Gorenstein dimension of the Frobenius endomorphism

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1 Motivation

Convention. All rings in this talk are Noetherian and p denotes a positive prime integer.

Notation. For a local homomorphism $\varphi\colon R \to S$ let ${}^{\varphi}S$ denote the additive Abelian group S with R-module structure coming from φ.

When R is a ring of characteristic p, the Frobenius endomorphism $\varphi\colon R \to R$ is given by $r \mapsto r^p$. The n-fold composition φ^n of φ with itself is given by $r \mapsto r^{p^n}$. The Frobenius module is ${}^{\varphi}R$.

The Theme. For a local ring of characteristic p, the homological properties of the residue field are similar to those of the Frobenius module.

Recall. For a local ring $(R, \mathfrak{m}, k)$ the following conditions are equivalent. [Auslander-Buchsbaum-Serre]

(a) R is regular.

(b) $\operatorname{pd}_R(M) < \infty$ for each finite R-module M.

(c) $\operatorname{pd}_R(k) < \infty$.

When R has characteristic p, let φ be the Frobenius endomorphism of R. Conditions (a)–(c) are equivalent to the following.

(d) ${}^{\varphi^n}R$ is flat for some (resp., every) integer $n \geq 1$. [Kunz]

(e) $\operatorname{fd}(\varphi^n) < \infty$ for some (resp., every) integer $n \geq 1$. [Rodicio]

Here fd is flat dimension and $\operatorname{fd}(\varphi^n) = \operatorname{fd}_R({}^{\varphi^n}R)$.

Question. What about the Gorenstein property?

Need a substitute for the projective dimension.

Definition. [Auslander-Bridger] Let R be a local ring and $(-)^* = \mathrm{Hom}_R(-, R)$. A finite R-module G is *totally reflexive* if

(i) The natural map $G \to G^{**}$ is an isomorphism, and

(ii) $\mathrm{Ext}^i_R(G, R) = 0 = \mathrm{Ext}^i_R(G^*, R)$ for each integer $i \geq 1$.

(E.g., finitely generated projective modules are totally reflexive.) A finite R-module M has *finite G-dimension* if there exists a resolution $0 \to G_n \to \cdots \to G_0 \to M \to 0$ with each G_i totally reflexive, and $\text{G-dim}_R(M)$ is the infimum of n for which such a resolution exists.

Properties. Let $(R, \mathfrak{m}, k)$ be a local ring and N a finite R-module.

- There is an inequality $\text{G-dim}_R(N) \leq \mathrm{pd}_R(N)$ with equality when $\mathrm{pd}_R(N) < \infty$.
- (AB-formula) When $\text{G-dim}_R(N) < \infty$, one has

$$\text{G-dim}_R(N) = \mathrm{depth}(R) - \mathrm{depth}_R(N).$$

- The following conditions are equivalent.
  (a) R is Gorenstein.
  (b) $\text{G-dim}_R(M) < \infty$ for each finite R-module M.
  (c) $\text{G-dim}_R(k) < \infty$.

With The Theme in mind, what are the analogs of the results of Kunz and Rodicio for G-dimension?

Danger! The G-dimension is only defined (here) for finite modules. In general, the Frobenius is not finitely generated.

To overcome this, use the technology of Cohen factorizations, as constructed by Avramov-Foxby-Herzog, and exploit the extensive investigation of their Gorenstein properties by Avramov-Foxby .

Instead of focusing exclusively on the Frobenius, it is convenient to develop the theory for a general local homomorphism.

2 G-dim over a local homomorphism

Notation. Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local homomorphism and M a finite S-module. When $\iota\colon (S,\mathfrak{n}) \to (\widehat{S},\widehat{n})$ is the natural map into the $\mathfrak{n}$-adic completion, set $\grave{\varphi} = \iota\varphi\colon R \to \widehat{S}$ and $\widehat{M} = M \otimes_S \widehat{S}$.

Definition. A *Cohen factorization* of $\grave{\varphi}$ is a diagram of local homomorphisms $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ where $\grave{\varphi} = \varphi'\dot{\varphi}$, with $\dot{\varphi}$ flat, R' complete, $R'/\mathfrak{m}R'$ regular, and φ' surjective. Set $\operatorname{edim}(\dot{\varphi}) = \operatorname{edim}(R'/\mathfrak{m}R')$.

Fact. A Cohen factorization of $\grave{\varphi}$ exists.

The point. Replace the nonfinte map φ with the very finite map φ'.

Theorem. (I-S-W) *The quantity* $\operatorname{G-dim}_{R'}(\widehat{M}) - \operatorname{edim}(\dot{\varphi})$ *is independent of the choice of Cohen factorization.*

Definition. $\operatorname{G-dim}_{\varphi}(M) := \operatorname{G-dim}_{R'}(\widehat{M}) - \operatorname{edim}(\dot{\varphi})$ and $\operatorname{G-dim}(\varphi) := \operatorname{G-dim}_{\varphi}(S)$.

Properties. Let $\psi\colon R \to T$ be a local homomorphism and N a finite T-module.

- If N is finite over R, then $\text{G-dim}_\psi(N) = \text{G-dim}_R(N)$.
- (AB-formula) If $\text{G-dim}_\psi(N) < \infty$, then

$$\text{G-dim}_\psi(N) = \operatorname{depth}(R) - \operatorname{depth}_T(N).$$

- The following conditions are equivalent.
  (a) R is Gorenstein.
  (b) For every local homomorphism $\varphi\colon R \to S$ and every finite S-module M, $\text{G-dim}_\varphi(M) < \infty$.
  (c) There exists a local homomorphism $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and an ideal I of S such that $\varphi(\mathfrak{m}) \subseteq I$ and $\text{G-dim}_\varphi(S/I) < \infty$.
- If $\varphi\colon R \to S$ is a local homomorphism and M a finite S-module isomorphic to N over R, then $\text{G-dim}_\varphi(M)$ and $\text{G-dim}_\psi(N)$ are simultaneously finite, though they may not be equal.

3 G-dimension over the Frobenius

Theorem. (I-S-W) *For a local ring R of charcteristic p with Frobenius endomorphism φ, the following conditions are equivalent.*

(a) R is Gorenstein.

(b) $\operatorname{G-dim}(\varphi^n) < \infty$ *for some/each integer $n \geq 1$.*

Sketch of the proof of "(b) $\Longrightarrow$ (a)". Assume that $\operatorname{G-dim}(\varphi^n) < \infty$ for a fixed integer $n \geq 1$. Pass to the completion $\widehat{R}$ to assume that R has a dualizing complex D^R.

Claim. $\operatorname{G-dim}(\varphi^{sn}) < \infty$ for all integers $s \geq 1$.

It is not known in general whether the composition of local homomorphisms of finite G-dimension has finite G-dimension. However, for powers of an endomorphism of a local ring, it is true. To see why, we use more of Avramov-Foxby.

Let $\psi\colon (S,\mathfrak{n}) \to (S',\mathfrak{n}')$ be a local homomorphism of complete rings with $\text{G-dim}(\psi) < \infty$. Let $I_S(t)$ be the Bass series of S:

$$I_S(t) = \sum_{i\geq 0} \mu_i(S)t^i.$$

Then $\mu_i(S) = \beta_i(D^S)$ and S is Gorenstein if and only if $I_S(t)$ is a polynomial. Avramov-Foxby define the Bass series of ψ, which is a formal Laurent series $I_\psi(t)$ such that

$$I_{S'}(t) = I_S(t)I_\psi(t).$$

The homomorphism ψ is *quasi-Gorenstein* at $\mathfrak{n}'$ if $\text{G-dim}(\psi) < \infty$ and $I_\psi(t)$ is a Laurent polynomial. Equivalently, ψ is quasi-Gorenstein at $\mathfrak{n}'$ if $D^S \otimes_S^{\mathbf{L}} S'$ is a dualizing complex for S'.

Fact. The composition of quasi-Gorenstein homomorphisms is quasi-Gorenstein.

Back to the Frobenius. We have assumed that $\text{G-dim}(\varphi^n) < \infty$.

The equality $I_R(t) = I_R(t)I_{\varphi^n}(t)$ implies that $I_{\varphi^n}(t) = 1$, so that φ^n is quasi-Gorenstein at $\mathfrak{m}$. Thus, φ^{sn} is quasi-Gorenstein at $\mathfrak{m}$ for each $s \geq 1$. In particular, $\text{G-dim}(\varphi^{sn}) < \infty$ for each $s \geq 1$.

Claim. R is Gorenstein.

It suffices to show that $\text{pd}_R(D^R) < \infty$. To check this, we use a variation of a theorem of Koh-Lee.

For a complex X, let $\sup(X) = \sup\{i \mid \text{H}_i(X) \neq 0\}$.

Proposition. Let X be a complex of R-modules with $\text{H}(X)$ finite. If $\sup(X \otimes^{\mathbf{L}}_R {}^{\varphi^m}R) < \infty$ for infinitely many $m \geq 1$, then $\text{pd}_R(X) < \infty$.

Since φ^{sn} is quasi-Gorenstein at $\mathfrak{m}$ for each $s \geq 1$, the complex $D^R \otimes^{\mathbf{L}}_R {}^{\varphi^{sn}}R$ is dualizing for R. In particular, $\sup(D^R \otimes^{\mathbf{L}}_R {}^{\varphi^{sn}}R)$ is finite, so the proposition implies that $\text{pd}(D^R) < \infty$. $\square$

Takahashi-Yoshino have proved this when R is F-finite. So it is worth mentioning a generalization.

Theorem. (I-S-W) *Let $(R, \mathfrak{m})$ be a local ring with a local endomorphism φ such that $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some $i \geq 1$. The following conditions are equivalent.*

(a) R is Gorenstein.

(b) $\operatorname{G-dim}(\varphi^n) < \infty$ *for some/each integer $n \geq 1$.*

(c) There exists a finite R-module $M \neq 0$ such that $\operatorname{pd}_R(M) < \infty$ *and* $\operatorname{G-dim}_{\varphi^n}(M) < \infty$ *for some $n \geq 1$.*

The implication "(c) $\Longrightarrow$ (b)" is a consequence of the following theorem which is proved using the Amplitude Inequality of Iversen and Foxby-Iyengar.

Theorem. (I-S-W) Let $\varphi\colon R \to S$ be a local homomorphism and M a nonzero finitely generated S-module with $\operatorname{pd}_S(M) < \infty$. Then

$$\operatorname{G-dim}_\varphi(M) = \operatorname{G-dim}(\varphi) + \operatorname{pd}_S(M).$$

In particular, $\operatorname{G-dim}_\varphi(M) < \infty$ if and only if $\operatorname{G-dim}(\varphi) + \operatorname{pd}_S(M)$.