, the augmented resolution W^+ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. We claim that it is also $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Indeed, since $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, we have $M \perp \mathcal{W}$, and the condition $\mathcal{W} \subseteq \mathcal{X}$ implies that $W_i \perp \mathcal{W}$ for each $i \geqslant 0$. A standard induction argument yields the claim.

We claim that $\mathcal{W} \perp M$. As in the proof of Lemma 3.3, the object $\operatorname{Coker}(\partial^W_i)$ is in $\mathcal{G}(\mathcal{W})$ for all $i \gg 0$. Hence, Fact 1.14 implies that $\mathcal{W} \perp \operatorname{Coker}(\partial^W_i)$ for all $i \gg 0$. Since $\mathcal{W} \perp W_i$ for all i, a standard induction argument yields the claim.

Since $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, the object M admits a proper $\mathcal{W}$-coresolution $M \xrightarrow{\simeq} \widetilde{W}$. Hence, the augmented coresolution ${}^+\widetilde{W}$ is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. A standard induction argument, using the conditions $\mathcal{W} \perp M$ and $\mathcal{W} \perp \widetilde{W}_i$, shows that ${}^+\widetilde{W}$ is also $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact.

Splice the resolutions W and $\widetilde{W}$ to construct the following exact sequence in $\mathcal{W}$

$$\widehat{W} = \cdots \xrightarrow{\partial^W_2} W_1 \xrightarrow{\partial^W_1} W_0 \longrightarrow \widetilde{W}_0 \xrightarrow{\partial^{\widetilde{W}}_0} \widetilde{W}_{-1} \xrightarrow{\partial^{\widetilde{W}}_{-1}} \cdots$$

such that $M \cong \operatorname{Coker}(\partial^{\widehat{W}}_1) = \operatorname{Coker}(\partial^W_1)$. Since W^+ and ${}^+\widetilde{W}$ are $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact, it follows that $\widehat{W}$ is a complete resolution of M, so M is in $\mathcal{G}(\mathcal{W})$, by definition. $\square$

The following characterizations of $\operatorname{res}\overline{\mathcal{W}}$ and $\operatorname{cores}\overline{\mathcal{V}}$ are akin to [3, (3.1)].

Theorem 3.6. *Assume that $\mathcal{W}$ is closed under kernels of epimorphisms and that $\mathcal{W} \perp \mathcal{W}$. Assume that $\mathcal{V}$ is closed under cokernels of monomorphisms and $\mathcal{V} \perp \mathcal{V}$.*

(a) *An object $M \in \mathcal{A}$ admits a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ (such that each α_n is a split surjection) if and only if $\mathcal{G}(\mathcal{W})$-$\operatorname{pd}(M) < \infty$. Hence, we have $\operatorname{res}\overline{\mathcal{W}} = \operatorname{res}\widehat{\mathcal{G}(\mathcal{W})}$, so the category $\operatorname{res}\overline{\mathcal{W}}$ is closed under direct summands and satisfies the two-of-three property.*
(b) *An object $N \in \mathcal{A}$ admits a Tate $\mathcal{V}$-coresolution $N \xrightarrow{\delta} V \xrightarrow{\beta} S$ (such that each β_n is a split injection) if and only if $\mathcal{G}(\mathcal{V})$-$\operatorname{id}(N) < \infty$. Hence, we have $\operatorname{cores}\overline{\mathcal{V}} = \operatorname{cores}\widehat{\mathcal{G}(\mathcal{V})}$, so the category $\operatorname{cores}\overline{\mathcal{V}}$ is closed under direct summands and satisfies the two-of-three property.*

Proof. The desired equivalences follow from Lemmas 3.3 and 3.4, using Fact 1.14. The properties of $\operatorname{res}\overline{\mathcal{W}}$ and $\operatorname{cores}\overline{\mathcal{V}}$ follow from [2, (3.4), (3.5)]. $\square$

The next result contains Theorem A from the introduction.

Theorem 3.7. *Let R be a commutative ring, and let C be a semidualizing R-module. With $\mathcal{P}_C = \mathcal{P}_C(R)$ and $\mathcal{I}_C = \mathcal{I}_C(R)$, one has $\operatorname{res}\widehat{\mathcal{G}(\mathcal{P}_C)} = \operatorname{res}\overline{\mathcal{P}_C}$ and $\operatorname{cores}\widehat{\mathcal{G}(\mathcal{I}_C)} = \operatorname{cores}\overline{\mathcal{I}_C}$. Also, the categories $\operatorname{res}\widehat{\mathcal{G}(\mathcal{P}_C)}$ and $\operatorname{cores}\widehat{\mathcal{G}(\mathcal{I}_C)}$ are closed under direct summands and satisfy the two-of-three property.*

Proof. Fact 2.6 implies that $\mathcal{P}_C(R)$ satisfies the hypotheses of Theorem 3.6(a) and that $\mathcal{I}_C(R)$ satisfies the hypotheses of Theorem 3.6(b). $\square$

The next result is the key for well-definedness and functoriality of Tate cohomology. The proof is almost identical to that of [3, (5.3)].

Lemma 3.8. *Let M, M', N, N' be objects in $\mathcal{A}$. Assume that M and M' admit Tate $\mathcal{W}$-resolutions $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ and $T' \xrightarrow{\alpha'} W' \xrightarrow{\gamma'} M'$, and that N and N' admit Tate $\mathcal{V}$-coresolutions $N \xrightarrow{\delta} V \xrightarrow{\beta} S$ and $N' \xrightarrow{\delta'} V' \xrightarrow{\beta'} S'$.*

(a) *For each morphism $f : M \to M'$ there is a morphism $\overline{f} : W \to W'$, unique up to homotopy, making the right-most square in the next diagram commute*

$$\begin{array}{ccccc} T & \xrightarrow{\alpha} & W & \xrightarrow{\gamma} & M \\ {\scriptstyle\widehat{f}}\downarrow & & {\scriptstyle\overline{f}}\downarrow & & {\scriptstyle f}\downarrow \\ T' & \xrightarrow{\alpha'} & W' & \xrightarrow{\gamma'} & M' \end{array}$$

and for each such $\overline{f}$ there exists a morphism $\widehat{f} : T \to T'$, unique up to homotopy, making the left-most square in the diagram commute up to homotopy. If f is an isomorphism, then $\overline{f}$ and $\widehat{f}$ are homotopy equivalences.

(b) *For each morphism $g : N \to N'$ there is a morphism $\overline{g} : V \to V'$, unique up to homotopy, making the left-most square in the next diagram commute*

$$\begin{array}{ccccc} N & \xrightarrow{\delta} & V & \xrightarrow{\beta} & S \\ {\scriptstyle\widehat{g}}\downarrow & & {\scriptstyle\overline{g}}\downarrow & & {\scriptstyle f}\downarrow \\ N' & \xrightarrow{\delta'} & V' & \xrightarrow{\beta'} & S' \end{array}$$

and for each such $\overline{g}$ there exists a morphism $\widehat{g} : S \to S'$, unique up to homotopy, making the right-most square in the diagram commute up to homotopy. If g is an isomorphism, then $\overline{g}$ and $\widehat{g}$ are homotopy equivalences.

What follows is a horseshoe lemma for Tate (co)resolutions like [3, (5.5)]. The proof is similar to that of [3, (5.5)], but it is different enough to merit inclusion.

Lemma 3.9. *Assume that $\mathcal{W}$ is closed under kernels of epimorphisms and that $\mathcal{W} \perp \mathcal{W}$. Assume that $\mathcal{V}$ is closed under cokernels of monomorphisms and $\mathcal{V} \perp \mathcal{V}$.*

(a) *Fix an exact sequence $0 \to M' \xrightarrow{\xi} M \xrightarrow{\zeta} M'' \to 0$ in $\mathcal{A}$ that is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Assume that M' and M'' admit Tate $\mathcal{W}$-resolutions $T' \xrightarrow{\alpha'} W' \xrightarrow{\gamma'} M'$ and $T'' \xrightarrow{\alpha''} W'' \xrightarrow{\gamma''} M''$ such that α'_n and α''_n*

are split surjections for all $n \in \mathbb{Z}$ and isomorphisms for each $n \geqslant d$. Then M admits a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ such that α_n is an isomorphism for each $n \geqslant d$ and such that there is a commutative diagram of morphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T' & \xrightarrow{\widehat{\xi}} & T & \xrightarrow{\widehat{\zeta}} & T'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\
0 & \longrightarrow & W' & \xrightarrow{\bar{\xi}} & W & \xrightarrow{\bar{\zeta}} & W'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\
0 & \longrightarrow & M' & \xrightarrow{\xi} & M & \xrightarrow{\zeta} & M'' & \longrightarrow & 0
\end{array} \tag{3.9.1}$$

wherein the top two rows are degreewise split exact.

(b) *Fix an exact sequence $0 \to N' \xrightarrow{\rho} N \xrightarrow{\tau} N'' \to 0$ in $\mathcal{A}$ that is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{V})$-exact. Assume that N' and N'' admit Tate $\mathcal{V}$-coresolutions $N' \xrightarrow{\delta'} V' \xrightarrow{\beta'} S'$ and $N'' \xrightarrow{\delta''} V'' \xrightarrow{\beta''} S''$ such that β'_n and β''_n are split injections for all $n \in \mathbb{Z}$ and isomorphisms for each $n \leqslant d$. Then N admits a Tate $\mathcal{V}$-coresolution $N \xrightarrow{\delta} V \xrightarrow{\beta} S$ such that β_n is an isomorphism for each $n \leqslant d$ and such that there is a commutative diagram of morphisms*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \xrightarrow{\rho} & N & \xrightarrow{\tau} & N'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta''} & & \\
0 & \longrightarrow & V' & \xrightarrow{\bar{\rho}} & V & \xrightarrow{\bar{\tau}} & V'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta''} & & \\
0 & \longrightarrow & S' & \xrightarrow{\widehat{\rho}} & S & \xrightarrow{\widehat{\tau}} & S'' & \longrightarrow & 0
\end{array}$$

wherein the bottom two rows are degreewise split exact.

Proof. We prove part (a); the proof of (b) is dual. The lower half of the diagram (3.9.1) is constructed in the relative horseshoe lemma [17, (1.9.a)]. Note that we have $W_n = W'_n \oplus W''_n$ for each $n \in \mathbb{Z}$, and $\bar{\xi}_n = \left(\begin{smallmatrix} \mathrm{id}_{W'_n} \\ 0 \end{smallmatrix}\right)$ and $\bar{\zeta}_n = (0 \ \mathrm{id}_{W''_n})$. Furthermore, we have $\partial^W_n = \left(\begin{smallmatrix} \partial^{W'}_n & f_n \\ 0 & \partial^{W''}_n \end{smallmatrix}\right)$ for some $f_n \in \operatorname{Hom}_{\mathcal{A}}(W''_n, W'_{n-1})$; and the equation $\partial^W_n \partial^W_{n+1} = 0$ implies that

$$\partial^{W'}_n f_{n+1} + f_n \partial^{W''}_{n+1} = 0. \tag{3.9.2}$$

We set $T_n = T'_n \oplus T''_n$ for each $n \in \mathbb{Z}$, and $\widehat{\xi}_n = \left(\begin{smallmatrix} \mathrm{id}_{T'_n} \\ 0 \end{smallmatrix}\right)$ and $\widehat{\zeta}_n = (0 \ \mathrm{id}_{T''_n})$.

The proof will be complete once we construct morphisms $g_n \in \operatorname{Hom}_{\mathcal{A}}(T''_n, T'_{n-1})$ and $h_n \in \operatorname{Hom}_{\mathcal{A}}(T''_n, W'_n)$ for each $n \in \mathbb{Z}$ such that

$$\partial^{T'}_n g_{n+1} + g_n \partial^{T''}_{n+1} = 0, \tag{3.9.3}$$

$$h_n \partial^{T''}_{n+1} = \partial^{W'}_{n+1} h_{n+1} + f_{n+1} \alpha''_{n+1} - \alpha'_n g_{n+1}. \tag{3.9.4}$$

Indeed, once this is done we set

$$\partial_n^T = \begin{pmatrix} \partial_n^{T'} & g_n \\ 0 & \partial_n^{T''} \end{pmatrix} \quad \text{and} \quad \alpha_n = \begin{pmatrix} \alpha'_n & h_n \\ 0 & \alpha''_n \end{pmatrix}.$$

Using Eq. (3.9.3), it is straightforward to show that ∂^T makes T into a chain complex such that $\widehat{\xi}$ and $\widehat{\zeta}$ are chain maps. Similarly, Eq. (3.9.4) implies that α is a chain map. Since the matrices defining these maps are upper-triangular, it follows readily that the diagram (3.9.1) commutes, using the fact that the horizontal maps in the top two rows are the canonical injections and surjections. Since α'_n and α''_n are isomorphisms for each $n \geqslant d$, the snake lemma implies that α_n is an isomorphism for each $n \geqslant d$. Similarly, α'_n and α''_n are surjections for each $n \in \mathbb{Z}$, the snake lemma implies that α_n is a surjection for each $n \in \mathbb{Z}$. Finally, the fact that $\mathcal{W}$ is closed under kernels of epimorphisms implies that each $\operatorname{Ker}(\alpha_n) \in \mathcal{W}$; so, the condition $\mathcal{W} \perp \mathcal{W}$ implies that each α_n is a split surjection. Since the top row $\mathbb{T}$ of (3.9.1) is degreewise split exact, the sequence $\operatorname{Hom}_{\mathcal{A}}(U, \mathbb{T})$ is exact for each $U \in \mathcal{W}$. Since $\operatorname{Hom}_{\mathcal{A}}(U, T')$ and $\operatorname{Hom}_{\mathcal{A}}(U, T'')$ are exact, a long exact sequence argument shows that $\operatorname{Hom}_{\mathcal{A}}(U, T)$ is also exact. In summary, we conclude that T is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact, and a similar argument shows that it is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact.

The assumption that α'_n and α''_n are isomorphisms for each $n \geqslant d$ implies that

$$\operatorname{Coker}(\partial_{d+1}^{W'}) \cong \operatorname{Coker}(\partial_{d+1}^{T'}) \in \mathcal{G}(\mathcal{W}) \quad \text{and} \quad \operatorname{Coker}(\partial_{d+1}^{W''}) \cong \operatorname{Coker}(\partial_{d+1}^{T''}) \in \mathcal{G}(\mathcal{W}).$$

The exact sequence of complexes

$$0 \to W'_{\geqslant d} \to W_{\geqslant d} \to W''_{\geqslant d} \to 0$$

has associated long exact sequence

$$0 \to \operatorname{Coker}(\partial_{d+1}^{W'}) \to \operatorname{Coker}(\partial_{d+1}^{W}) \to \operatorname{Coker}(\partial_{d+1}^{W''}) \to 0. \tag{3.9.5}$$

Fact 1.14 implies that $\mathcal{G}(\mathcal{W})$ is closed under extensions, so $\operatorname{Coker}(\partial_{d+1}^{W}) \in \mathcal{G}(\mathcal{W})$.

For each $n > d$ set $g_n = (\alpha'_{n-1})^{-1} f_n \alpha''_n$. For each $n > d$, this yields

$$\begin{aligned} g_n \partial_{n+1}^{T''} &= (\alpha'_{n-1})^{-1} f_n \alpha''_n \partial_{n+1}^{T''} = (\alpha'_{n-1})^{-1} f_n \partial_{n+1}^{W''} \alpha''_{n+1} \\ &= -(\alpha'_{n-1})^{-1} \partial_n^{W'} f_{n+1} \alpha''_{n+1} = -(\alpha'_{n-1})^{-1} \partial_n^{W'} \alpha'_n (\alpha'_n)^{-1} f_{n+1} \alpha''_{n+1} \\ &= -(\alpha'_{n-1})^{-1} \alpha'_{n-1} \partial_n^{T'} (\alpha'_n)^{-1} f_{n+1} \alpha''_{n+1} = -\partial_n^{T'} (\alpha'_n)^{-1} f_{n+1} \alpha''_{n+1} \\ &= -\partial_n^{T'} g_{n+1}. \end{aligned}$$

The first, fourth, and sixth equalities are by definition; the second one holds because α'' is a chain map; the third one is from Eq. (3.9.2); and the fifth one holds because α' is a chain map. This implies that (3.9.3) is satisfied for each $n > d$. Thus, we have constructed the complex $T_{\geqslant d}$ and a degreewise split exact sequence

$$0 \to T'_{\geqslant d} \xrightarrow{\widehat{\xi}_{\geqslant d}} T_{\geqslant d} \xrightarrow{\widehat{\zeta}_{\geqslant d}} T''_{\geqslant d} \to 0. \tag{3.9.6}$$

For $n \geqslant d$, set $h_n = 0$. One checks readily that our choices for g_n and h_n satisfy (3.9.4) for all $n > d$, and that α_n is an isomorphism for $n \geqslant d$. In particular, we have $\operatorname{Coker}(\partial_{d+1}^T) \cong \operatorname{Coker}(\partial_{d+1}^W)$. The sequence (3.9.5) is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact because $\operatorname{Ext}^1_{\mathcal{A}}(\operatorname{Coker}(\partial_{d+1}^{W''}), \mathcal{W}) = 0$; see Fact 1.14. Hence, the relative horseshoe lemma [17, (1.9.b)] yields a commutative diagram of morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Coker}(\partial^{T'}_{d+1}) & \longrightarrow & \operatorname{Coker}(\partial^{T}_{d+1}) & \longrightarrow & \operatorname{Coker}(\partial^{T''}_{d+1}) & \longrightarrow & 0 \\ & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ 0 & \longrightarrow & T'_{<d} & \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_{T'_{<d}} \\ 0 \end{smallmatrix}\right)} & T_{<d} & \xrightarrow{\left(\begin{smallmatrix} 0 & \mathrm{id}_{T''_{<d}} \end{smallmatrix}\right)} & T''_{<d} & \longrightarrow & 0. \end{array} \tag{3.9.7}$$

Splice $T_{\geqslant d}$ and $T_{<d}$ along $\operatorname{Coker}(\partial^T_{d+1})$ to form T. Note that the differential on T is of the form $\partial^T_n = \left(\begin{smallmatrix} \partial^{T'}_n & g_n \\ 0 & \partial^{T''}_n \end{smallmatrix}\right)$ and the equation $\partial^T_n \partial^T_{n+1} = 0$ implies that (3.9.3) holds for all $n \in \mathbb{Z}$. It remains to build the h_n for $n < d$ such that (3.9.4) holds for all $n \leqslant d$. We generate the remaining homomorphisms by descending induction on n, for which the base case ($n > d$) has already been addressed with $h_n = 0$.

By induction, we assume that h_{n+1} has been constructed and we find h_n. Using the fact that T'' is $\operatorname{Hom}_{\mathcal{A}}(-, W'_n)$-exact, it suffices to show that the homomorphism $\partial^{W'}_{n+1} h_{n+1} + f_{n+1}\alpha''_{n+1} - \alpha'_n g_{n+1}$ is a cycle in $\operatorname{Hom}_{\mathcal{A}}(T'', W'_n)_{n+1}$. This is done in the following sequence wherein the first, third, and fifth equalities are routine:

$$\begin{aligned} &\big(\partial^{W'}_{n+1} h_{n+1} + f_{n+1}\alpha''_{n+1} - \alpha'_n g_{n+1}\big)\partial^{T''}_{n+2} \\ &\quad = \partial^{W'}_{n+1} h_{n+1} \partial^{T''}_{n+2} + f_{n+1}\alpha''_{n+1}\partial^{T''}_{n+2} - \alpha'_n g_{n+1}\partial^{T''}_{n+2} \\ &\quad = \partial^{W'}_{n+1}\big(\partial^{W'}_{n+2} h_{n+2} + f_{n+2}\alpha''_{n+2} - \alpha'_{n+1} g_{n+2}\big) \\ &\qquad + f_{n+1}\alpha''_{n+1}\partial^{T''}_{n+2} - \alpha'_n g_{n+1}\partial^{T''}_{n+2} \\ &\quad = \partial^{W'}_{n+1}\partial^{W'}_{n+2} h_{n+2} + \partial^{W'}_{n+1} f_{n+2}\alpha''_{n+2} - \partial^{W'}_{n+1}\alpha'_{n+1} g_{n+2} \\ &\qquad + f_{n+1}\alpha''_{n+1}\partial^{T''}_{n+2} - \alpha'_n g_{n+1}\partial^{T''}_{n+2} \\ &\quad = 0 + \partial^{W'}_{n+1} f_{n+2}\alpha''_{n+2} - \alpha'_n\partial^{T'}_{n+1} g_{n+2} + f_{n+1}\partial^{W''}_{n+2}\alpha''_{n+2} - \alpha'_n g_{n+1}\partial^{T''}_{n+2} \\ &\quad = \big(\partial^{W'}_{n+1} f_{n+2} + f_{n+1}\partial^{W''}_{n+2}\big)\alpha''_{n+2} - \alpha'_n\big(\partial^{T'}_{n+1} g_{n+2} + g_{n+1}\partial^{T''}_{n+2}\big). \end{aligned}$$

The second equality follows because h_{n+1} satisfies Eq. (3.9.4); the fourth one follows as α' and α'' are morphisms and W' is a complex. The last expression in this sequence vanishes by (3.9.2) and (3.9.3). This completes the proof. □

The next result provides strict resolutions, as in [3, (3.8)], for use in Theorem 4.10. Note that Lemma 3.4 provides Tate resolutions satisfying the hypotheses.

Lemma 3.10. *Assume that $\mathcal{W}$ is closed under direct summands. Let $f : M \to M'$ be a morphism in $\operatorname{res}\overline{\mathcal{W}}$, and let $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ and $T' \xrightarrow{\alpha'} W' \xrightarrow{\gamma'} M'$ be Tate $\mathcal{W}$-resolutions such that $\operatorname{Coker}(\partial^T_1), \operatorname{Coker}(\partial^{T'}_1) \in \mathcal{X}$ and such that α_n and α'_n are split surjections for all n.*

(a) *There exists a degreewise split exact sequence of $\mathcal{A}$-complexes*

$$0 \to \Sigma^{-1} X \to \widetilde{T} \to W \to 0$$

where $\widetilde{T} = (T_{\geqslant 0})^+$, and satisfying the following conditions:

- *X is a bounded strict $\mathcal{W}\mathcal{X}$-resolution of M,*
- *$\widetilde{T}_n = 0$ for each $n < -1$,*
- *$\widetilde{T}_n$ is in $\mathcal{W}$ for each $n \geqslant 0$, and*
- *$\widetilde{T}$ is exact,*
- *$\widetilde{T}_{-1}$ is in $\mathcal{X}$,*
- *$\widetilde{T}_{\geqslant 0} \cong T_{\geqslant 0}$.*

(b) *There exists a commutative diagram of morphisms of $\mathcal{A}$-complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^{-1}X & \longrightarrow & \widetilde{T} & \longrightarrow & W & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \Sigma^{-1}f^*} & & \Big\downarrow{\scriptstyle \widetilde{f}} & & \Big\downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & \Sigma^{-1}X' & \longrightarrow & \widetilde{T}' & \longrightarrow & W' & \longrightarrow & 0 \end{array}$$

wherein each row is an exact sequence as in part (a)*, the morphisms f^* and $\overline{f}$ are lifts of f, and $\widetilde{f}$ is induced by a lift of f.*

Proof. (a) The hard truncation $T_{\geqslant 0}$ is a proper $\mathcal{W}$-resolution of $\operatorname{Coker}(\partial_1^T)$. Set $\widetilde{T} = (T_{\geqslant 0})^+$. The morphism $\alpha : T \to W$ is a degreewise split surjection, and it follows that the induced morphism $\nu : \widetilde{T} \to W$ is a degreewise split surjection. Setting $X = \Sigma \operatorname{Ker}(\nu)$, yields a degreewise split exact sequence of the desired form. Since $\widetilde{T}$ is exact, the associated long exact sequence shows that X is a resolution of M. Since α_n is an isomorphism for $n \gg 0$, we conclude that X is bounded. As α_n is a split surjection for each n, we have $X_n \in \mathcal{W}$ for each $n \geqslant 1$. Since $X_0 \cong \operatorname{Coker}(\partial_1^T) \in \mathcal{X}$, it follows that X is a bounded strict $\mathcal{W}\mathcal{X}$-resolution of M.

(b) Lemma 3.8(a) yields the following commutative diagram

$$\begin{array}{ccccc} T & \xrightarrow{\alpha} & W & \xrightarrow{\gamma} & M \\ \Big\downarrow{\scriptstyle \widehat{f}} & & \Big\downarrow{\scriptstyle \overline{f}} & & \Big\downarrow{\scriptstyle f} \\ T' & \xrightarrow{\alpha'} & W' & \xrightarrow{\gamma'} & M' \end{array}$$

of morphisms of $\mathcal{A}$-complexes. Using the definitions $\widetilde{T} = (T_{\geqslant 0})^+$ and $\widetilde{T}' = (T'_{\geqslant 0})^+$, it is straightforward to show that $\widehat{f}$ induces a morphism $\widetilde{f} : \widetilde{T} \to \widetilde{T}'$ that makes the next diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^{-1}X & \longrightarrow & \widetilde{T} & \longrightarrow & W & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \widetilde{f}} & & \Big\downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & \Sigma^{-1}X' & \longrightarrow & \widetilde{T}' & \longrightarrow & W' & \longrightarrow & 0. \end{array}$$

From the conditions $X = \Sigma \operatorname{Ker}(\nu)$ and $X' = \Sigma \operatorname{Ker}(\nu')$ it is straightforward to show that $\widetilde{f}$ induces a morphism f^* making the desired diagram commute.

By definition, $\overline{f}$ is a lift of f. Since $\widetilde{T}$ and $\widetilde{T}'$ are exact, the morphism $\widetilde{f}$ is a quasiisomorphism. Using the induced diagrams on long exact sequences, one readily shows that these facts imply that f^* is a lift of f. $\square$

The proof of the next result is dual to the previous proof.

Lemma 3.11. *Assume that $\mathcal{V}$ is closed under direct summands. Let $g : N \to N'$ be a morphism in* cores $\overline{\mathcal{V}}$*, and let $N \xrightarrow{\delta} V \xrightarrow{\beta} L$ and $N' \xrightarrow{\delta'} V' \xrightarrow{\beta'} L'$ be Tate $\mathcal{V}$-coresolutions such that* $\operatorname{Ker}(\partial_0^L), \operatorname{Ker}(\partial_0^{L'}) \in \mathcal{Y}$ *and such that β_n and β'_n are split surjections for all n.*

(a) *There exists a degreewise split exact sequence of $\mathcal{A}$-complexes*

$$0 \to V \to \widetilde{S} \to \Sigma Y \to 0$$

where $\widetilde{S} = (S_{\geqslant 0})^{+}$, *and satisfying the following conditions*:

- Y *is a bounded strict* $\mathcal{YV}$*-coresolution of* N,
- $\widetilde{S}_n = 0$ *for each* $n > 1$,
- $\widetilde{S}_n$ *is in* $\mathcal{V}$ *for each* $n \leqslant 0$, *and*
- $\widetilde{S}$ *is exact,*
- $\widetilde{S}_1$ *is in* $\mathcal{Y}$,
- $\widetilde{S}_{\leqslant 0} \cong S_{\leqslant 0}$.

(b) *There exists a commutative diagram of morphisms of* $\mathcal{A}$*-complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & \widetilde{S} & \longrightarrow & \Sigma Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle \widetilde{g}} & & \downarrow{\scriptstyle \Sigma g^*} & & \\ 0 & \longrightarrow & V' & \longrightarrow & \widetilde{S}' & \longrightarrow & \Sigma Y' & \longrightarrow & 0 \end{array}$$

wherein each row is an exact sequence as in part (a), *the morphisms* $\bar{g}$ *and* g^* *are lifts of* g, *and* $\widetilde{g}$ *is induced by a lift of* g. □

We end this section with two examples. The first one shows that, even when $\mathcal{W}$ is a projective generator and an injective cogenerator for $\mathcal{X}$, one may have $\mathcal{X} \subsetneq \mathcal{G}(\mathcal{W})$.

Example 3.12. Let R be a commutative noetherian local ring with residue field k. Let $\mathcal{W}$ denote the category of finite rank free R-modules. Let $\mathcal{X}$ denote the category of finitely generated R-modules G in $\mathcal{GP}(R)$ with finite complexity, that is, such that the sequence of Betti numbers $\{\beta_i^R(G)\}$ is bounded above by a polynomial in i. (The category $\mathcal{X}$ was studied by Gerko [10].) It is straightforward to show that $\mathcal{W} \subseteq \mathcal{X} \subseteq \mathcal{G}(\mathcal{W})$ and that $\mathcal{W}$ is a projective generator and an injective cogenerator for $\mathcal{X}$. Furthermore, if R is artinian and Gorenstein, then $k \in \mathcal{G}(\mathcal{W})$. If R is not a complete intersection, then $k \notin \mathcal{X}$ because k has infinite complexity, so we have $\mathcal{X} \subsetneq \mathcal{G}(\mathcal{W})$ in this case.

Our next example shows that some categories are not perfectly suited for studying in this context.

Example 3.13. Let R be a commutative noetherian ring. An R-module G is *strongly Gorenstein projective* if it is in $\mathcal{GP}(R)$ with complete projective resolution that is periodic of period 1, that is, of the form $\cdots \xrightarrow{\partial} P \xrightarrow{\partial} P \xrightarrow{\partial} P \xrightarrow{\partial} \cdots$. These modules were introduced by Bennis and Mahdou [4] who prove that an R-module is in $\mathcal{GP}(R)$ if and only if it is a direct summand of a strongly Gorenstein projective R-module. Let $\mathcal{GP}^s(R)$ denote the category of strongly Gorenstein projective modules. Then we have $\mathcal{P}(R) \subseteq \mathcal{GP}^s(R) \subseteq \mathcal{GP}(R)$, and $\mathcal{P}(R)$ is a projective generator and an injective cogenerator for $\mathcal{GP}^s(R)$.

On the surface, it looks as though our results should apply to the category $\mathcal{X} = \mathcal{GP}^s(R)$. However, this category is not closed under direct summands in general (see [4, (3.11)]) so it is not exact and many our results do not apply. For instance, in Lemma 3.4, we can conclude that each strongly Gorenstein projective R-module M admits a Tate $\mathcal{P}(R)$-resolution $T \to W \to M$; however, we cannot conclude directly that $\mathrm{Ker}(\partial_i^T)$ is strongly Gorenstein projective.

4. Foundations of Tate cohomology

This section contains fundamental results on Tate cohomology functors, including the proof of Theorem B.

Definition 4.1. Let M, M', N, N' be objects in $\mathcal{A}$ equipped with homomorphisms $f: M \to M'$ and $g: N \to N'$. If M admits a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$, define the *nth Tate cohomology group* $\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M, N)$ as

$$\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M, N) = \mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(T, N)\big)$$

for each integer n. If M' also admits a Tate $\mathcal{W}$-resolution $T' \xrightarrow{\alpha} W' \xrightarrow{\gamma} M'$, then let $\widehat{f}$ be as in Lemma 3.8 and define

$$\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(f,N) = \mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(\widehat{f},N)\big) : \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}\big(M',N\big) \to \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,N),$$
$$\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,g) = \mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(T,g)\big) : \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,N) \to \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}\big(M,N'\big).$$

The following *comparison homomorphisms*

$$\varepsilon^n_{\mathcal{W}\mathcal{A}}(M,N) = \mathrm{H}_{-n}\big(\mathrm{Hom}(\alpha,N)\big) : \mathrm{Ext}^n_{\mathcal{W}\mathcal{A}}(M,N) \to \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,N)$$

make the next diagram commute for each integer n

$$\begin{array}{ccccc}
\mathrm{Ext}^n_{\mathcal{W}\mathcal{A}}(M',N) & \xrightarrow{\mathrm{Ext}^n_{\mathcal{W}\mathcal{A}}(f,N)} & \mathrm{Ext}^n_{\mathcal{W}\mathcal{A}}(M,N) & \xrightarrow{\mathrm{Ext}^n_{\mathcal{W}\mathcal{A}}(M,g)} & \mathrm{Ext}^n_{\mathcal{W}\mathcal{A}}(M,N') \\
\big\downarrow{\scriptstyle \varepsilon^n_{\mathcal{W}\mathcal{A}}(M',N)} & & \big\downarrow{\scriptstyle \varepsilon^n_{\mathcal{W}\mathcal{A}}(M,N)} & & \big\downarrow{\scriptstyle \varepsilon^n_{\mathcal{W}\mathcal{A}}(M,N')} \\
\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M',N) & \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(f,N)} & \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,N) & \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,g)} & \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,N').
\end{array}$$

On the other hand, if N admits a Tate $\mathcal{V}$-coresolution $N \xrightarrow{\delta} V \xrightarrow{\beta} S$, define the *nth Tate cohomology group* $\widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,N)$ as

$$\widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,N) = \mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(M,S)\big)$$

for each integer n. If N' also admits a Tate $\mathcal{V}$-coresolution $N' \xrightarrow{\delta'} V' \xrightarrow{\beta'} S'$, then let $\widehat{g}$ be as in Lemma 3.8 and define

$$\widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(f,N) = \mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(f,S)\big) : \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}\big(M',N\big) \to \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,N),$$
$$\widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,g) = \mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(M,\widehat{g})\big) : \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,N) \to \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}\big(M,N'\big).$$

The following *comparison homomorphisms*

$$\varepsilon^n_{\mathcal{A}\mathcal{V}}(M,N) = \mathrm{H}_{-n}\big(\mathrm{Hom}(M,\beta)\big) : \mathrm{Ext}^n_{\mathcal{A}\mathcal{V}}(M,N) \to \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,N)$$

make the next diagram commute for each integer n

$$\begin{array}{ccccc}
\mathrm{Ext}^n_{\mathcal{A}\mathcal{V}}(M',N) & \xrightarrow{\mathrm{Ext}^n_{\mathcal{A}\mathcal{V}}(f,N)} & \mathrm{Ext}^n_{\mathcal{A}\mathcal{V}}(M,N) & \xrightarrow{\mathrm{Ext}^n_{\mathcal{A}\mathcal{V}}(M,g)} & \mathrm{Ext}^n_{\mathcal{A}\mathcal{V}}(M,N') \\
\big\downarrow{\scriptstyle \varepsilon^n_{\mathcal{A}\mathcal{V}}(M',N)} & & \big\downarrow{\scriptstyle \varepsilon^n_{\mathcal{A}\mathcal{V}}(M,N)} & & \big\downarrow{\scriptstyle \varepsilon^n_{\mathcal{A}\mathcal{V}}(M,N')} \\
\widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M',N) & \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(f,N)} & \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,N) & \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,g)} & \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M,N').
\end{array}$$

Fact 4.2. Let R be a commutative ring, and assume that $\mathcal{W}$ and $\mathcal{V}$ are subcategories of $\mathcal{A} = \mathcal{M}(R)$. Let M, M', N, N' be R-modules equipped with R-module homomorphisms $f : M \to M'$ and $g : N \to N'$. If M admits a Tate $\mathcal{W}$-resolution, then each group $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M,N)$ is an R-module, and the comparison maps $\varepsilon^n_{\mathcal{W}\mathcal{A}}(M,N)$ are R-module homomorphisms. If M' also admits a Tate $\mathcal{W}$-resolution, then the maps $\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(f,N)$ and $\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(M,g)$ are R-module homomorphisms. Similar comments hold for $\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}$ and $\varepsilon^n_{\mathcal{AV}}(M,N)$.

Fact 4.3. Parts (a) and (b) of Lemma 3.8 show that

$$\widehat{\operatorname{Ext}}^n_{\mathcal{WA}} : \operatorname{res}\overline{\mathcal{W}} \times \mathcal{A} \to \mathcal{A}b \quad \text{and} \quad \widehat{\operatorname{Ext}}^n_{\mathcal{AV}} : \mathcal{A} \times \operatorname{cores}\overline{\mathcal{V}} \to \mathcal{A}b$$

are well-defined bifunctors and that

$$\varepsilon^n_{\mathcal{WA}} : \operatorname{Ext}^n_{\mathcal{WA}}|_{\operatorname{res}\overline{\mathcal{W}}\times\mathcal{A}} \to \widehat{\operatorname{Ext}}^n_{\mathcal{WA}} \varepsilon^n_{\mathcal{AV}} : \operatorname{Ext}^n_{\mathcal{AV}}|_{\mathcal{A}\times\operatorname{cores}\overline{\mathcal{V}}} \to \widehat{\operatorname{Ext}}^n_{\mathcal{AV}}$$

are natural transformations, independent of resolutions and liftings.

Notation 4.4. Let R be a commutative ring, and let C be a semidualizing R-module. We abbreviate as follows:

$$\begin{aligned} \widehat{\operatorname{Ext}}_{\mathcal{P}_C} &= \widehat{\operatorname{Ext}}_{\mathcal{P}_C(R)\mathcal{M}(R)}, & \widehat{\operatorname{Ext}}_{\mathcal{I}_C} &= \widehat{\operatorname{Ext}}_{\mathcal{M}(R)\mathcal{I}_C(R)}, \\ \operatorname{Ext}_{\mathcal{P}_C} &= \operatorname{Ext}_{\mathcal{P}_C(R)\mathcal{M}(R)}, & \operatorname{Ext}_{\mathcal{I}_C} &= \operatorname{Ext}_{\mathcal{M}(R)\mathcal{I}_C(R)}, \\ \operatorname{Ext}_{\mathcal{G}(\mathcal{P}_C)} &= \operatorname{Ext}_{\mathcal{G}(\mathcal{P}_C(R))\mathcal{M}(R)}, & \operatorname{Ext}_{\mathcal{G}(\mathcal{I}_C)} &= \operatorname{Ext}_{\mathcal{M}(R)\mathcal{G}(\mathcal{I}_C(R))}. \end{aligned}$$

The next result shows that objects with finite homological dimensions have vanishing Tate cohomology, as in [3, (5.2)]. See Theorems 5.2 and 5.4 for converses.

Proposition 4.5. *Let M and N be objects in $\mathcal{A}$, and assume $\mathcal{W} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{V}$.*

(a) *If $\mathcal{W}$-pd$(M) < \infty$, then $\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(M,-) = 0$ and $\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(-,M) = 0$ for all n.*
(b) *If $\mathcal{V}$-id$(N) < \infty$, then $\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(-,N) = 0$ and $\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(N,-) = 0$ for all n.*

Proof. We prove part (a); the proof of (b) is dual. Assume that $\mathcal{W}$-pd$(M) < \infty$. The vanishing $\widehat{\operatorname{Ext}}_{\mathcal{WA}}(M,-) = 0$ follows from Remark 3.2, since we have a Tate $\mathcal{W}$-resolution of M of the form $0 \to W \to M$. The vanishing $\widehat{\operatorname{Ext}}_{\mathcal{WA}}(-,M) = 0$ follows from the last part of Fact 1.13 since, for each complete $\mathcal{W}$-resolution T', the complex $\operatorname{Hom}_{\mathcal{A}}(T',M)$ is exact. □

Our next results provide long exact sequences for Tate cohomology. They are proved like [3, (5.4), (5.6)], using Lemma 3.9.

Lemma 4.6. *Let M be an object in $\operatorname{res}\overline{\mathcal{W}}$, and let N be an object in $\operatorname{cores}\overline{\mathcal{V}}$. Consider an exact sequence in $\mathcal{A}$*

$$\mathbb{L} = 0 \to L' \xrightarrow{f'} L \xrightarrow{f} L'' \to 0.$$

(a) *If the sequence $\mathbb{L}$ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact, then there is a long exact sequence*

$$\begin{aligned} \cdots \to \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(M,L') &\xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(M,f')} \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(M,L) \\ &\xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(M,f)} \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(M,L'') \xrightarrow{\widehat{\eth}^n_{\mathcal{WA}}(M,\mathbb{L})} \widehat{\operatorname{Ext}}^{n+1}_{\mathcal{WA}}(M,L') \xrightarrow{\widehat{\operatorname{Ext}}^{n+1}_{\mathcal{WA}}(M,f')} \cdots \end{aligned}$$

that is natural in M and $\mathbb{L}$, and is compatible with the long exact sequence in relative cohomology via the comparison maps $\varepsilon^n_{\mathcal{WA}}$ from Definition 4.1.

(b) *If the sequence* $\mathbb{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{V})$*-exact, then there is a long exact sequence*

$$\cdots \to \widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(L'',N) \xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(f,N)} \widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(L,N) \xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(f',N)} \widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(L',N) \xrightarrow{\widehat{\eth}^n_{\mathcal{AV}}(\mathbb{L},N)} \widehat{\operatorname{Ext}}^{n+1}_{\mathcal{AV}}(L'',N) \xrightarrow{\widehat{\operatorname{Ext}}^{n+1}_{\mathcal{AV}}(f,N)} \cdots$$

that is natural in N and $\mathbb{L}$*, and is compatible with the long exact sequence in relative cohomology via the comparison maps* $\varepsilon^n_{\mathcal{AV}}$ *from Definition 4.1.*

Lemma 4.7. *Let M and N be objects in* $\mathcal{A}$*, and assume that* $\mathcal{W}\perp\mathcal{W}$ *and* $\mathcal{V}\perp\mathcal{V}$*. Consider an exact sequence in* $\mathcal{A}$

$$\mathbb{L} = 0 \to L' \xrightarrow{f'} L \xrightarrow{f} L'' \to 0.$$

(a) *Assume that* $\mathcal{W}$ *is closed under kernels of epimorphisms, the objects* L, L', L'' *are in* $\operatorname{res}\overline{\mathcal{W}}$*, and the sequence* $\mathbb{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$*-exact. Then there is a long exact sequence*

$$\cdots \to \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(L'',N) \xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(f,N)} \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(L,N) \xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(f',N)} \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(L',N) \xrightarrow{\widehat{\eth}^n_{\mathcal{WA}}(\mathbb{L},N)} \widehat{\operatorname{Ext}}^{n+1}_{\mathcal{WA}}(L'',N) \xrightarrow{\widehat{\operatorname{Ext}}^{n+1}_{\mathcal{WA}}(f,N)} \cdots$$

that is natural in N and $\mathbb{L}$*, and is compatible with the long exact sequence in relative cohomology via the comparison maps* $\varepsilon^n_{\mathcal{WA}}$*.*

(b) *Assume that* $\mathcal{V}$ *is closed under cokernels of monomorphisms, the objects* L, L', L'' *are in* $\operatorname{cores}\overline{\mathcal{V}}$*, and the sequence* $\mathbb{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{V})$*-exact. Then there is a long exact sequence*

$$\cdots \to \widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(M,L') \xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(M,f')} \widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(M,L) \xrightarrow{\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(M,f)} \widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(M,L'') \xrightarrow{\widehat{\eth}^n_{\mathcal{AV}}(M,\mathbb{L})} \widehat{\operatorname{Ext}}^{n+1}_{\mathcal{AV}}(M,L') \xrightarrow{\widehat{\operatorname{Ext}}^{n+1}_{\mathcal{AV}}(M,f')} \cdots$$

that is natural in M and $\mathbb{L}$*, and is compatible with the long exact sequence in relative cohomology via the comparison maps* $\varepsilon^n_{\mathcal{AV}}$*.* □

The next two lemmas allow us to dimension-shift with Tate cohomology. They have similar proofs, as do the other natural invariants.

Lemma 4.8. *Assume that* $\mathcal{W}\perp\mathcal{W}$*, and consider an exact sequence in* $\mathcal{A}$

$$\mathbb{L} = 0 \to L' \to L \to L'' \to 0$$

that is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$*-exact and such that* $L \in \operatorname{res}\widehat{\mathcal{W}}$*.*

(a) *The natural transformation* $\widehat{\eth}^n(-,\mathbb{L}) : \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(-,L'') \xrightarrow{\cong} \widehat{\operatorname{Ext}}^{n+1}_{\mathcal{WA}}(-,L')$ *is an isomorphism of functors for each* $n \in \mathbb{Z}$*.*

(b) *If* $\mathcal{W}$ *is closed under kernels of epimorphisms and* $L', L'' \in \operatorname{res}\overline{\mathcal{W}}$*, then the natural transformation* $\widehat{\eth}^n(\mathbb{L},-) : \widehat{\operatorname{Ext}}^n_{\mathcal{WA}}(L',-) \xrightarrow{\cong} \widehat{\operatorname{Ext}}^{n+1}_{\mathcal{WA}}(L'',-)$ *is an isomorphism of functors for each* $n \in \mathbb{Z}$*.*

Proof. (a) Use the long exact sequence from Lemma 4.6(a) with the vanishing $\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(-,L)=0$ from Proposition 4.5(a).

(b) Our hypotheses guarantee that the functors and transformation under consideration are defined. Now use the long exact sequence from Lemma 4.7(b) with the vanishing $\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(L,-)=0$ from Proposition 4.5(a). $\square$

Lemma 4.9. *Assume that $\mathcal{V}\perp\mathcal{V}$, and consider an exact sequence in $\mathcal{A}$*

$$\mathbb{L}=0\to L'\to L\to L''\to 0$$

that is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{V})$-exact and such that $L\in\operatorname{cores}\widehat{\mathcal{V}}$.

(a) *The natural transformation $\widehat{\eth}^n(\mathbb{L},-):\widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(L',-)\xrightarrow{\cong}\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{AV}}(L'',-)$ is an isomorphism of functors for each $n\in\mathbb{Z}$.*
(b) *If $\mathcal{V}$ is closed under cokernels of monomorphisms and $L',L''\in\operatorname{cores}\overline{\mathcal{V}}$, then the natural transformation $\widehat{\eth}^n(-,\mathbb{L}):\widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(-,L'')\xrightarrow{\cong}\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{AV}}(-,L')$ is an isomorphism of functors for each $n\in\mathbb{Z}$.* $\square$

Next, we connect relative and Tate cohomology via a long exact sequence.

Theorem 4.10. *Assume that $\mathcal{X}$ is exact and closed under kernels of epimorphisms. Assume that $\mathcal{W}$ is closed under direct summands and is both an injective cogenerator and a projective generator for $\mathcal{X}$. Fix objects $M\in\operatorname{res}\widehat{\mathcal{X}}$ and $N\in\mathcal{A}$, and set $d=\mathcal{X}\text{-pd}(M)$. There is a long exact sequence*

$$\begin{aligned}
0\to \mathrm{Ext}^1_{\mathcal{XA}}(M,N)&\xrightarrow{\vartheta^1_{\mathcal{XWA}}(M,N)}\mathrm{Ext}^1_{\mathcal{WA}}(M,N)\xrightarrow{\varepsilon^1_{\mathcal{WA}}(M,N)}\widehat{\mathrm{Ext}}^1_{\mathcal{WA}}(M,N)\\
\to \mathrm{Ext}^2_{\mathcal{XA}}(M,N)&\xrightarrow{\vartheta^n_{\mathcal{XWA}}(M,N)}\mathrm{Ext}^2_{\mathcal{WA}}(M,N)\xrightarrow{\varepsilon^n_{\mathcal{WA}}(M,N)}\widehat{\mathrm{Ext}}^2_{\mathcal{WA}}(M,N)\\
\cdots\to \mathrm{Ext}^d_{\mathcal{XA}}(M,N)&\xrightarrow{\vartheta^d_{\mathcal{XWA}}(M,N)}\mathrm{Ext}^d_{\mathcal{WA}}(M,N)\xrightarrow{\varepsilon^d_{\mathcal{WA}}(M,N)}\widehat{\mathrm{Ext}}^d_{\mathcal{WA}}(M,N)\to 0
\end{aligned}$$

that is natural in M and N, and the next maps are isomorphisms for each $n>d$

$$\varepsilon^n_{\mathcal{WA}}(M,N):\mathrm{Ext}^n_{\mathcal{WA}}(M,N)\xrightarrow{\cong}\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M,N).$$

Proof. By Lemma 3.4(a) there is a Tate $\mathcal{W}$-resolution $T\to W\to M$ such that α_n is a split surjection for each n. Lemma 3.10(a) yields a degreewise split exact sequence of complexes

$$0\to\Sigma^{-1}X\to T'\to W\to 0 \tag{4.10.1}$$

wherein X is a bounded strict $\mathcal{WX}$-resolution of M and $T'_{\geqslant 0}\cong T_{\geqslant 0}$. In particular, there are isomorphisms for each $n\geqslant 1$

$$\mathrm{Ext}^n_{\mathcal{XA}}(M,N)\cong\mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(X,N)\big),\qquad \mathrm{Ext}^n_{\mathcal{WA}}(M,N)\cong\mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}(W,N)\big),$$
$$\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M,N)\cong\mathrm{H}_{-n}\big(\mathrm{Hom}_{\mathcal{A}}\big(T',N\big)\big).$$

Recall that $\mathrm{Ext}^n_{\mathcal{XA}}(M,N)=0$ for $n>d$. Apply the functor $\mathrm{Hom}_{\mathcal{A}}(-,N)$ to the sequence (4.10.1) and take the induced long exact sequence to obtain the desired long exact sequence and the isomorphisms.

To show that the long exact sequence is natural in N, let $g:N\to N'$ be a morphism in $\mathcal{A}$. Apply $\mathrm{Hom}_{\mathcal{A}}(-,g)$ to the sequence (4.10.1) to obtain the next commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(W,N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\widetilde{T},N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\Sigma^{-1}X,N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(W,N') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\widetilde{T},N') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\Sigma^{-1}X,N') & \longrightarrow & 0
\end{array}$$

which induces a commutative diagram of long exact sequences, as desired.

To show that the long exact sequence is natural in M, let $f : M \to M'$ be a morphism in $\mathcal{A}$. Apply $\mathrm{Hom}_{\mathcal{A}}(-,N)$ to the diagram from Lemma 3.10(b) to obtain the next commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(W',N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\widetilde{T}',N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\Sigma^{-1}X',N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(W,N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\widetilde{T},N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(\Sigma^{-1}X,N) & \longrightarrow & 0
\end{array}$$

which induces the desired commutative diagram of long exact sequences. □

4.11. *Proof of Theorem* B. Fact 2.6 shows that hypotheses of Theorem 4.10 are satisfied by $\mathcal{W} = \mathcal{P}_C(R)$ and $\mathcal{X} = \mathcal{G}(\mathcal{P}_C(R))$. □

The proofs of the next results are dual to those of Theorem 4.10 and Theorem B.

Theorem 4.12. *Assume that $\mathcal{Y}$ is exact and closed under cokernels of monomorphisms. Assume that $\mathcal{V}$ is closed under direct summands and is an injective cogenerator and a projective generator for $\mathcal{Y}$. Fix objects $M \in \mathcal{A}$ and $N \in \operatorname{cores}\widehat{\mathcal{Y}}$, and set $d = \mathcal{Y}\text{-id}(N)$. There is a long exact sequence*

$$\begin{aligned}
0 \to \mathrm{Ext}^1_{\mathcal{AY}}(M,N) &\xrightarrow{\vartheta^1_{\mathcal{AYV}}(M,N)} \mathrm{Ext}^1_{\mathcal{AV}}(M,N) \xrightarrow{\varepsilon^1_{\mathcal{AV}}(M,N)} \widehat{\mathrm{Ext}}^1_{\mathcal{AV}}(M,N) \\
\to \mathrm{Ext}^2_{\mathcal{AY}}(M,N) &\xrightarrow{\vartheta^n_{\mathcal{AYV}}(M,N)} \mathrm{Ext}^2_{\mathcal{AV}}(M,N) \xrightarrow{\varepsilon^n_{\mathcal{AV}}(M,N)} \widehat{\mathrm{Ext}}^2_{\mathcal{AV}}(M,N) \\
\cdots \to \mathrm{Ext}^d_{\mathcal{AY}}(M,N) &\xrightarrow{\vartheta^d_{\mathcal{AYV}}(M,N)} \mathrm{Ext}^d_{\mathcal{AV}}(M,N) \xrightarrow{\varepsilon^d_{\mathcal{AV}}(M,N)} \widehat{\mathrm{Ext}}^d_{\mathcal{AV}}(M,N) \to 0
\end{aligned}$$

that is natural in M and N, and the next maps are isomorphisms for each $n > d$

$$\varepsilon^n_{\mathcal{AV}}(M,N) : \mathrm{Ext}^n_{\mathcal{AV}}(M,N) \xrightarrow{\cong} \widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(M,N).$$

Corollary 4.13. *Let R be a commutative ring, and let C be a semidualizing R-module. Let M and N be R-modules, and assume that $d = \mathcal{G}(\mathcal{I}_C)\text{-id}_R(N) < \infty$. There is a long exact sequence that is natural in M and N*

$$\begin{aligned}
0 \to \mathrm{Ext}^1_{\mathcal{G}(\mathcal{I}_C)}(M,N) &\to \mathrm{Ext}^1_{\mathcal{I}_C}(M,N) \to \widehat{\mathrm{Ext}}^1_{\mathcal{I}_C}(M,N) \\
\to \mathrm{Ext}^2_{\mathcal{G}(\mathcal{I}_C)}(M,N) &\to \mathrm{Ext}^2_{\mathcal{I}_C}(M,N) \to \widehat{\mathrm{Ext}}^2_{\mathcal{I}_C}(M,N) \\
\cdots \to \mathrm{Ext}^d_{\mathcal{G}(\mathcal{I}_C)}(M,N) &\to \mathrm{Ext}^d_{\mathcal{I}_C}(M,N) \to \widehat{\mathrm{Ext}}^d_{\mathcal{I}_C}(M,N) \to 0
\end{aligned}$$

and there are isomorphisms $\mathrm{Ext}^n_{\mathcal{I}_C}(M,N) \xrightarrow{\cong} \widehat{\mathrm{Ext}}^n_{\mathcal{I}_C}(M,N)$ for each $n > d$.

5. Vanishing of Tate cohomology

This section focuses on the interplay between finiteness of homological dimensions and vanishing of Tate cohomology. It contains the proof of Theorem C. We begin with a result that compares to [3, (5.9)], though the proof is different.

Lemma 5.1. *Assume that $\mathcal{W}$ is closed under direct summands, and let $M \in \mathcal{G}(\mathcal{W})$. If $\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0$ or $\widehat{\operatorname{Ext}}^0_{\mathcal{A}\mathcal{W}}(M, M) = 0$, then M is in $\mathcal{W}$.*

Proof. We prove the case where $\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0$; the proof of the other case is dual. From Remark 3.2 there is a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ such that α_n is an isomorphism for all $n \geqslant 0$. This induces the second and third isomorphisms in the following sequence

$$\operatorname{Im}\bigl(\partial_0^T\bigr) \underset{\cong}{\xleftarrow{\sigma}} \operatorname{Coker}\bigl(\partial_1^T\bigr) \underset{\cong}{\xrightarrow{\overline{\alpha_0}}} \operatorname{Coker}\bigl(\partial_1^W\bigr) \underset{\cong}{\xrightarrow{\overline{\gamma_0}}} M.$$

The first isomorphism comes from the exactness of T. It is straightforward to show that the left-most rectangle in the following diagram commutes

$$\begin{array}{ccccc}
W_0 & \underset{\cong}{\xleftarrow{\alpha_0}} & T_0 & \xrightarrow{\partial_0^T} & T_{-1} \\
\downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \epsilon} \\
M & \underset{\cong}{\xleftarrow{\overline{\gamma_0}\,\overline{\alpha_0}}} & \operatorname{Coker}(\partial_1^T) & \underset{\cong}{\xrightarrow{\sigma}} & \operatorname{Im}(\partial_0^T).
\end{array}$$

Here, the morphisms π and ϵ are the natural surjection and injection, respectively, and it follows that the right-most rectangle also commutes. This diagram provides a monomorphism $f = \epsilon\sigma(\overline{\gamma_0}\,\overline{\alpha_0})^{-1} : M \hookrightarrow T_{-1}$ such that

$$f\gamma_0\alpha_0 = \partial_0^T. \tag{5.1.1}$$

The vanishing hypothesis

$$0 = \widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = \mathrm{H}_0\bigl(\operatorname{Hom}_{\mathcal{A}}(T, M)\bigr)$$

implies that every chain map $T \to M$ is null-homotopic. In particular, the chain map $T \xrightarrow{\gamma\alpha} M$ is null-homotopic with homotopy s as in the next diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_2^T} & T_1 & \xrightarrow{\partial_1^T} & T_0 & \xrightarrow{\partial_0^T} & T_{-1} & \xrightarrow{\partial_{-1}^T} & \cdots \\
& & \downarrow{\scriptstyle \gamma_1\alpha_1} & \swarrow{\scriptstyle s_0=0} & \downarrow{\scriptstyle \gamma_0\alpha_0} & \swarrow{\scriptstyle s_{-1}} & \downarrow{\scriptstyle \gamma_{-1}\alpha_{-1}} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots.
\end{array}$$

This yields a morphism $s_{-1} : T_{-1} \to M$ such that

$$\gamma_0\alpha_0 = s_{-1}\partial_0^T. \tag{5.1.2}$$

Combine (5.1.2) and (5.1.1) to obtain the following sequence

$$s_{-1} f \gamma_0 \alpha_0 = s_{-1} \partial_0^T = \gamma_0 \alpha_0 = \mathrm{id}_M \gamma_0 \alpha_0$$

and use the fact that $\gamma_0 \alpha_0$ is surjective to conclude that $s_{-1} f = \mathrm{id}_M$. Thus, the morphism $f : M \to T_{-1}$ is a split monomorphism. Since $\mathcal{W}$ is closed under direct summands and T_{-1} is in $\mathcal{W}$, it follows that M is in $\mathcal{W}$, as desired. □

The next result contains a partial converse Proposition 4.5(a), as in [3, (5.9)].

Theorem 5.2. *Assume that $\mathcal{W}$ is exact and closed under kernels of epimorphisms and that $\mathcal{W} \perp \mathcal{W}$. For an object $M \in \mathrm{res}\,\widehat{\mathcal{G}(\mathcal{W})}$, the next conditions are equivalent:*

(i) $\mathcal{W}\text{-pd}(M) < \infty$;
(ii) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ *for each (equivalently, for some) $n \in \mathbb{Z}$;*
(iii) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, -) = 0$ *for each (equivalently, for some) $n \in \mathbb{Z}$; and*
(iv) $\widehat{\mathrm{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0$.

Proof. Fact 1.14 yields a $\mathcal{W}\mathcal{G}(\mathcal{W})$-hull

$$0 \to M \to K \to M^{(-1)} \to 0 \tag{5.2.1}$$

that is, an exact sequence with $K \in \mathrm{res}\,\widehat{\mathcal{W}}$ and $M^{(-1)} \in \mathcal{G}(\mathcal{W})$; see Definition 1.7. Fact 1.14 implies that $\mathcal{W} \perp M$, so the sequence (5.2.1) is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. From the assumption $\mathcal{W} \perp \mathcal{W}$, we conclude that $\mathcal{W} \perp \mathrm{res}\,\widehat{\mathcal{W}}$. In particular, we have $\mathcal{W} \perp K$, and a standard argument implies that $\mathcal{W} \perp M^{(-1)}$.

Fact 1.14 shows that Lemma 3.4(a) applies to the category $\mathcal{X} = \mathcal{G}(\mathcal{W})$. So, the object $M \in \mathrm{res}\,\widehat{\mathcal{G}(\mathcal{W})}$ admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\gamma} M$.

The implication (i) $\Rightarrow$ (ii) follows from Proposition 4.5(a).

(ii) $\Rightarrow$ (iv). Assume that $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ for some $n \in \mathbb{Z}$. If $n = 0$, then condition (iv) follows immediately.

Assume next that $n < 0$. Set $M^{(0)} = M$ and $M^{(i)} = \mathrm{Im}(\partial_i^W)$ for each $i \geqslant 1$. The next exact sequences are $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact because W is a proper resolution

$$0 \to M^{(i)} \to W_{i-1} \to M^{(i-1)} \to 0. \tag{5.2.2}$$

Since $M^{(0)}, W_i \in \mathrm{res}\,\overline{\mathcal{W}}$, induction on i implies that each $M^{(i)}$ is in $\mathrm{res}\,\overline{\mathcal{W}}$ by Corollary 3.6(a). Repeated application of Lemma 4.8(b) yields the isomorphisms in the following sequence

$$\widehat{\mathrm{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = \widehat{\mathrm{Ext}}^0_{\mathcal{W}\mathcal{A}}\big(M^{(0)}, M\big) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}\big(M^{(-n)}, M\big) = 0$$

while the vanishing is by hypothesis.

Assume next $n > 0$. The object $M^{(-1)}$ from (5.2.1) is in $\mathcal{G}(\mathcal{W})$. For $i \leqslant -2$ use the complete $\mathcal{W}$-resolution of $M^{(-1)}$ to construct exact sequences

$$0 \to M^{(i+1)} \to W_i \to M^{(i)} \to 0$$

with $W_i \in \mathcal{W}$ and $M^{(i)} \in \mathcal{G}(\mathcal{W})$. Since the complete $\mathcal{W}$-resolution of $M^{(-1)}$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact, the same is true of each of these sequences. A standard argument shows that $\mathcal{W} \perp M^{(i)}$ for each $i \leqslant 2$. Repeated application of Lemma 4.8(b) yields the isomorphisms in the following sequence

$$\widehat{\mathrm{Ext}}^0_{\mathcal{WA}}(M,M)\cong\widehat{\mathrm{Ext}}^1_{\mathcal{WA}}\big(M^{(-1)},M\big)\cong\cdots\cong\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}\big(M^{(-n)},M\big)=0$$

while the vanishing is by hypothesis.

The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are verified similarly.

(iv) $\Rightarrow$ (i). Assume $\widehat{\mathrm{Ext}}^0_{\mathcal{WA}}(M,M)=0$ and again consider the $\mathcal{WX}$-hull (5.2.1). The isomorphisms in the following sequence are from Lemma 4.8,

$$\widehat{\mathrm{Ext}}^0_{\mathcal{WA}}\big(M^{(-1)},M^{(-1)}\big)\cong\widehat{\mathrm{Ext}}^{-1}_{\mathcal{WA}}\big(M,M^{(-1)}\big)\cong\widehat{\mathrm{Ext}}^0_{\mathcal{WA}}(M,M)=0$$

while the vanishing is by hypothesis. Since $M^{(-1)}\in\mathcal{G}(\mathcal{W})$, Lemma 5.1 implies that $M^{(-1)}\in\mathcal{W}$. Since K is in $\operatorname{res}\widehat{\mathcal{W}}$, the exact sequence (5.2.1) implies that $M\in\operatorname{res}\widehat{\mathcal{W}}$, using [2, (3.5)]. $\square$

5.3. *Proof of Theorem* C. Theorem 5.2 applies to $\mathcal{W}=\mathcal{P}_C(R)$ by Fact 2.6. $\square$

The proofs of the next results are dual to those of Theorem 5.2 and Theorem C.

Theorem 5.4. *Assume that $\mathcal{V}$ is exact and closed under cokernels of monomorphisms and that $\mathcal{V}\perp\mathcal{V}$. For each $M\in\operatorname{cores}\widehat{\mathcal{G}(\mathcal{V})}$, the following are equivalent*:

(i) $\mathcal{V}$-$\mathrm{id}(M)<\infty$;
(ii) $\widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(-,M)=0$ *for each (equivalently, for some) $n\in\mathbb{Z}$*;
(iii) $\widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(M,-)=0$ *for each (equivalently, for some) $n\in\mathbb{Z}$; and*
(iv) $\widehat{\mathrm{Ext}}^0_{\mathcal{AV}}(M,M)=0$.

Corollary 5.5. *Let R be a commutative ring, and let C be a semidualizing R-module. For an R-module M with $\mathcal{G}(\mathcal{I}_C)$-$\mathrm{id}_R(M)<\infty$, the following are equivalent*:

(i) $\mathcal{I}_C$-$\mathrm{id}_R(M)<\infty$;
(ii) $\widehat{\mathrm{Ext}}^n_{\mathcal{I}_C}(M,-)=0$ *for each (equivalently, for some) $n\in\mathbb{Z}$*;
(iii) $\widehat{\mathrm{Ext}}^n_{\mathcal{I}_C}(-,M)=0$ *for each (equivalently, for some) $n\in\mathbb{Z}$; and*
(iv) $\widehat{\mathrm{Ext}}^0_{\mathcal{I}_C}(M,M)=0$.

The next two results compare to [3, (7.2)] and [17, (4.8)].

Corollary 5.6. *Assume that $\mathcal{X}$ is exact and closed under kernels of epimorphisms. Assume that $\mathcal{W}$ is closed under direct summands and kernels of epimorphisms. Assume that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$. Let M be an object of $\mathcal{A}$ with $d=\mathcal{X}$-$\mathrm{pd}(M)<\infty$. The next conditions are equivalent*:

(i) $\mathcal{W}$-$\mathrm{pd}(M)<\infty$;
(ii) *The natural transformation $\vartheta^i_{\mathcal{XWA}}(M,-):\mathrm{Ext}^i_{\mathcal{XA}}(M,-)\xrightarrow{\cong}\mathrm{Ext}^i_{\mathcal{WA}}(M,-)$ is an isomorphism for each i; and*
(iii) *The natural transformation $\vartheta^i_{\mathcal{XWA}}(M,-):\mathrm{Ext}^i_{\mathcal{XA}}(M,-)\xrightarrow{\cong}\mathrm{Ext}^i_{\mathcal{WA}}(M,-)$ is an isomorphism either for two successive values of i with $1\leqslant i<d$ or for a single value of i with $i\geqslant d$.*

Proof. The implication (i) $\Rightarrow$ (ii) is in [17, (4.8)], and (ii) $\Rightarrow$ (iii) is trivial.

For (iii) $\Rightarrow$ (i), we consider three cases.

Case 1: The natural transformations $\vartheta^i_{\mathcal{XWA}}(M,-)$ and $\vartheta^{i+1}_{\mathcal{XWA}}(M,-)$ are isomorphisms where $1\leqslant i<d-1$. In this case, use the long exact sequence in Theorem 4.10 to conclude that $\widehat{\mathrm{Ext}}^i_{\mathcal{WA}}(M,-)=0$. The conclusion $\mathcal{W}$-$\mathrm{pd}(M)<\infty$ then follows from Theorem 5.2.

Case 2: The natural transformation $\vartheta^d_{\mathcal{XWA}}(M,-)$ is an isomorphism. As in Case 1, we conclude that $\widehat{\mathrm{Ext}}^d_{\mathcal{WA}}(M,-)=0$ and hence $\mathcal{W}$-pd$(M)<\infty$.

Case 3: The natural transformation $\vartheta^i_{\mathcal{XWA}}(M,-)$ is an isomorphism for some $i>d$. Our assumption yields the isomorphism in the next sequence

$$\mathrm{Ext}^i_{\mathcal{WA}}(M,-)\cong \mathrm{Ext}^i_{\mathcal{XA}}(M,-)=0$$

while the vanishing is from [17, (4.5.b)] since $i>d=\mathcal{X}$-pd(M). From [17, (4.5.a)] we conclude that $\mathcal{W}$-pd$(M)<i<\infty$. □

Our next result augments the previous one in the special case $\mathcal{X}=\mathcal{G}(\mathcal{W})$.

Corollary 5.7. *Assume that $\mathcal{W}\perp\mathcal{W}$ and that $\mathcal{W}$ is closed under direct summands and kernels of epimorphisms. Let M be an object of $\mathcal{A}$ with $d=\mathcal{G}(\mathcal{W})$-pd$(M)<\infty$. The following conditions are equivalent*:

(i) $\mathcal{W}$-pd$(M)<\infty$;
(ii) *The transformation $\vartheta^i_{\mathcal{G}(\mathcal{W})\mathcal{WA}}(-,M):\mathrm{Ext}^i_{\mathcal{G}(\mathcal{W})\mathcal{A}}(-,M)\xrightarrow{\cong}\mathrm{Ext}^i_{\mathcal{WA}}(-,M)$ is an isomorphism on* $\mathrm{res}\,\overline{\mathcal{W}}$ *for each i; and*
(iii) *The transformation $\vartheta^i_{\mathcal{G}(\mathcal{W})\mathcal{WA}}(-,M):\mathrm{Ext}^i_{\mathcal{G}(\mathcal{W})\mathcal{A}}(-,M)\xrightarrow{\cong}\mathrm{Ext}^i_{\mathcal{WA}}(-,M)$ is an isomorphism on* $\mathrm{res}\,\overline{\mathcal{W}}$ *either for a single value of i with $i\geqslant d$ or for two successive values of i with $1\leqslant i<d$.*

Proof. First note that $\mathrm{res}\,\overline{\mathcal{W}}\subseteq\mathrm{res}\,\widetilde{\mathcal{W}}$ by Remark 3.1. Furthermore, we have $\mathrm{res}\,\overline{\mathcal{W}}=\mathrm{res}\,\widehat{\mathcal{G}(\mathcal{W})}\subseteq\mathrm{res}\,\widetilde{\mathcal{G}(\mathcal{W})}$ by Theorem 3.6(a) and [17, (3.3.b)]. The implication (i) ⇒ (ii) now follows from [17, (4.10)]. The implication (ii) ⇒ (iii) is trivial, and (iii) ⇒ (i) follows as in the proof of Corollary 5.6. □

The proofs of the last two results of this section are dual to the previous two.

Corollary 5.8. *Assume that $\mathcal{Y}$ is exact and closed under cokernels of monomorphisms. Assume that $\mathcal{V}$ is closed under direct summands and cokernels of monomorphisms. Assume that $\mathcal{V}$ is both an injective cogenerator and a projective generator for $\mathcal{Y}$. Let N be an object of $\mathcal{A}$ with $d=\mathcal{Y}$-id$(N)<\infty$. The following conditions are equivalent*:

(i) $\mathcal{V}$-id$(N)<\infty$;
(ii) *The natural transformation $\vartheta^i_{\mathcal{AYV}}(-,N):\mathrm{Ext}^i_{\mathcal{AY}}(-,N)\cong\mathrm{Ext}^i_{\mathcal{AV}}(-,N)$ is an isomorphism for each i; and*
(iii) *The natural transformation $\vartheta^i_{\mathcal{AYV}}(-,N):\mathrm{Ext}^i_{\mathcal{AY}}(-,N)\cong\mathrm{Ext}^i_{\mathcal{AV}}(-,N)$ is an isomorphism either for a single value of i with $i\geqslant d$ or for two successive values of i with $1\leqslant i<d$.*

Corollary 5.9. *Assume that $\mathcal{V}\perp\mathcal{V}$ and that $\mathcal{V}$ is closed under direct summands and cokernels of monomorphisms. Let M be an object of $\mathcal{A}$ with $d=\mathcal{G}(\mathcal{V})$-id$(M)<\infty$. The following conditions are equivalent*:

(i) $\mathcal{V}$-id$(M)<\infty$;
(ii) *The transformation $\vartheta^i_{\mathcal{AG}(\mathcal{V})\mathcal{V}}(N,-):\mathrm{Ext}^i_{\mathcal{AG}(\mathcal{V})}(N,-)\cong\mathrm{Ext}^i_{\mathcal{AV}}(N,-)$ is an isomorphism on* $\mathrm{cores}\,\overline{\mathcal{V}}$ *for each i; and*
(iii) *The transformation $\vartheta^i_{\mathcal{AG}(\mathcal{V})\mathcal{V}}(N,-):\mathrm{Ext}^i_{\mathcal{AG}(\mathcal{V})}(N,-)\cong\mathrm{Ext}^i_{\mathcal{AV}}(N,-)$ is an isomorphism on* $\mathrm{cores}\,\overline{\mathcal{V}}$ *either for a single value of i with $i\geqslant d$ or for two successive values of i with $1\leqslant i<d$.*

6. Balance for Tate cohomology

We begin this section with its main result, which implies Theorem D; see (6.2).

Theorem 6.1. *Assume that $\mathcal{W} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{V}$ and $\mathcal{G}(\mathcal{W}) \perp \mathcal{V}$ and $\mathcal{W} \perp \mathcal{G}(\mathcal{V})$. Assume that $\mathcal{W}$ is closed under kernels of epimorphisms and direct summands and that $\mathcal{V}$ is closed under cokernels of monomorphisms and direct summands. Assume also that $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0 = \operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \operatorname{cores}\widehat{\mathcal{V}})$. For all $M \in \operatorname{res}\widehat{\mathcal{G}(\mathcal{W})}$ and all $N \in \operatorname{cores}\widehat{\mathcal{G}(\mathcal{V})}$ and all $n \geqslant 1$, we have*

$$\widehat{\operatorname{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\operatorname{Ext}}^n_{\mathcal{A}\mathcal{V}}(M, N).$$

If, in addition, we have $\operatorname{res}\widehat{\mathcal{W}} = \operatorname{cores}\widehat{\mathcal{V}}$, then this isomorphism holds for all $n \in \mathbb{Z}$.

Proof. We begin by noting that [17, (6.4)] implies that $\mathcal{W} \perp \operatorname{cores}\widehat{\mathcal{G}(\mathcal{V})}$ and $\operatorname{res}\widehat{\mathcal{G}(\mathcal{W})} \perp \mathcal{V}$. Theorem 3.6(a) yields a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \to M$ such that α_n is a split surjection for each $n \in \mathbb{Z}$. Lemma 3.10(a) provides a degreewise split exact sequence of complexes

$$0 \to \Sigma^{-1}X \to \widetilde{T} \to W \to 0 \tag{6.1.1}$$

wherein X is a bounded strict $\mathcal{W}\mathcal{G}(\mathcal{W})$-resolution of M, $\widetilde{T}$ is exact, $\widetilde{T}_n = 0$ for each $n < -1$, $\widetilde{T}_{-1}$ is in $\mathcal{G}(\mathcal{W})$, $\widetilde{T}_n$ is in $\mathcal{W}$ for each $n \geqslant 0$, and $\widetilde{T}_{\geqslant 0} \cong T_{\geqslant 0}$. In particular, there are isomorphisms for each $n \geqslant 1$

$$\widehat{\operatorname{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, -) \cong \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, -)\big). \tag{6.1.2}$$

Similarly, let $N \xrightarrow{\delta} V \xrightarrow{\beta} L$ be a Tate $\mathcal{V}$-coresolution such that each β_n is a split monomorphism, and consider a degreewise split exact sequence of complexes

$$0 \to V \to \widetilde{S} \xrightarrow{\eta} \Sigma Y \to 0 \tag{6.1.3}$$

wherein Y is a bounded strict $\mathcal{G}(\mathcal{V})\mathcal{V}$-coresolution, $\widetilde{S}$ is exact, $\widetilde{S}_n = 0$ for each $n > 1$, $\widetilde{S}_1$ is in $\mathcal{G}(\mathcal{V})$, $\widetilde{S}_n$ is in $\mathcal{V}$ for each $n \leqslant 0$, and $\widetilde{S}_{\leqslant 0} \cong S_{\leqslant 0}$. In particular, there are isomorphisms for each $n \geqslant 1$

$$\widehat{\operatorname{Ext}}^n_{\mathcal{A}\mathcal{W}}(-, N) \cong \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(-, \widetilde{S})\big). \tag{6.1.4}$$

The proof will be complete in the case $n \geqslant 1$ once we verify the quasiisomorphisms in the following sequence wherein the isomorphism in the middle is standard

$$\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, N) \simeq \operatorname{Hom}_{\mathcal{A}}\big(\widetilde{T}, \Sigma^{-1}\widetilde{S}\big) \cong \operatorname{Hom}_{\mathcal{A}}(\Sigma\widetilde{T}, \widetilde{S}) \simeq \operatorname{Hom}_{\mathcal{A}}(M, \widetilde{S}). \tag{6.1.5}$$

Indeed, this provides the second isomorphism in the following sequence

$$\widehat{\operatorname{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, N) \cong \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, N)\big) \cong \mathrm{H}_{-n}\big(\operatorname{Hom}_{\mathcal{A}}(M, \widetilde{S})\big) \cong \widehat{\operatorname{Ext}}^n_{\mathcal{A}\mathcal{V}}(M, N)$$

for each $n \geqslant 1$, while the first and third isomorphisms are from (6.1.2) and (6.1.4).

We claim that the complex $\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, \Sigma^{-1}V)$ is exact. To see this, note that the condition $\mathcal{G}(\mathcal{W}) \perp \mathcal{V}$ implies that $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(T'_i, V_j) = 0$ for all indices i and j. Since $\widetilde{T}$ is bounded below, a standard argument implies that $\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, V_j)$ is exact for each index j, and similarly it follows that $\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, V)$ is exact. We conclude that $\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, \Sigma^{-1}V) \cong \Sigma^{-1}\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, V)$ is also exact, as claimed.

Now, apply $\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, \Sigma^{-1}(-))$ to the degreewise split exact sequence (6.1.3) to obtain the next exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}\big(\widetilde{T}, \Sigma^{-1}V\big) \to \operatorname{Hom}_{\mathcal{A}}\big(\widetilde{T}, \Sigma^{-1}\widetilde{S}\big) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, \Sigma^{-1}\eta)} \operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, Y) \to 0.$$

The exactness of $\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, \Sigma^{-1}V)$ established above shows that the morphism $\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, \Sigma^{-1}\eta)$ is a quasiisomorphism. From [17, (6.6.b)] we know that the first morphism in the following sequence is a quasiisomorphism

$$\operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, N) \xrightarrow[\simeq]{\operatorname{Hom}_{\mathcal{A}}(\widetilde{T},\delta)} \operatorname{Hom}_{\mathcal{A}}(\widetilde{T}, Y) \xleftarrow[\simeq]{\operatorname{Hom}_{\mathcal{A}}(\widetilde{T},\Sigma^{-1}\eta)} \operatorname{Hom}_{\mathcal{A}}\big(\widetilde{T}, \Sigma^{-1}\widetilde{S}\big).$$

Combined together, these yield the first quasiisomorphism in (6.1.5); the second one is dual. This completes the proof when $n \geqslant 1$.

For the remainder of the proof, assume that $n < 1$ and that $\operatorname{res}\widehat{\mathcal{W}} = \operatorname{cores}\widehat{\mathcal{V}}$. Fix a $\mathcal{W}\mathcal{G}(\mathcal{W})$-hull

$$0 \to M \to K \to M' \to 0 \tag{6.1.6}$$

that is, an exact sequence in $\mathcal{A}$ with $K \in \operatorname{res}\widehat{\mathcal{W}} = \operatorname{cores}\widehat{\mathcal{V}}$ and $M' \in \mathcal{G}(\mathcal{W})$; see Definition 1.7. We proceed by descending induction on n. The base case $n \geqslant 1$ has already been established. Assuming that the desired isomorphisms hold with index $n+1$, we have the second isomorphism in the next sequence

$$\operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M, N) \cong \operatorname{Ext}^{n+1}_{\mathcal{W}\mathcal{A}}\big(M', N\big) \cong \operatorname{Ext}^{n+1}_{\mathcal{A}\mathcal{V}}\big(M', N\big) \cong \operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}(M, N).$$

The first isomorphism is from Lemma 4.8(b), and the third isomorphism is from Lemma 4.9(a). This completes the proof. □

6.2. *Proof of Theorem* D. We need to check that the categories $\mathcal{W} = \mathcal{P}_B(R)$ and $\mathcal{V} = \mathcal{I}_{B^\dagger}(R)$ satisfy the hypotheses of Theorem 6.1. We have $\mathcal{P}_B(R) \perp \mathcal{P}_B(R)$ and $\mathcal{I}_{B^\dagger}(R) \perp \mathcal{I}_{B^\dagger}(R)$; see Fact 2.6. The conditions $\mathcal{G}(\mathcal{P}_B(R)) \perp \mathcal{I}_{B^\dagger}(R)$ and $\mathcal{P}_B(R) \perp \mathcal{G}(\mathcal{I}_{B^\dagger}(R))$ are from [17, (6.16)]. The fact that $\mathcal{P}_B(R)$ is closed under kernels of epimorphisms and direct summands, and that $\mathcal{I}_{B^\dagger}(R)$ is closed under cokernels of monomorphisms and direct summands is in Fact 2.6. We have

$$\operatorname{Ext}^{\geqslant 1}_{\mathcal{P}_B}\big(\operatorname{res}\widehat{\mathcal{P}_B(R)}, \mathcal{I}_{B^\dagger}(R)\big) = 0 = \operatorname{Ext}^{\geqslant 1}_{\mathcal{I}_{B^\dagger}}\big(\mathcal{P}_B(R), \operatorname{cores}\widehat{\mathcal{I}_{B^\dagger}(R)}\big)$$

from [17, (6.15)]. Finally, when R is noetherian and C is dualizing for R, we have $\operatorname{res}\widehat{\mathcal{P}_B(R)} = \operatorname{cores}\widehat{\mathcal{I}_{B^\dagger}(R)}$ by Lemma 2.7. □

Corollary 6.3. *Let R be a commutative ring, and let M and N be R-modules such that $\mathcal{G}\mathcal{P}\text{-}\mathrm{pd}_R(M) < \infty$ and $\mathcal{G}\mathcal{I}\text{-}\mathrm{id}_R(N) < \infty$. For each $n \geqslant 1$, we have*

$$\widehat{\operatorname{Ext}}^n_{\mathcal{P}}(M, N) \cong \widehat{\operatorname{Ext}}^n_{\mathcal{I}}(M, N).$$

When R is Gorenstein, this isomorphism holds for all $n \in \mathbb{Z}$.

Proof. One readily checks that the categories $\mathcal{W} = \mathcal{P}(R)$ and $\mathcal{V} = \mathcal{I}(R)$ satisfy the hypotheses of Theorem 6.1: the relative Ext-vanishing follows from the balance $\operatorname{Ext}_{\mathcal{P}} \cong \operatorname{Ext} \cong \operatorname{Ext}_{\mathcal{I}}$ on $\mathcal{M}(R) \times \mathcal{M}(R)$, and the other hypotheses are standard. □

We conclude with two applications of Theorems 5.2 and 6.1.

Theorem 6.4. *If $\mathcal{W}$ and $\mathcal{V}$ satisfy the hypotheses of Theorem* 6.1, *then there are containments* $\operatorname{res}\widehat{\mathcal{G}(\mathcal{W})} \cap \operatorname{cores}\widehat{\mathcal{V}} \subseteq \operatorname{res}\widehat{\mathcal{W}}$ *and* $\operatorname{cores}\widehat{\mathcal{G}(\mathcal{V})} \cap \operatorname{res}\widehat{\mathcal{W}} \subseteq \operatorname{cores}\widehat{\mathcal{V}}$.

Proof. We verify the first containment; the second one is verified dually. Fix an object $M \in \operatorname{res}\widehat{\mathcal{G}(\mathcal{W})} \cap \operatorname{cores}\widehat{\mathcal{V}}$. The object M admits a $\mathcal{W}\mathcal{X}$-hull

$$0 \to M \to K \to X \to 0.$$

By assumption, we have $K \in \operatorname{res}\widehat{\mathcal{W}}$ and $X \in \mathcal{G}(\mathcal{W})$. The condition $\mathcal{W} \perp \mathcal{G}(\mathcal{W})$ from Fact 1.14 shows that $\mathcal{W} \perp M$, so the displayed sequence is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Lemma 4.8(b) yields the first isomorphism in the next sequence

$$\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) \cong \widehat{\operatorname{Ext}}^1_{\mathcal{W}\mathcal{A}}(X, M) \cong \widehat{\operatorname{Ext}}^1_{\mathcal{A}\mathcal{V}}(X, M) = 0.$$

The second isomorphism is from Theorem 6.1, and the vanishing is from Theorem 5.4. Hence, Theorem 5.2 implies $\mathcal{W}\text{-pd}(M) < \infty$, as desired. □

From this we recover some of the main results of [20].

Corollary 6.5. *Let R be a commutative ring, and let C be a semidualizing R-module. Let M be an R-module.*

(a) *If $\mathcal{GP}_C\text{-pd}_R(M) < \infty$ and $\operatorname{id}_R(M) < \infty$, then $\mathcal{P}_C\text{-pd}_R(M) < \infty$.*
(b) *If $\mathcal{GI}\text{-id}_R(M) < \infty$ and $\mathcal{P}_C\text{-pd}_R(M) < \infty$, then $\operatorname{id}_R(M) < \infty$.*
(c) *If $\mathcal{GP}\text{-pd}_R(M) < \infty$ and $\mathcal{I}_C\text{-id}_R(M) < \infty$, then $\operatorname{pd}_R(M) < \infty$.*
(d) *If $\mathcal{GI}_C\text{-id}_R(M) < \infty$ and $\operatorname{pd}_R(M) < \infty$, then $\mathcal{I}_C\text{-id}_R(M) < \infty$.*

Proof. We prove part (a); the other parts are similar or easier. Assume that $\mathcal{GP}_C\text{-pd}_R(M) < \infty$ and $\operatorname{id}_R(M) < \infty$. The finiteness of $\operatorname{id}_R(M)$ implies that $M \in \mathcal{B}_C(R)$, by Fact 2.6. Hence, the condition $\mathcal{GP}_C\text{-pd}_R(M) < \infty$ works with Lemma 2.9 to imply that $\mathcal{G}(\mathcal{P}_C)\text{-pd}_R(M) < \infty$. Now apply Theorem 6.4 with $\mathcal{W} = \mathcal{P}_C(R)$ and $\mathcal{V} = \mathcal{I}(R)$ to conclude that $\mathcal{P}_C\text{-pd}_R(M) < \infty$. □

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