Gorenstein dimension of the Frobenius endomorphism

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1 Motivation

Convention. All rings in this talk are Noetherian and $p$ denotes a positive prime integer.

Notation. For a local homomorphism $\phi: R \to S$ let $\phi S$ denote the additive Abelian group $S$ with $R$-module structure coming from $\phi$.

When $R$ is a ring of characteristic $p$, the Frobenius endomorphism $\phi: R \to R$ is given by $r \mapsto r^p$. The $n$-fold composition $\phi^n$ of $\phi$ with itself is given by $r \mapsto r^{p^n}$. The Frobenius module is $\phi R$.

The Theme. For a local ring of characteristic $p$, the homological properties of the residue field are similar to those of the Frobenius module.
Recall. For a local ring \((R, \mathfrak{m}, k)\) the following conditions are equivalent. [Auslander-Buchsbaum-Serre]

(a) \(R\) is regular.

(b) \(\text{pd}_R(M) < \infty\) for each finite \(R\)-module \(M\).

(c) \(\text{pd}_R(k) < \infty\).

When \(R\) has characteristic \(p\), let \(\varphi\) be the Frobenius endomorphism of \(R\). Conditions (a)–(c) are equivalent to the following.

(d) \(\varphi^n R\) is flat for some (resp., every) integer \(n \geq 1\). [Kunz]

(e) \(\text{fd}(\varphi^n) < \infty\) for some (resp., every) integer \(n \geq 1\). [Rodicio]

Here \(\text{fd}\) is flat dimension and \(\text{fd}(\varphi^n) = \text{fd}_R(\varphi^n R)\).

Question. What about the Gorenstein property?

Need a substitute for the projective dimension.
Definition. [Auslander-Bridger] Let \( R \) be a local ring and \((-)^* = \text{Hom}_R(-, R)\). A finite \( R \)-module \( G \) is *totally reflexive* if

(i) The natural map \( G \to G^{**} \) is an isomorphism, and

(ii) \( \text{Ext}^i_R(G, R) = 0 = \text{Ext}^i_R(G^*, R) \) for each integer \( i \geq 1 \).

(E.g., finitely generated projective modules are totally reflexive.)

A finite \( R \)-module \( M \) has *finite \( G \)-dimension* if there exists a resolution \( 0 \to G_n \to \cdots \to G_0 \to M \to 0 \) with each \( G_i \) totally reflexive, and \( \text{G-dim}_R(M) \) is the infimum of \( n \) for which such a resolution exists.

Properties. Let \((R, \mathfrak{m}, k)\) be a local ring and \( N \) a finite \( R \)-module.

- There is an inequality \( \text{G-dim}_R(N) \leq \text{pd}_R(N) \) with equality when \( \text{pd}_R(N) < \infty \).

- (AB-formula) When \( \text{G-dim}_R(N) < \infty \), one has

\[
\text{G-dim}_R(N) = \text{depth}(R) - \text{depth}_R(N).
\]
• The following conditions are equivalent.

(a) \( R \) is Gorenstein.
(b) \( \text{G-dim}_R(M) < \infty \) for each finite \( R \)-module \( M \).
(c) \( \text{G-dim}_R(k) < \infty \).

With The Theme in mind, what are the analogs of the results of Kunz and Rodicio for G-dimension?

Danger! The G-dimension is only defined (here) for finite modules. In general, the Frobenius is not finitely generated.

To overcome this, use the technology of Cohen factorizations, as constructed by Avramov-Foxby-Herzog, and exploit the extensive investigation of their Gorenstein properties by Avramov-Foxby.

Instead of focusing exclusively on the Frobenius, it is convenient to develop the theory for a general local homomorphism.
2 G-dim over a local homomorphism

Notation. Let $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism and $M$ a finite $S$-module. When $\iota: (S, \mathfrak{n}) \to (\hat{S}, \hat{\mathfrak{n}})$ is the natural map into the $\mathfrak{n}$-adic completion, set $\hat{\varphi} = \iota \varphi: R \to \hat{S}$ and $\hat{M} = M \otimes_S \hat{S}$.

Definition. A Cohen factorization of $\varphi$ is a diagram of local homomorphisms $R \xrightarrow{\hat{\varphi}} R' \xrightarrow{\varphi'} S$ where $\varphi = \varphi' \hat{\varphi}$, with $\hat{\varphi}$ flat, $R'$ complete, $R'/\mathfrak{m}R'$ regular, and $\varphi'$ surjective. Set $\text{edim}(\hat{\varphi}) = \text{edim}(R'/\mathfrak{m}R')$.

Fact. A Cohen factorization of $\varphi$ exists.

The point. Replace the nonfinte map $\varphi$ with the very finite map $\varphi'$.

Theorem. (I-S-W) The quantity $\text{G-dim}_{R'}(\hat{M}) - \text{edim}(\hat{\varphi})$ is independent of the choice of Cohen factorization.

Definition. $\text{G-dim}_{\varphi}(M) := \text{G-dim}_{R'}(\hat{M}) - \text{edim}(\hat{\varphi})$ and $\text{G-dim}(\varphi) := \text{G-dim}_{\varphi}(S')$. 
Properties. Let $\psi: R \to T$ be a local homomorphism and $N$ a finite $T$-module.

- If $N$ is finite over $R$, then $\text{G-dim}_\psi(N) = \text{G-dim}_R(N)$.
- (AB-formula) If $\text{G-dim}_\psi(N) < \infty$, then
  $$\text{G-dim}_\psi(N) = \text{depth}(R) - \text{depth}_T(N).$$
- The following conditions are equivalent.
  (a) $R$ is Gorenstein.
  (b) For every local homomorphism $\varphi: R \to S$ and every finite $S$-module $M$, $\text{G-dim}_\varphi(M) < \infty$.
  (c) There exists a local homomorphism $\varphi: (R, m) \to (S, n)$ and an ideal $I$ of $S$ such that $\varphi(m) \subseteq I$ and $\text{G-dim}_\varphi(S/I) < \infty$.
- If $\varphi: R \to S$ is a local homomorphism and $M$ a finite $S$-module isomorphic to $N$ over $R$, then $\text{G-dim}_\varphi(M)$ and $\text{G-dim}_\psi(N)$ are simultaneously finite, though they may not be equal.
3 G-dimension over the Frobenius

Theorem. (I-S-W) For a local ring $R$ of characteristic $p$ with Frobenius endomorphism $\varphi$, the following conditions are equivalent.

(a) $R$ is Gorenstein.

(b) $G\text{-dim}(\varphi^n) < \infty$ for some/each integer $n \geq 1$.

Sketch of the proof of “(b) $\implies$ (a)”. Assume that $G\text{-dim}(\varphi^n) < \infty$ for a fixed integer $n \geq 1$. Pass to the completion $\hat{R}$ to assume that $R$ has a dualizing complex $D^R$.

Claim. $G\text{-dim}(\varphi^{sn}) < \infty$ for all integers $s \geq 1$.

It is not known in general whether the composition of local homomorphisms of finite G-dimension has finite G-dimension. However, for powers of an endomorphism of a local ring, it is true. To see why, we use more of Avramov-Foxby.
Let $\psi: (S, n) \to (S', n')$ be a local homomorphism of complete rings with $\text{G-dim}(\psi) < \infty$. Let $I_S(t)$ be the Bass series of $S$:

$$I_S(t) = \sum_{i \geq 0} \mu_i(S) t^i.$$ 

Then $\mu_i(S) = \beta_i(D^S)$ and $S$ is Gorenstein if and only if $I_S(t)$ is a polynomial. Avramov-Foxby define the Bass series of $\psi$, which is a formal Laurent series $I_\psi(t)$ such that

$$I_{S'}(t) = I_S(t) I_\psi(t).$$

The homomorphism $\psi$ is quasi-Gorenstein at $n'$ if $\text{G-dim}(\psi) < \infty$ and $I_\psi(t)$ is a Laurent polynomial. Equivalently, $\psi$ is quasi-Gorenstein at $n'$ if $D^S \otimes_S^L S'$ is a dualizing complex for $S'$. 

Fact. The composition of quasi-Gorenstein homomorphisms is quasi-Gorenstein.

Back to the Frobenius. We have assumed that $\text{G-dim}(\varphi^n) < \infty$. 
The equality $I_R(t) = I_R(t)I_{\varphi^n}(t)$ implies that $I_{\varphi^n}(t) = 1$, so that $\varphi^n$ is quasi-Gorenstein at $\mathfrak{m}$. Thus, $\varphi^{sn}$ is quasi-Gorenstein at $\mathfrak{m}$ for each $s \geq 1$. In particular, $\text{G-dim}(\varphi^{sn}) < \infty$ for each $s \geq 1$.

Claim. $R$ is Gorenstein.

It suffices to show that $\text{pd}_R(D^R) < \infty$. To check this, we use a variation of a theorem of Koh-Lee.

For a complex $X$, let $\text{sup}(X) = \sup\{i \mid H_i(X) \neq 0\}$.

Proposition. Let $X$ be a complex of $R$-modules with $H(X)$ finite. If $\text{sup}(X \otimes^L_R \varphi^m R) < \infty$ for infinitely many $m \geq 1$, then $\text{pd}_R(X) < \infty$.

Since $\varphi^{sn}$ is quasi-Gorenstein at $\mathfrak{m}$ for each $s \geq 1$, the complex $D^R \otimes^L_R \varphi^{sn} R$ is dualizing for $R$. In particular, $\text{sup}(D^R \otimes^L_R \varphi^{sn} R)$ is finite, so the proposition implies that $\text{pd}(D^R) < \infty$.

Takahashi-Yoshino have proved this when $R$ is F-finite. So it is worth mentioning a generalization.
Theorem. (I-S-W) Let \((R, \mathfrak{m})\) be a local ring with a local endomorphism \(\varphi\) such that \(\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2\) for some \(i \geq 1\). The following conditions are equivalent.

(a) \(R\) is Gorenstein.

(b) \(\text{G-dim}(\varphi^n) < \infty\) for some/each integer \(n \geq 1\).

(c) There exists a finite \(R\)-module \(M \neq 0\) such that \(\text{pd}_R(M) < \infty\) and \(\text{G-dim}_{\varphi^n}(M) < \infty\) for some \(n \geq 1\).

The implication “(c) \(\implies\) (b)” is a consequence of the following theorem which is proved using the Amplitude Inequality of Iversen and Foxby-Iyengar.

Theorem. (I-S-W) Let \(\varphi: R \to S\) be a local homomorphism and \(M\) a nonzero finitely generated \(S\)-module with \(\text{pd}_S(M) < \infty\). Then

\[
\text{G-dim}_{\varphi}(M) = \text{G-dim}(\varphi) + \text{pd}_S(M).
\]

In particular, \(\text{G-dim}_{\varphi}(M) < \infty\) if and only if \(\text{G-dim}(\varphi) + \text{pd}_S(M)\).