SEMIDUALIZING MODULES

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Abstract. Here is a survey of some aspects of semidualizing modules, theory and applications.

Contents

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Introduction

Semidualizing modules have been found to be useful for several constructions in homological algebra. Introduced by Foxby, Vasconcelos, Golod, and Wakamatsu. Studied by Reiten, White, Sather-Wagstaff, Holm, Avramov, Foxby, Christensen, Frankild, Danes, Iranians

In this article we survey some aspects of the theory and application of these modules. For unexplained terminology, the reader is encouraged to consult Matsumura and Bruns and Herzog.

We make no attempt to be exhaustive. For instance, we omit relative cohomology, Wakamatsu and who knows what else. The point is to motivate the reader with applications and to make the reader familiar enough with the tools to be able to read more in the literature.

In a sense, the most natural setting for this material is in the derived category where semidualizing modules are special cases of the semidualizing complexes, just as dualizing modules are special cases of dualizing complexes. however, this is not necessarily the most natural place for someone who is not familiar with the derived category to learn about the material. For readers who are comfortable with the derived category, I recommend the papers of Avramov and Foxby [4], Christensen [7] and Gerko [12] along with papers that reference these, e.g., [9, 10, 11, 18].

Good exercises: check that the diagrams actually commute, and prove the results and parts of results that are left unproved. Of course, any detail that is not explained should be verified.

Throughout, let R be a commutative noetherian ring.[1]

1. Prelude

We begin by surveying some of the "classical" aspects of homological commutative algebra, which will motivate the definition of semidualizing modules. We will focus in this section on finitely generated modules, although there are versions of these theories for non-finitely generated modules (and for chain complexes), in an attempt to keep things accessible. Note also that this section does not adhere to the original chronology of the research.

1.1. **Projective Dimension.** Let M be a finitely generated R-module. In a sense, the nicest R-modules are the free modules and, more generally, the projective modules. Most modules are not projective. (For instance, when R is a local ring, every R-module is projective if and only if R is a field.) However, there is a process by which one can "approximate" M by projective R-modules.

Specifically, there is a finitely generated projective R-module P_0 equipped with a surjection $\tau\colon P_0 \to M$. If M is not projective, then $M_1 = \operatorname{Ker}(\tau) \neq 0$; this "syzygy module" can be thought of as the error from the approximation of M by P_0. The module M_1 may or may not be projective, but we can approximate it by a projective R-module as we did with M.

Indeed, since R is noetherian and P_0 is finitely generated, the submodule $M_1 \subseteq P_0$ is also finitely generated. Repeat the above procedure inductively to find surjections $\tau_{i+1}\colon P_{i+1} \to M_{i+1}$ for each $i \geqslant 0$ where P_{i+1} is projective and $M_{i+1} =$

[1]While neither of these conditions is necessary for the study of semidualizing modules, these assumptions allow for a certain amount of simplicity in our hypotheses.

$\mathrm{Ker}(\tau_i) \subseteq P_i$. Composing the surjections τ_{i+1} with the inclusions $M_{i+1} \subseteq P_i$, we obtain the following exact sequence

$$P^+ = \cdots \xrightarrow{\partial^P_{i+1}} P_i \xrightarrow{\partial^P_i} P_{i-1} \xrightarrow{\partial^P_{i-1}} \cdots P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \to 0$$

which we call an *augmented projective resolution* of M. The *projective resolution* of M associated to P^+ is the sequence obtained by truncating:

$$P = \cdots \xrightarrow{\partial^P_{i+1}} P_i \xrightarrow{\partial^P_i} P_{i-1} \xrightarrow{\partial^P_{i-1}} \cdots P_1 \xrightarrow{\partial^P_1} P_0 \to 0.$$

Note that P is not in general exact. Indeed, one has $\mathrm{Ker}(\partial^P_i) = \mathrm{Im}(\partial^P_{i+1})$ for each $i \geqslant 1$, but $\mathrm{Coker}(\partial^P_1) \cong M$, and so P is exact if and only if $M = 0$. (One might say that P is "acyclic", but we will not use this term because it means different things to different people.) We say that P is a *free resolution* of M when each P_i is free. Note that, when R is local, an R-module is free if and only if it is projective, and so the notions of projective resolution and free resolution are the same in this setting.

If M admits a projective resolution P such that $P_i = 0$ for $i \gg 0$, then we say that M has finite projective dimension. More specifically, the *projective dimension* of M is the shortest such resolution:

$$\mathrm{pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid P_n \neq 0\} \mid P \text{ is a projective resolution of } M\}.$$

Modules with finite projective dimension are quite special, as we will see below. One need not look far to find modules of finite projective dimension: Hilbert's famous Syzygy Theorem [16] says that, when k is a field, every finitely generated module over the polynomial ring $k[X_1, \ldots, x_d]$ has projective dimension at most d. In the local setting, this is a sort of precursor to the famous theorem of Auslander, Buchsbaum [3] and Serre [19]:

Theorem 1.1.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of Krull dimension d. The following conditions are equivalent:*

(i) *R is regular, that is, the maximal ideal $\mathfrak{m}$ can be generated by d elements;*
(ii) *$\mathrm{pd}_R(M) < \infty$ for each finitely generated R-module;*
(iii) *$\mathrm{pd}_R(k) < \infty$.*

One important application of this result is the solution of the localization problem for regular local rings: If R is a regular local ring and $\mathfrak{p} \subset R$ is a prime ideal, then the localization $R_\mathfrak{p}$ is also regular.

Theorem 1.1.1 substantiates the following maxim: to understand a ring is to understand its modules. If you like, the nicer the ring, the nicer its modules, and conversely. We shall see this maxim in action in numerous places below. One could say, as I often do, that module theory is representation theory for rings, with the modules taking the place of representations. This is backwards, though, since representation theory is, in fact, nothing other than the module theory of group rings.

Another feature of the projective dimension is the "Auslander-Buchsbaum formula" [3]:

Theorem 1.1.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. If M is an R-module of finite projective dimension, then $\mathrm{pd}_R(M) = \mathrm{depth}(R) - \mathrm{depth}_R(M)$; in particular, if $M \neq 0$, then $\mathrm{depth}_R(M) \leqslant \mathrm{depth}(R)$.*

Here, the "depth" of M is the length of the longest M-regular sequence in $\mathfrak{m}$; this can be expressed homologically as

$$\operatorname{depth}_R(M) = \inf\{i \geqslant 0 \mid \operatorname{Ext}^i_R(k, M) \neq 0\}.$$

And $\operatorname{depth}(R) = \operatorname{depth}_R(M)$.

Note that this result shows how to find modules of infinite projective dimension; just find a module $M \neq 0$ with $\operatorname{depth}_R(M) > \operatorname{depth}(R)$. For instance, when k is a field, the ring $R = k[\![X, Y]\!]/(X^2, XY)$ has depth 0 and the module $M = R/XR \cong k[\![X]\!]$ has depth 1.

1.2. **Complete Intersection Dimension.** The class of regular local rings is not stable under specialization: if $(R, \mathfrak{m})$ is a regular local ring and $x \in \mathfrak{m}$ is an R-regular element, then R/xR may not be a regular local ring. This corresponds to the geometric fact that a hypersurface in a smooth variety need not be smooth. In a sense, this is unfortunate. However, it leads to our next class of rings.

Definition 1.2.1. A local ring $(R, \mathfrak{m})$ is a *complete intersection* if its $\mathfrak{m}$-adic completion $\widehat{R}$ has the form $\widehat{R} \cong Q/(\mathbf{x})Q$ where Q is a regular local ring and $\mathbf{x}$ is a Q-regular sequence.

Recall that Cohen's Structure Theorem [8] guarantees that the completion of any local ring is a homomorphic image of a regular local ring. Since the completion of a regular local ring is regular, it follows that every regular local ring is a complete intersection. It is straightforward to show that the class of complete intersection rings is closed under specialization. Furthermore, this definition of complete intersection is independent of the choice of regular local ring surjecting onto $\widehat{R}$: a theorem of Grothendieck [14, (19.3.2)] says that, if R is a complete intersection and $\pi \colon A \to Q$ is a ring epimorphism where A is a regular local ring, then $\operatorname{Ker}(\pi)$ is generated by an A-regular sequence.

Avramov, Gasharov and Peeva [5] introduced the *complete intersection dimension* of a finitely generated R-module M, in part, to find and study modules whose free resolutions do not grow too quickly. For the sake of simplicity, we only discuss this invariant when R is local. Recall that a ring homomorphism of local rings $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is *local* when $\mathfrak{m}S \subseteq \mathfrak{n}$.

Definition 1.2.2. Let $(R, \mathfrak{m})$ be a local ring. A *quasi-deformation* of R is a diagram of local ring homomorphisms

$$R \xrightarrow{\rho} R' \xleftarrow{\tau} Q$$

where ρ is flat and τ is surjective with kernel generated by a Q-regular sequence.

A finitely generated R-module M has *finite complete intersection dimension* when there exists a quasi-deformation $R \to R' \leftarrow Q$ such that $\operatorname{pd}_Q(R' \otimes_R M)$ is finite; specifically, we have

$$\operatorname{CI-dim}_R(M) = \inf\{\operatorname{pd}_Q(R' \otimes_R M) - \operatorname{pd}_Q(R') \mid R \to R' \leftarrow Q \text{ quasi-deformation}\}.$$

When R is a local complete intersection, it follows readily from Theorem 1.1.1 that every R-module has finite complete intersection dimension: write $\widehat{R} \cong Q/(\mathbf{x})Q$ where Q is a regular local ring and $\mathbf{x}$ is a Q-regular sequence and use the quasi-deformation $R \to \widehat{R} \leftarrow Q$. Moreover, Avramov, Gasharov and Peeva [5] show that the complete intersection dimension satisfies properties like those in Theorems 1.1.1 and 1.1.2:

Theorem 1.2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring of Krull dimension d. The following conditions are equivalent:*

(i) *R is a complete intersection;*
(ii) $\text{CI-dim}_R(M) < \infty$ *for each finitely generated R-module;*
(iii) $\text{CI-dim}_R(k) < \infty$.

Theorem 1.2.4 (AB-formula)**.** *Let R be a local ring and M a finitely generated R-module. If $R \to R' \leftarrow Q$ is a quasi-deformation such that* $\text{pd}_Q(R' \otimes_R M) < \infty$*, then* $\text{pd}_Q(R' \otimes_R M) - \text{pd}_Q(R') = \text{depth}(R) - \text{depth}_R(M)$*. If* $\text{CI-dim}_R(M) < \infty$*, then* $\text{CI-dim}_R(M) = \text{depth}(R) - \text{depth}_R(M)$*; in particular, if $M \neq 0$, then* $\text{depth}_R(M) \leqslant \text{depth}(R)$.

The "AB" in the AB-formula stands for Auslander-Buchsbaum, naturally, and also Auslander-Bridger, as we shall see below. As a consequence of the AB-formula, we see that the complete intersection dimension is a refinement of the projective dimension.

Corollary 1.2.5. *Let R be a local ring and M a finitely generated R-module. There is an inequality* $\text{CI-dim}_R(M) \leqslant \text{pd}_R(M)$*, with equality when* $\text{pd}_R(M) < \infty$.

Proof. Assume without loss of generality that $\text{pd}_R(M) < \infty$. Using the trivial quasi-deformation $R \to R \leftarrow R$, it is straightforward to see that M has finite complete intersection dimension. Theorems 1.1.2 and 1.2.4 show that $\text{CI-dim}_R(M) = \text{pd}_R(M)$, as desired. $\square$

Using the work of Cohen [8] and Grothendieck [13], Avramov, Gasharov and Peeva [5] show one can exert a certain amount of control on the structure of quasi-deformations:

Proposition 1.2.6. *Let $(R, \mathfrak{m})$ be a local ring and M an R-module of finite complete intersection dimension. Then there exists a quasi-deformation $R \xrightarrow{\rho} R' \leftarrow Q$ such that $\text{pd}_Q(R' \otimes_R M) < \infty$ and such that Q is complete with algebraically closed residue field and such that the closed fibre $R'/\mathfrak{m}R'$ is artinian (hence, Cohen-Macaulay).*

We shall see in Theorem 7.0.3 below how semidualizing modules allow us to improve Proposition 1.2.6.

Here is an open question that I would very much like to answer. Note that the corresponding result for modules of finite projective dimension is well-known.

Question 1.2.7. Let R be a local ring and consider an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of finitely generated R-modules. If two of the modules M_i have finite complete intersection dimension, must the third one also?

If one of the M_i has finite projective dimension, then Question 1.2.7 is readily answered in the affirmative. In particular, every module of finite complete intersection dimension has a bounded resolution by modules of complete intersection dimension zero, namely, an appropriate truncation of a projective resolution. On the other hand, it is not known whether M must have finite complete intersection dimension if it has a bounded resolution by modules of complete intersection dimension zero. Indeed, this is equivalent to one of the implications in Question 1.2.7.

1.3. **G-Dimension.** It is well known that R is always projective as an R-module. It is natural to ask whether it is always self-injective, i.e., injective as an R-module. The answer is "no" in general because, for instance, if a local ring R has a finitely generated injective module, then R must be artinian. One can hope to remedy this by asking whether R has finite injective dimension as an R-module, that is, when does there exist an exact sequence

$$0 \to R \to I_0 \to I_1 \to \cdots \to I_d \to 0$$

where each I_j is an injective R-module? Once again, the answer is "no" because, for instance, if a local ring R has a finitely generated module of finite injective dimension, then R must be Cohen-Macaulay.

Definition 1.3.1. A local ring R is *Gorenstein* if it has finite injective dimension as an R-module.

These rings are named after the famous group theorist Daniel Gorenstein who showed that the group algebra $k[G]$ of a finite group G is always self-injective.

It can be shown, using techniques of Auslander, Buchsbaum [3] and Serre [19], that every finitely generated module over a regular local ring has finite injective dimension; hence, every regular local ring is Gorenstein. Furthermore, the class of Gorenstein rings is stable under specialization, and so every complete intersection is also Gorenstein. Thus we have the implications

$$\text{regular} \implies \text{complete intersection} \implies \text{Gorenstein} \implies \text{Cohen-Macaulay}.$$

Auslander and Bridger [1, 2] introduced the *G-dimension* of a finitely generated module, in part, to give a module-theoretic characterization of Gorenstein rings like that in Theorem 1.1.1. Like the projective dimension, it is defined in terms of resolutions by certain modules with good homological properties, the "totally reflexive" modules. To define these modules, we need the natural biduality map.

Definition 1.3.2. Let M and N be R-modules. The natural biduality map

$$\delta_N^M : N \to \operatorname{Hom}_R(\operatorname{Hom}_R(N, M), M)$$

is the R-module homomorphism given by $\delta_N^M(n)(\psi) = \psi(n)$.

As the name suggests, a totally reflexive module is a reflexive module with some additional properties. The addition properties have to do with the vanishing of the Ext-modules that arise from the biduality map.

Definition 1.3.3. An R-module G is *totally reflexive* if it satisfies the following conditions:

(1) G is finitely generated over R,
(2) $\operatorname{Ext}_R^{\geqslant 1}(G, R) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(G, R), R)$, and
(3) The natural biduality map $\delta_G^R \colon G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G, R), R)$ is an isomorphism.

We let $\mathcal{G}(R)$ denote the class of all totally reflexive R-modules.

Example 1.3.4. Every finitely generated projective R-module is totally reflexive; see Proposition 4.1.2.

Definition 1.3.5. Let M be a finitely generated R-module. An *augmented G-resolution* of M is an exact sequence

$$G^+ = \cdots \xrightarrow{\partial^G_{i+1}} G_i \xrightarrow{\partial^G_i} G_{i-1} \xrightarrow{\partial^G_{i-1}} \cdots G_1 \xrightarrow{\partial^G_1} G_0 \xrightarrow{\tau} M \to 0$$

wherein each G_i is totally reflexive. The *G-resolution* of M associated to G^+ is the sequence obtained by truncating:

$$G^+ = \cdots \xrightarrow{\partial^G_{i+1}} G_i \xrightarrow{\partial^G_i} G_{i-1} \xrightarrow{\partial^G_{i-1}} \cdots G_1 \xrightarrow{\partial^G_1} G_0 \to 0$$

Since every finitely generated R-module has a resolution by finitely generated projective R-modules and every finitely generated projective R-module is totally reflexive, it follows that every finitely generated R-module has a G-resolution.

Definition 1.3.6. Let M be a finitely generated R-module. If M admits a G-resolution G such that $G_i = 0$ for $i \gg 0$, then we say that M has finite G-dimension. More specifically, the *G-dimension* of M is the shortest such resolution:

$$\text{G-dim}_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_n \neq 0\} \mid G \text{ is a G-resolution of } M\}.$$

Auslander and Bridger [1, 2] show that the G-dimension satisfies properties like those in Theorems 1.1.1 and 1.1.2:

Theorem 1.3.7. *Let $(R, \mathfrak{m}, k)$ be a local ring of Krull dimension d. The following conditions are equivalent:*

(i) *R is Gorenstein;*
(ii) $\text{G-dim}_R(M) < \infty$ *for each finitely generated R-module;*
(iii) $\text{G-dim}_R(k) < \infty$.

Theorem 1.3.8 (AB-formula)**.** *Let R be a local ring and M a finitely generated R-module. If* $\text{G-dim}_R(M) < \infty$*, then* $\text{G-dim}_R(M) = \text{depth}(R) - \text{depth}_R(M)$*; in particular, if $M \neq 0$, then* $\text{depth}_R(M) \leqslant \text{depth}(R)$.

Corollary 1.3.9. *Let R be a local ring and M a finitely generated R-module. There are inequalities* $\text{G-dim}_R(M) \leqslant \text{CI-dim}_R(M) \leqslant \text{pd}_R(M)$*, with equality to the left of any finite quantity.*

Here is another open question that I would very much like to answer. It is a special case (though, equivalent to the general case) of Avramov and Foxby's Composition Question for ring homomorphisms of finite G-dimension [4, (4.8)]. Note that it is straightforward to answer the corresponding result for homomorphisms of finite projective dimension in the affirmative. The analogue for complete intersection dimension is also open.

Question 1.3.10. Let $R \to S \to T$ be surjective local ring homormophisms. If $\text{G-dim}_R(S)$ and $\text{G-dim}_S(T)$ are finite, must $\text{G-dim}_R(T)$ also be finite?

We shall see in Theorem 8.0.4 below how semidualizing modules allow us to give a partial answer to Question 1.3.10.

1.4. **Other Dualities.** One can modify Definition 1.3.3 to consider dualities with respect to modules other than R. However, not every class of modules which arises in this way is well-suited for building a homological dimension. We shall see next that, in a sense, the best class of modules arise from considering dualities with respect to semidualizing modules.

Definition 1.4.1. Let C be a finitely generated R-module. An R-module G is *totally C-reflexive* if it satisfies the following conditions:

(1) G is finitely generated over R,
(2) $\operatorname{Ext}_R^{\geqslant 1}(G,C) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(G,C),C)$, and
(3) The natural biduality map $\delta_G^C\colon G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G,C),G)$ is an isomorphism.

We let $\mathcal{G}_C(R)$ denote the class of all totally C-reflexive R-modules.

Now we are finally ready to define the main players of this article.

Definition 1.4.2. Let C be an R-module. The *homothety morphism* for C is the R-linear map $\chi_C^R\colon R \to \operatorname{Hom}_R(C,C)$ given by $r \mapsto [c \mapsto rc]$.

Definition 1.4.3. The R-module C is *semidualzing* if it satisfies the following conditions:

(1) C is finitely generated;
(2) The homothety morphism $\chi_C^R\colon R \to \operatorname{Hom}_R(C,C)$ is an isomorphism; and
(3) $\operatorname{Ext}_R^i(C,C) = 0$ for all $i > 0$.

The R-module C is *dualizing* if it is semidualizing and has finite injective dimension.

Here is what we mean when we say that duality with respect to a semidualizing module is, in a sense, best. We shall see in Proposition 4.1.2 below that the conditions in this result are equivalent to *every* finitely generated projective R-module being totally C-reflexive.

Proposition 1.4.4. *Let C be a finitely generated R-module. The following conditions are equivalent:*

(i) C is a semidualizing R-module;
(ii) R is a totally C-reflexive R-module;
(iii) C is a totally C-reflexive R-module and $\operatorname{Ann}_R(C) = 0$.

Proof. Let $f\colon \operatorname{Hom}_R(R,C) \to C$ be the isomorphism given by $f(\psi) = \psi(1)$. It is readily shown that the following diagrams commute:

(1.4.4.1)
$$\begin{tikzcd} R \arrow[rr, "\chi_C^R"] \arrow[drr, "\delta_R^C"'] & & \operatorname{Hom}_R(C,C) \arrow[d, "\cong"', "\operatorname{Hom}_R(f,C)"] \\ & & \operatorname{Hom}_R(\operatorname{Hom}_R(R,C),C) \end{tikzcd}$$

(1.4.4.2)
$$\begin{tikzcd} C \arrow[rr, "\delta_C^C"] \arrow[drr, "\cong", "f"'] & & \operatorname{Hom}_R(\operatorname{Hom}_R(C,C),C) \arrow[d, "\operatorname{Hom}_R(\chi_C^R,C)"] \\ & & \operatorname{Hom}_R(R,C). \end{tikzcd}$$

(i) $\Longleftrightarrow$ (ii). For $i \geqslant 1$, we have $\operatorname{Ext}_R^i(R,C) = 0$ because R is projective, and
$$\operatorname{Ext}_R^i(\operatorname{Hom}_R(R,C),C) \cong \operatorname{Ext}_R^i(C,C).$$
In particular, we have $\operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(R,C),C) = 0$ if and only if $\operatorname{Ext}_R^{\geqslant 1}(C,C) = 0$. Furthermore, diagram (1.4.4.1) shows that δ_R^C is an isomorphism if and only if χ_C^R is an isomorphism. Thus C is semidualizing if and only if R is totally C-reflexive.

(i) $\Longrightarrow$ (iii). Assume that C is semidualizing. The isomorphism $\mathrm{Hom}_R(C,C) \cong R$ implies $\mathrm{Ann}_R(C) \subseteq \mathrm{Ann}_R(R) = 0$.

We next show that C is totally C-reflexive. For $i \geqslant 1$, we have

$$\mathrm{Ext}^i_R(C,C) = 0$$

because C is semidualizing, and also

$$\mathrm{Ext}^i_R(\mathrm{Hom}_C(C,C),C) \cong \mathrm{Ext}^i_R(R,C) = 0.$$

Using the fact that χ^R_C is an isomorphism, diagram (1.4.4.2) shows that δ^C_C is an isomorphism, and so C is totally C-reflexive.

(iii) $\Longrightarrow$ (i). Assume that C is a totally C-reflexive R-module and $\mathrm{Ann}_R(C) = 0$. Note that it follows that $\mathrm{Supp}_R(C) = V(\mathrm{Ann}_R(C)) = V(0) = \mathrm{Spec}(R)$. It follows that $\mathrm{Ext}^i_R(C,C) = 0$, so it remains to show that χ^R_C is an isomorphism. We have $0 = \mathrm{Ann}_R(C) = \mathrm{Ker}(\chi^R_C)$, and so χ^R_C is injective. Set $N = \mathrm{Coker}(\chi^R_C)$ and consider the exact sequence

$$0 \to R \xrightarrow{\chi^R_C} \mathrm{Hom}_R(C,C) \to N \to 0.$$

The associated long exact sequence in $\mathrm{Ext}_R(-,C)$ begins as follows

$$0 \to \mathrm{Hom}_R(N,C) \to \mathrm{Hom}_R(\mathrm{Hom}_R(C,C),C) \xrightarrow[\cong]{\mathrm{Hom}_R(\chi^R_C,C)} \mathrm{Hom}_R(R,C).$$

The fact that $\mathrm{Hom}_R(\chi^R_C,C)$ is an isomorphism follows from diagram (1.4.4.2) because C is totally C-reflexive. We conclude that $\mathrm{Hom}_R(N,C) = 0$. The next piece of the long exact sequence has the following form

$$\mathrm{Hom}_R(\mathrm{Hom}_R(C,C),C) \xrightarrow{\cong} \mathrm{Hom}_R(R,C) \to \mathrm{Ext}^1_R(N,C) \to \underbrace{\mathrm{Ext}^1_R(\mathrm{Hom}_R(C,C),C)}_{=0}$$

and so $\mathrm{Ext}^1_R(N,C) = 0$. Other pieces of the long exact sequence have the form

$$\underbrace{\mathrm{Ext}^{i-1}_R(R,C)}_{=0} \to \mathrm{Ext}^i_R(N,C) \to \underbrace{\mathrm{Ext}^i_R(\mathrm{Hom}_R(C,C),C)}_{=0}$$

and it follows that $\mathrm{Ext}^i_R(N,C) = 0$ for all $i \geqslant 0$.

We will be done once we show that $N = 0$, so suppose that $N \neq 0$ and let $\mathfrak{p} \in \mathrm{Supp}_R(N)$. It follows that $\mathfrak{p} \in \mathrm{Spec}(R) = \mathrm{Supp}_R(C)$, and so $C_\mathfrak{p}, N_\mathfrak{p}$ are nonzero finitely generated $R_\mathfrak{p}$-modules. From [17, (16.6)] it follows that there exists some $i \geqslant 0$ such that

$$0 \neq \mathrm{Ext}^i_{R_\mathfrak{p}}(N_\mathfrak{p}, C_\mathfrak{p}) = \mathrm{Ext}^i_R(N,C)_\mathfrak{p} = 0$$

which is clearly a contradiction. $\square$

Question 1.4.5. Let C be a finitely generated R-module and assume that C is totally C-reflexive. Does it follow that C is a semidualizing R-module? In other words, does it follow that $\mathrm{Ann}_R(C) = 0$?

2. Semidualizing Basics

In this section, we survey the basic properties of semidualizing modules. We begin with some examples, the first of which is readily verified.

2.1. **Examples.**

Example 2.1.1. A free R-module of rank 1 is semidualizing. A free R-module is semidualizing if and only if it has rank 1. The module R^1 is dualizing for R if and only if R is Gorenstein.

Here is a loose discussion of one of the most important examples.

Remark 2.1.2. Dualizing modules were introduced by Grothendieck [15] for the study of local cohomology. While every ring R admits a semidualzing module, namely R, the existence of a dualizing module is much more restrictive: R admits a dualizing module if and only if it is Cohen-Macaulay and a homomorphic image of a Gorenstein ring. We sketch the ideas of the proof here as they will come up in the sequel. Furthermore, it shows (at least in the abstract) how to build examples of dualizing modules. See [6, Ch. 3] for more details.

First, assume that R admits a dualzing module D. In particular, D is a finitely generated R-module of finite injective dimension, so R is Cohen-Macaulay. [ref] Consider the *trivial extension* of R by D (also known as the *idealization* of D) denoted $R \ltimes D$. As an R-module, we have $R \ltimes D = R \oplus D$. And we endow $R \ltimes D$ with a ring structure given by $(r,d)(r'd') = (rr', rd' + r'd)$. This makes $R \ltimes D$ into a (commutative noetherian) ring. Furthermore, there is a commutative diagram of ring homomorphisms

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "\mathrm{id}_R"'] & R \ltimes D \arrow[d, "g"] \\ & R \end{tikzcd}$$

where $f(r) = (r, 0)$ and $g(r,d) = r$. The ring $R \ltimes D$ is Gorenstein [ref] and g is surjective, so R is a homomorphic image of the Gorenstein ring $R \ltimes D$.

Conversely, assume that R is Cohen-Macaulay and a homomorphic image of a Gorenstein ring S, say $R \cong S/\mathfrak{a}$. If $\dim(R) = \dim(S)$, then $\mathrm{Ext}^i_S(R,S) = 0$ for all $i \neq 0$, and the R-module

$$\mathrm{Hom}_S(R,S) \cong \mathrm{Hom}_S(S/\mathfrak{a}, S) \cong \{s \in S \mid s\mathfrak{a} = 0\}$$

is dualizing for R. More generally, set $d = \dim(S) - \dim(R) \geqslant 0$; then $\mathrm{Ext}^i_S(R,S) = 0$ for all $i \neq d$, and the R-module $\mathrm{Ext}^d_S(R,S)$ is dualizing for R. [ref] In practice, one can reduce this construction to the case $d = 0$ by replacing S with $S/(y_1, \ldots, y_d)S$ where $y_1, \ldots, y_d$ is an S-regular sequence contained in $\mathfrak{a}$. More generally, if S is only assumed to be Cohen-Macaulay with a dualizing module E and R is only assumed to be module finite over S, then $\mathrm{Ext}^i_S(R,S) = 0$ for all $i \neq d = \dim(S) - \dim(R) \geqslant 0$ and R has a dualizing module $D = \mathrm{Ext}^d_S(R,E)$.

Here is an honest example.

Example 2.1.3. Let k be a field.

The ring $R_1 = k[X_1, \ldots, X_m]/(X_1^{r_1}, \ldots, X_m^{r_m})$ is local, Gorenstein and artinian. It follows (ref below) that R_1 has exactly one semidualzing module, up to isomorphism, namely R_1^1.

Consider the ring

$$R_2 = k[X_1, \ldots, X_m]/(X_1, \ldots, X_m)^2 \cong k[\![X_1, \ldots, X_m]\!]/(X_1, \ldots, X_m)^2$$

which is artinian (hence Cohen-Macaulay) and local with maximal ideal $\mathfrak{m} = (X_1, \ldots, X_m)R_2$; it is Gorenstein if and only if $m = 1$. It is a homomorphic image of the Gorenstein local rings $k[\![X_1, \ldots, X_m]\!]$ and $S = k[\![X_1, \ldots, X_m]\!]/(X_1^2, \ldots, X_m^2)$. In particular, R_2 has a dualizing module D. (Because $\mathfrak{m}^2 = 0$, we also know that R_2 has exactly two semidualzing modules, up to isomorphism, namely R_1^1 and D; see [ref below]). Let $x_i \in S$ denote the residue of X_i. Setting $\mathfrak{n} = (x_1, \ldots, x_m)S$, the maximal ideal of S, we have $R_2 \cong S/\mathfrak{n}^2$. Because $\dim(S) = 0 = \dim(R_2)$, Remark 2.1.2 implies

$$\begin{aligned} D &\cong \{s \in S \mid s\mathfrak{n}^2 = 0\} \\ &= \{s \in S \mid s x_i x_j = 0 \text{ for all } i \neq j\} \\ &= (\textstyle\prod_{j \neq i} x_j \mid i = 1, \ldots, m)S. \end{aligned}$$

The next result shows one way to build new semidualizing modules from old ones. We will see other methods below. Recall that a ring R is *essentially of finite type* over another ring Q if R is isomorphic to a ring of the form $S^{-1}Q[X_1, \ldots, X_m]/\mathfrak{a}$. Also, recall that, if R_1 and R_2 are essentially of finite type over Q, then the ring $R_1 \otimes_Q R_2$ is essentially of finite type over Q, and hence is noetherian.

Proposition 2.1.4. *Let R_1 and R_2 be rings essentially of finite type over a field k. For $i = 1, 2$ let C_i be a semidualizing R_i-module. Then the $R_1 \otimes_k R_2$-module $C_1 \otimes_k C_2$ is semidualizing.*

Proof. One checks readily that $C_1 \otimes_k C_2$ is a finitely generated $R_1 \otimes_k R_2$-module via the action $(r_1 \otimes r_2)(c_1 \otimes c_2) = (r_1 c_1) \otimes (r_2 c_2)$. The Künneth formula

$$\operatorname{Ext}^i_{R_1 \otimes_k R_2}(C_1 \otimes_k C_2, C_1 \otimes_k C_2) \cong \operatorname{Ext}^i_{R_1}(C_1, C_1) \otimes_k \operatorname{Ext}^i_{R_2}(C_2, C_2)$$

implies $\operatorname{Ext}^i_{R_1 \otimes_k R_2}(C_1 \otimes_k C_2, C_1 \otimes_k C_2) = 0$ for $i \geqslant 1$. Furthermore, there is a commutative diagram

$$\begin{tikzcd} R_1 \otimes_k R_2 \arrow[r, "\chi^{R_1 \otimes_k R_2}_{C_1 \otimes_k C_2}"] \arrow[dr, "\chi^{R_1}_{C_1} \otimes_k \chi^{R_2}_{C_2}"', "\cong"] & \operatorname{Hom}_{R_1 \otimes_k R_2}(C_1 \otimes_k C_2, C_1 \otimes_k C_2) \arrow[d, "\cong"] \\ & \operatorname{Hom}_{R_1}(C_1, C_1) \otimes_k \operatorname{Hom}_{R_2}(C_2, C_2). \end{tikzcd}$$

Hence, the homothety map $\chi^{R_1 \otimes_k R_2}_{C_1 \otimes_k C_2}$ is an isomorphism. □

We will see below (using reflexivity) that, if $C_1 \not\cong C_1'$ or $C_2 \not\cong C_2'$, then $C_1 \otimes_k C_2 \not\cong C_1' \otimes_k C_2'$.

One can iterate this process for tensor-product rings $R_1 \otimes_k R_2 \otimes_k \cdots \otimes_k R_n$. In particular, this process yields 2^n distinct semidualizing modules on the n-fold tensor product of rings of the form $k[X_1, \ldots, X_m]/(X_1, \ldots, X_m)^2$.

2.2. **Basic Properties.**

Proposition 2.2.1. *Let C be a semidualzing R-module. One has* $\operatorname{Supp}_R(C) = \operatorname{Spec}(R)$ *and* $\operatorname{Ass}_R(C) = \operatorname{Ass}_R(R)$ *and* $\dim_R(C) = \dim(R)$ *and* $\operatorname{Ann}_R(C) = 0$. *In particular, an element $x \in R$ is R-regular if and only if it is C-regular.*

Proof. The equality $\mathrm{Ann}_R(C) = 0$ is shown in Proposition 1.4.4. This implies $\mathrm{Supp}_R(C) = V(\mathrm{Ann}_R(C)) = V(0) = \mathrm{Spec}(R)$, and the equality $\dim_R(C) = \dim(R)$ follows directly. The isomorphism $\mathrm{Hom}_R(C,C) \cong R$ implies

$$\begin{aligned}\mathrm{Ass}_R(R) &= \mathrm{Ass}_R(\mathrm{Hom}_R(C,C)) = \mathrm{Supp}_R(C) \cap \mathrm{Ass}_R(C)\\ &= \mathrm{Spec}(R) \cap \mathrm{Ass}_R(C) = \mathrm{Ass}_R(C).\end{aligned}$$

For the last statement, note that $xC = C$ if and only if x is a unit. Indeed, one implication is immediate. For the nontrivial implication, assume that $xC = C$. It follows that, for each maximal ideal $\mathfrak{m} \subset R$, we have $xC_{\mathfrak{m}} = C_{\mathfrak{m}}$; since $C_{\mathfrak{m}} \neq 0$, Nakayama's lemma implies that x represents a unit in $R_{\mathfrak{m}}$ and so $x \notin \mathfrak{m}$. Since this is so for each maximal ideal, we conclude that x is a unit.

Assume for the rest of the proof that x is not a unit in R. Then x is a non-zerodivisor on R if and only if $x \notin \cup_{\mathfrak{p}\in\mathrm{Ass}_R(R)}\mathfrak{p} = \cup_{\mathfrak{p}\in\mathrm{Ass}_R(C)}\mathfrak{p}$, that is, if and only if x is a non-zerodivisor on C. □

Proposition 2.2.2. *Let $\varphi\colon R \to S$ be a flat ring homomorphism, and let C be a finitely generated R-module. If C is a semidualzing R-module, then $C \otimes_R S$ is a semidualizing S-module. The converse holds when φ is faithfully flat.*

Proof. The S-module $C \otimes_R S$ is finitely generated because C is a finitely generated R-module. For $i \geqslant 1$, we have

$$\mathrm{Ext}^i_S(C \otimes_R S, C \otimes_R S) \cong \mathrm{Ext}^i_R(C,C) \otimes_R S.$$

If $\mathrm{Ext}^i_R(C,C) = 0$, then this shows that $\mathrm{Ext}^i_S(C \otimes_R S, C \otimes_R S) = 0$. The converse holds when φ is faithfully flat.

Finally, there is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\chi^R_C \otimes_R S"] \arrow[d, "\chi^S_{C\otimes_R S}"'] & \mathrm{Hom}_R(C,C) \otimes_R S \arrow[dl, "\cong"] \\ \mathrm{Hom}_S(C \otimes_R S, C \otimes_R S). \end{tikzcd}$$

If the homothety map χ^R_C is an isomorphism, then so is $\chi^R_C \otimes_R S$, and so the diagram shows that $\chi^S_{C\otimes_R S}$ is an isomorphism. Conversely, if $\chi^S_{C\otimes_R S}$ is an isomorphism, then the diagram shows that $\chi^R_C \otimes_R S$ is an isomorphism; if we also assume that φ is faithfully flat, then χ^R_C is an isomorphism. This yields desired result. □

Here is a compliment to part of Proposition 2.2.2.

Proposition 2.2.3. *Let A be a finitely generated R-module. The following conditions are equivalent:*

(i) *A is a semidualizing R-module;*
(ii) *$U^{-1}A$ is a semidualizing $U^{-1}R$-module for each multiplicatively closed subset $U \subset R$;*
(iii) *$A_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p} \subset R$;*
(iv) *$A_{\mathfrak{m}}$ is a semidualizing $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subset R$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is in Proposition 2.2.2; use the flat homomorphism $R \to U^{-1}R$. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are straightforward.

(iv) $\implies$ (i). Assume that $A_\mathfrak{m}$ is a semidualizing $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subset R$. For each $i \geqslant 1$ and each $\mathfrak{m}$ this provides the vanishing

$$\operatorname{Ext}^i_R(A,A)_\mathfrak{m} \cong \operatorname{Ext}^i_{R_\mathfrak{m}}(A_\mathfrak{m},A_\mathfrak{m}) = 0$$

where the isomorphism is standard because A is finitely generated and R is noetherian. Since this is so for each maximal ideal $\mathfrak{m}$, we conclude that $\operatorname{Ext}^i_R(A,A) = 0$ for each $i \geqslant 1$.

Furthermore, for each $\mathfrak{m}$ there is a commutative diagram

$$\begin{tikzcd}
R_\mathfrak{m} \arrow[rr, "\chi^{R_\mathfrak{m}}_{A_\mathfrak{m}}", "\cong"'] \arrow[d, "(\chi^R_A)_\mathfrak{m}"'] & & \operatorname{Hom}_{R_\mathfrak{m}}(A_\mathfrak{m},A_\mathfrak{m}) \arrow[dll, "\cong"] \\
\operatorname{Hom}_R(A,A)_\mathfrak{m} & &
\end{tikzcd}$$

where the unspecified map is the natural isomorphism. It follows that $(\chi^R_A)_\mathfrak{m}$ is an isomorphism for each $\mathfrak{m}$, and so χ^R_A is an isomorphism. This implies that A is a semidualzing R-module, as desired. $\square$

Corollary 2.2.4. *Let C be a semidualizing R-module. If P is a finitely generated projective R-module of rank 1, then P and $C \otimes_R P$ are semidualizing R-modules.*

Proof. By assumption, we have $P_\mathfrak{m} \cong R_\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset R$. Since $R_\mathfrak{m}$ is a semidualizing $R_\mathfrak{m}$-module, Proposition 2.2.3 implies that P is a semidualizing R-module. Similarly, the isomorphisms

$$(C \otimes_R P)_\mathfrak{m} \cong C_\mathfrak{m} \otimes_{R_\mathfrak{m}} P_\mathfrak{m} \cong C_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m} \cong C_\mathfrak{m}$$

show that $C \otimes_R P$ is a semidualizing R-module. $\square$

Remark 2.2.5. With the notation of Proposition 2.2.2, assume that C is dualizing for R. While $S \otimes_R C$ will be semidualizing for S, it may not be dualizing for S. For example, let k be a field and let R be a non-Gorenstein local k-algebra; then k is dualizing for k, but $R \cong S \otimes_k k$ is not dualizing for R.

On the other hand, the $U^{-1}R$-module $U^{-1}C$ will be dualizing because of the inequality $\operatorname{id}_{U^{-1}R}(U^{-1}C) \leqslant \operatorname{id}_R(C) < \infty$.

We will see other situations where semidualizing R-modules give rise to semidualizing S-modules once we have built some more technology. For instance, we will show that, a R-regular sequence $\mathbf{x} = x_1, \ldots, x_n \in R$ is also C-regular and $C/\mathbf{x}C$ is a semidualizing $R/\mathbf{x}R$-module.

Proposition 2.2.6. *Let M be a nonzero R-module. If C is a semidualzing R-module, then* $\operatorname{Hom}_R(C,M) \neq 0$ *and* $C \otimes_R M \neq 0$.

Proof. Proposition 2.2.1 and Lemma A.2.1. $\square$

3. Foxby Classes

The more I learn about semidualizing modules, the more I realize how important the Foxby classes are.

3.1. **Basic Properties.**

Definition 3.1.1. Let M and N be R-modules. The natural evaluation map

$$\xi_N^M : M \otimes_R \operatorname{Hom}_R(M,N) \to N$$

is the R-module homomorphism given by $\xi_N^M(m \otimes \psi) = \psi(m)$. The natural map

$$\gamma_N^M : N \to \operatorname{Hom}_R(M, M \otimes_R N)$$

is the R-module homomorphism given by $\gamma_N^M(n)(m) = m \otimes n$.

The classes defined next are collectively known as Foxby classes. Avramov and Foxby [4] and Christensen [7]. Recall that the term "semidualizing" is defined in 1.4.3.

Definition 3.1.2. Let C be a semidualzing R-module. The *Auslander class* $\mathcal{A}_C(R)$ is the class of all R-modules M satisfying the following conditions:

(1) $\operatorname{Tor}^R_{\geqslant 1}(C,M) = 0 = \operatorname{Ext}_R^{\geqslant 1}(C, C \otimes_R M)$, and
(2) The natural map $\gamma_M^C : M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* $\mathcal{B}_C(R)$ is the class of all R-modules M satisfying the following conditions:

(1) $\operatorname{Ext}_R^{\geqslant 1}(C,M) = 0 = \operatorname{Tor}^R_{\geqslant 1}(C, \operatorname{Hom}_R(C,M))$, and
(2) The natural evaluation map $\xi_M^C : C \otimes_R \operatorname{Hom}_R(C,M) \to M$ is an isomorphism.

The following example is readily verified.

Example 3.1.3. In the case $C = R$: the classes $\mathcal{A}_R(R)$ and $\mathcal{B}_R(R)$ are both equal to the class of all R-module.

Here is one of the most frequently used properties of Foxby classes.

Proposition 3.1.4. *Let C be a semidualzing R-module, and consider an exact sequence of R-module homomorphisms*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0.$$

(a) *If two of the M_i are in $\mathcal{A}_C(R)$, then so is the third.*
(b) *If two of the M_i are in $\mathcal{B}_C(R)$, then so is the third.*

Proof. Assume first that $M_1, M_2 \in \mathcal{A}_C(R)$. Consider the long exact sequence in $\operatorname{Tor}_i^R(C,-)$ associated to the given sequence. Since $\operatorname{Tor}_{i-1}^R(C,M_1) = 0 = \operatorname{Tor}_i^R(C,M_2)$ for each $i > 1$, we see readily that $\operatorname{Tor}_i^R(C,M_3) = 0$ for each $i > 1$. For the remaining Tor-module, consider the following piece of the long exact sequence

$$0 \to \operatorname{Tor}_1^R(C,M_1) \to C \otimes_R M_3 \xrightarrow{C \otimes_R f} C \otimes_R M_2.$$

Apply $\operatorname{Hom}_R(C,-)$ to obtain the bottom exact sequence in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "\gamma^C_{M_1}"', "\cong"] & M_2 \arrow[d, "\gamma^C_{M_2}"', "\cong"] \\
\operatorname{Hom}(C, \operatorname{Tor}_1(C,M_3)) \arrow[r, hook] & \operatorname{Hom}(C, C \otimes M_1) \arrow[r, "\operatorname{Hom}(C, C \otimes f)"] & \operatorname{Hom}(C, C \otimes M_2).
\end{tikzcd}$$

The top row is from our original sequence. Since f is injective, it follows that $\operatorname{Hom}_R(C, C \otimes f)$ is also injective, and so $\operatorname{Hom}_R(C, \operatorname{Tor}_1^R(C, M_3)) \neq 0$. Now apply Proposition 2.2.6 to conclude $\operatorname{Tor}_1^R(C, M_3) \neq 0$.

It follows that we have an exact sequence

$$0 \to C \otimes_R M_1 \xrightarrow{C \otimes_R f} C \otimes_R M_2 \xrightarrow{C \otimes_R g} C \otimes_R M_3 \to 0.$$

Consider the associated long exact sequence in $\operatorname{Ext}_R^i(C, -)$. As above, it is straightforward to show that the vanishing $\operatorname{Ext}_R^{\geqslant 1}(C, C \otimes_R M_1) = 0 = \operatorname{Ext}_R^{\geqslant 1}(C, C \otimes_R M_2)$ implies $\operatorname{Ext}_R^{\geqslant 1}(C, C \otimes_R M_3) = 0$. Finally, the remainder of the long exact sequence fits into the bottom row of the next commutative diagram

$$\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[d, "\gamma^C_{M_1}"', "\cong"] & M_2 \arrow[r, two heads, "g"] \arrow[d, "\gamma^C_{M_2}"', "\cong"] & M_3 \arrow[d, "\gamma^C_{M_3}"'] \\
\operatorname{Hom}(C, C \otimes M_1) \arrow[r, "\operatorname{Hom}(C, C\otimes f)"] & \operatorname{Hom}(C, C \otimes M_2) \arrow[r, two heads, "\operatorname{Hom}(C, C\otimes g)"] & \operatorname{Hom}(C, C \otimes M_3)
\end{tikzcd}$$

and a diagram chase shows that $\gamma^C_{M_3}$ is an isomorphism.

The other cases are verified similarly. $\square$

Corollary 3.1.5. *Let C be a semidualzing R-module, and consider an exact sequence of R-module homomorphisms*

$$0 \to M_1 \to M_2 \to \cdots \to M_n \to 0.$$

(a) *If $M_i \in \mathcal{A}_C(R)$ for all $i \neq j$, then $M_j \in \mathcal{A}_C(R)$.*
(b) *If $M_i \in \mathcal{B}_C(R)$ for all $i \neq j$, then $M_j \in \mathcal{B}_C(R)$.*

Proof. By induction on n, using Proposition 3.1.4. $\square$

Here is another frequently cited property of Foxby classes. Recall that an R-module M has finite flat dimension when it admits a bounded resolution $0 \to F_n \to \cdots \to F_0 \to M \to 0$ by flat R-modules.

Proposition 3.1.6. *Let C be a semidualzing R-module.*

(a) *If M is an R-module of finite flat dimension, then $M \in \mathcal{A}_C(R)$.*
(b) *If M is an R-module of finite injective dimension, then $M \in \mathcal{B}_C(R)$.*

Proof. We verify part (a); part (b) is verified similarly. Applying Corollary 3.1.5 to a flat resolution

$$0 \to F_n \to \cdots \to F_0 \to M \to 0$$

we see that it suffices to show that each F_i is in $\mathcal{A}_C(R)$. Hence, we may assume that M is flat.

The Tor-vanishing conclusion is standard in this case. For the Ext-vanishing, let P be a free resolution of C such that each P_i is finitely generated. Lemma A.1.2 provides an isomorphism of complexes

$$\operatorname{Hom}_R(P, C \otimes_R M) \cong \operatorname{Hom}_R(P, C) \otimes_R M$$

and this yields the second isomorphism in the next sequence:

$$\begin{aligned}
\operatorname{Ext}_R^i(C, C \otimes_R M) &\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P, C \otimes_R M)) \cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P, C) \otimes_R M) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P, C)) \otimes_R M \cong \operatorname{Ext}_R^i(C, C) \otimes_R M.
\end{aligned}$$

The first and fourth isomorphisms are by definition, and the third isomorphism follows from the flatness of M. Since $\operatorname{Ext}^i_R(C,C)=0$ for all $i\geqslant 1$ by assumption, we have $\operatorname{Ext}^i_R(C,C\otimes_R M)=0$. For the evaluation map, use the following commutative diagram where the unspecified isomorphism is the natural isomorphism.

$$\begin{tikzcd} M \arrow[r, "\gamma^C_M"] \arrow[d, "\cong"] & \operatorname{Hom}_R(C,C\otimes_R M) \\ R\otimes_R M \arrow[r, "\chi^R_C\otimes_R M", "\cong"'] & \operatorname{Hom}_R(C,C)\otimes_R M. \arrow[u, "\cong"', "\omega_{CCM}"] \end{tikzcd}$$

See Definitions 1.4.2, 3.1.1 and A.1.1 for the other maps. The homothety map χ^R_C is an isomorphism by assumption, and ω_{CCM} is an isomoprhism by Lemma A.1.2. Thus γ^C_M is an isomoprhism. $\square$

Corollary 3.1.7. *Let C be a semidualzing R-module and let M be an R-module. Fix a flat (e.g., projective) resolution F of M and an injective resolution I of M.*

(a) *$M\in\mathcal{A}_C(R)$ if and only if $\operatorname{Im}(\partial^F_i)\in\mathcal{A}_C(R)$ for some (equivalently, every) $i\geqslant 0$.*
(b) *$M\in\mathcal{B}_C(R)$ if and only if $\operatorname{Im}(\partial^I_i)\in\mathcal{A}_C(R)$ for some (equivalently, every) $i\geqslant 0$.* $\square$

3.2. **Equivalences.** Here is Foxby equivalence.

Theorem 3.2.1. *Let C be a semidualzing R-module.*

(a) *An R-module M is in $\mathcal{A}_C(R)$ if and only if $C\otimes_R M$ is in $\mathcal{B}_C(R)$.*
(b) *An R-module M is in $\mathcal{B}_C(R)$ if and only if $\operatorname{Hom}_R(C,M)$ is in $\mathcal{A}_C(R)$.*
(c) *The functors $C\otimes_R -\colon \mathcal{A}_C(R)\to\mathcal{B}_C(R)$ and $\operatorname{Hom}_R(C,-)\colon\mathcal{B}_C(R)\to\mathcal{A}_C(R)$ are inverse equivalences of categories.*

Proof. (a) We begin by noting the readily verified equality

$$\xi^C_{C\otimes_R M}\circ(C\otimes_R\gamma^C_M)=\mathrm{id}_{C\otimes_R M}\colon C\otimes_R M\to C\otimes_R M. \tag{3.2.1.1}$$

For the first implication, assume that $M\in\mathcal{A}_C(R)$. By definition, this means that $\operatorname{Tor}^R_{\geqslant 1}(C,M)=0=\operatorname{Ext}^{\geqslant 1}_R(C,C\otimes_R M)$, and the natural map $\gamma^C_M\colon M\to \operatorname{Hom}_R(C,C\otimes_R M)$ is an isomorphism. To show that $C\otimes_R M$ is in $\mathcal{B}_C(R)$, we need to verify three conditions.

(1) $\operatorname{Ext}^{\geqslant 1}_R(C,C\otimes_R M)=0$: this is true by assumption.

(2) $\operatorname{Tor}^R_{\geqslant 1}(C,\operatorname{Hom}_R(C,C\otimes_R M))=0$: this follows from our assumptions because $\operatorname{Tor}^R_i(C,\operatorname{Hom}_R(C,C\otimes_R M))\cong\operatorname{Tor}^R_i(C,M)=0$ for each $i\geqslant 1$.

(3) the natural evaluation map $\xi^C_{C\otimes_R M}\colon C\otimes_R\operatorname{Hom}_R(C,C\otimes_R M)\to C\otimes_R M$ is an isomorphism: Since γ^C_M is an isomorphism by assumption, it follows that $C\otimes_R\gamma^C_M$ is an isomorphism. Equation (3.2.1.1) implies that $\xi^C_{C\otimes_R M}=(C\otimes_R\gamma^C_M)^{-1}$ is also an isomorphism.

For the converse, assume that $C\otimes_R M$ is in $\mathcal{B}_C(R)$. Since this implies, in particular, that $\operatorname{Ext}^{\geqslant 1}_R(C,C\otimes_R M)=0$, we need only check two conditions to show that $M\in\mathcal{A}_C(R)$:

(4) the natural map $\gamma^C_M\colon M\to\operatorname{Hom}_R(C,C\otimes_R M)$ is an isomorphism: By assumption, the map $\xi^C_{C\otimes_R M}$ is an isomorphism. Using equation (3.2.1.1) as in (3)

above, we conclude that $C \otimes_R \gamma_M^C$ is an isomorphism. Set $N = \operatorname{Coker}(\gamma_M^C)$ and consider the exact sequence

$$M \xrightarrow{\gamma_M^C} \operatorname{Hom}_R(C, C \otimes_R M) \to N \to 0.$$

The right-exactness of $C \otimes_R -$ yields the next exact sequence

$$C \otimes_R M \xrightarrow[\cong]{C \otimes_R \gamma_M^C} C \otimes_R \operatorname{Hom}_R(C, C \otimes_R M) \to C \otimes_R N \to 0.$$

It follows that $C \otimes_R N = 0$, and so $N = 0$ by Proposition 2.2.6. Hence γ_M^C is surjective. Set $K = \operatorname{Ker}(\gamma_M^C)$ and consider the exact sequence

$$0 \to K \to M \xrightarrow{\gamma_M^C} \operatorname{Hom}_R(C, C \otimes_R M) \to 0.$$

The long exact sequence in $\operatorname{Tor}^R(C, -)$ yields the next exact sequence

$$\underbrace{\operatorname{Tor}_1^R(C, \operatorname{Hom}_R(C, C \otimes_R M))}_{=0} \to C \otimes K \to C \otimes M \xrightarrow{\cong} C \otimes \operatorname{Hom}(C, C \otimes M) \to 0.$$

It follows that $C \otimes_R K = 0$, and so $K = 0$ by Proposition 2.2.6. Hence γ_M^C is also injective.

(5) $\operatorname{Tor}_{\geqslant 1}^R(C, M) = 0$: this follows from our assumptions along with (4) because $\operatorname{Tor}_i^R(C, M) \cong \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, C \otimes_R M)) = 0$ for all $i \geqslant 1$.

This completes the proof of part (a). The proof of part (b) is similar, and part (c) follows from parts (a) and (b). □

Corollary 3.2.2. *Let C be a semidualizing R-module.*

(a) *If M is a R-module of finite flat dimension (e.g., if M is flat or projective), then $C \otimes_R M \in \mathcal{B}_C(R)$. In particular $C \in \mathcal{B}_C(R)$.*

(b) *If M is a R-module of finite injective dimension (e.g., if M is injective), then $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C(R)$.*

(c) *M has finite flat dimension if and only if $C \otimes_R M$ admits a bounded resolution $0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to C \otimes_R M \to 0$ where each F_i is flat.*

(d) *M has finite projective dimension if and only if $C \otimes_R M$ admits a bounded resolution $0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_0 \to C \otimes_R M \to 0$ where each P_i is projective.*

(e) *M has finite injective dimension if and only if $\operatorname{Hom}_R(C, M)$ admits a bounded resolution $0 \to \operatorname{Hom}_R(C, M) \to \operatorname{Hom}_R(C, I_0) \cdots \to \operatorname{Hom}_R(C, I_n) \to 0$ where each I_j is injective.*

Proof. Part (a) follows from Proposition 3.1.6(a) and Theorem 3.2.1(a), and similarly for part (b).

(c) Assume first that M has finite flat dimension and fix a bounded resolution $0 \to F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$ where each F_i is flat. Proposition 3.1.6(a) implies that each module in this sequence is in $\mathcal{A}_C(R)$. Set $M_i = \operatorname{Im}(\partial_i)$ for each i. Since each M_i has finite flat dimension, it is in $\mathcal{A}_C(R)$. Hence, applying $C \otimes_R -$ to the exact sequence

$$0 \to M_{i+1} \to F_i \to M_i \to 0$$

yields an exact sequence

$$0 \to C \otimes_R M_{i+1} \to C \otimes_R F_i \to C \otimes_R M_i \to 0$$

and it follows that the sequence

$$0 \to C \otimes_R F_n \xrightarrow{C\otimes_R \partial_n} \cdots \xrightarrow{C\otimes_R \partial_1} C \otimes_R F_0 \xrightarrow{C\otimes_R \partial_0} C \otimes_R M \to 0$$

is exact, as desired.

For the converse, assume that $C \otimes_R M$ admits a bounded resolution

$$0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to C \otimes_R M \to 0 \tag{3.2.2.1}$$

where each F_i is flat. Since each $C \otimes_R F_i$ is in $\mathcal{B}_C(R)$ by part (a), Corollary 3.1.5(b) implies that $C \otimes_R M \in \mathcal{B}_C(R)$. Hence, Theorem 3.2.1(a) implies that $M \in \mathcal{A}_C(R)$. Since each module in the sequence (3.2.2.1) is in $\mathcal{B}_C(R)$, an argument as in the previous paragraph implies that the induced sequence in the bottom row of the following commutative diagram is exact:

$$\begin{tikzcd}
0 \arrow[r] & F_n \arrow[r] \arrow[d, "\cong"', "\gamma^C_{F_n}"] & \cdots \arrow[r] & F_0 \arrow[r] \arrow[d, "\cong"', "\gamma^C_{F_0}"] & M \arrow[r] \arrow[d, "\cong"', "\gamma^C_M"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(C, C \otimes F_n) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}(C, C \otimes F_0) \arrow[r] & \mathrm{Hom}(C, C \otimes M) \arrow[r] & 0
\end{tikzcd}$$

It follows that the top row is also exact and has the desired form.

Parts (d) and (e) are proved similarly. □

Here is a dualizing equivalence for Foxby classes. Recall that an injective R-module E is emphfaithfully injective if the functor $\mathrm{Hom}_R(-, E)$ is faithfully exact, that is, if it satisfies the following condition: a sequence S of R-module homomorphisms is exact if and only if the sequence $\mathrm{Hom}_R(S, E)$ is exact. This is equivalent to the following condition: for each R-module M, one has $M = 0$ if and only if $\mathrm{Hom}_R(M, E) = 0$.

Proposition 3.2.3. *Let C be a semidualizing R-module, let E be an injective R-module, and let M be an R-module.*

(a) *If M is in $\mathcal{B}_C(R)$, then $\mathrm{Hom}_R(M, E)$ is in $\mathcal{A}_C(R)$. The converse holds when E is faithfully injective.*
(b) *If M is in $\mathcal{A}_C(R)$, then $\mathrm{Hom}_R(M, E)$ is in $\mathcal{B}_C(R)$. The converse holds when E is faithfully injective.*

Proof. (a) We begin by recalling the following isomomorphism for each $i \geqslant 0$:

$$\mathrm{Tor}^R_i(C, \mathrm{Hom}_R(M, E)) \cong \mathrm{Hom}_R(\mathrm{Ext}^i_R(C, M), E). \tag{3.2.3.1}$$

To explain the isomorphism, let P be a projective resolution of C such that each P_i is finitely generated. The Hom-evaluation isomorphism from Lemma A.1.3 yields an isomorphism of complexes

$$P \otimes_R \mathrm{Hom}_R(M, E) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(P, M), E)$$

and this explains the second isomorphism in the next sequence

$$\begin{aligned}
\mathrm{Tor}^R_i(C, \mathrm{Hom}_R(M, E)) &\cong \mathrm{H}_i(P \otimes_R \mathrm{Hom}_R(M, E)) \\
&\cong \mathrm{H}_i(\mathrm{Hom}_R(\mathrm{Hom}_R(P, M), E)) \\
&\cong \mathrm{Hom}_R(\mathrm{H}_i(\mathrm{Hom}_R(P, M)), E) \\
&\cong \mathrm{Hom}_R(\mathrm{Ext}^i_R(C, M), E).
\end{aligned}$$

The first and fourth isomorphisms are by definition, and the third isomorphism follows from the injectivity of E. One concludes that, if $\operatorname{Ext}^i_R(C,M)=0$, then $\operatorname{Tor}^R_i(C,\operatorname{Hom}_R(M,E))=0$; and the converse holds when E is faithfully injective.

Thus, for the rest of the proof we assume the vanishing conditions $\operatorname{Ext}^{\geqslant 1}_R(C,M)=0=\operatorname{Tor}^R_{\geqslant 1}(C,\operatorname{Hom}_R(M,E))$. An argument similar to the previous paragraph explains the isomorphism $C\otimes_R\operatorname{Hom}_R(M,E)\cong\operatorname{Hom}_R(\operatorname{Hom}_R(C,M),E)$ and hence the first isomorphism in the next sequence

$$\begin{aligned}\operatorname{Ext}^i_R(C,C\otimes_R\operatorname{Hom}_R(M,E))&\cong\operatorname{Ext}^i_R(C,\operatorname{Hom}_R(\operatorname{Hom}_R(C,M),E))\\&\cong\operatorname{Hom}_R(\operatorname{Tor}^R_i(C,\operatorname{Hom}_R(C,M)),E).\end{aligned}$$

The second isomorphism follows as in the previous paragraph, using Hom-tensor adjointness in place of Hom-evaluation. From these isomorphisms, one concludes that, if $\operatorname{Tor}^R_i(C,\operatorname{Hom}_R(C,M))=0$, then $\operatorname{Ext}^i_R(C,C\otimes_R\operatorname{Hom}_R(M,E))=0$; and the converse holds when E is faithfully injective.

It is routine to show that the following diagram commutes

$$\begin{tikzcd}
\operatorname{Hom}_R(M,E) \arrow[r, "\gamma^C_{\operatorname{Hom}(M,E)}"] \arrow[d, "\operatorname{Hom}(\xi^C_M,E)"'] & \operatorname{Hom}_R(C,C\otimes_R\operatorname{Hom}_R(M,E)) \arrow[d, "\operatorname{Hom}(C,\theta_{CME})"', "\cong"] \\
\operatorname{Hom}_R(C\otimes_R\operatorname{Hom}_R(C,M),E) & \operatorname{Hom}_R(C,\operatorname{Hom}_R(\operatorname{Hom}_R(C,M),E)). \arrow[l, "\cong"']
\end{tikzcd}$$

where the unspecified isomorphism is Hom-tensor adjointness. From this, one sees that, if ξ^C_M is an isomorphism, then so is $\gamma^C_{\operatorname{Hom}(M,E)}$; and the converse holds when E is faithfully injective.

From these three steps, the desired implications are established.

The proof of (b) is similar. □

The next two results are proved similarly to the previous one.

Proposition 3.2.4. *Let C be a semidualizing R-module, let F be a flat R-module, and let M be an R-module.*

(a) *If M is in $\mathcal{A}_C(R)$, then $M\otimes_R F$ is in $\mathcal{A}_C(R)$. The converse holds when F is faithfully flat.*
(b) *If M is in $\mathcal{B}_C(R)$, then $M\otimes_R F$ is in $\mathcal{B}_C(R)$. The converse holds when F is faithfully flat.* □

Proposition 3.2.5. *Let C be a semidualizing R-module, let P be a projective R-module, and let M be an R-module.*

(a) *If M is in $\mathcal{A}_C(R)$, then $\operatorname{Hom}_R(P,M)$ is in $\mathcal{A}_C(R)$. The converse holds when F is faithfully projective.*
(b) *If M is in $\mathcal{B}_C(R)$, then $\operatorname{Hom}_R(P,M)$ is in $\mathcal{B}_C(R)$. The converse holds when F is faithfully projective.* □

Proposition 3.2.6. *Let C be a semidualizing R-module, and let M be an R-module.*

(a) *The module M is semidualizing and in $\mathcal{A}_C(R)$ if and only if $C\otimes_R M$ is semidualizing and in $\mathcal{B}_C(R)$.*
(b) *The module M is semidualizing and in $\mathcal{B}_C(R)$ if and only if $\operatorname{Hom}_R(C,M)$ is semidualizing and in $\mathcal{A}_C(R)$.*

Proof. (a) Theorem 3.2.1(a) tells us that $M \in \mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(R)$, so assume that $M \in \mathcal{A}_C(R)$. Using this assumptions and arguing as in the proof of Proposition 3.2.3 we have isomorphisms

$$\operatorname{Ext}^i_R(M,M) \cong \operatorname{Ext}^i_R(C \otimes_R \operatorname{Hom}_R(C,M)) \cong \operatorname{Ext}^i(\operatorname{Hom}_R(C,M),\operatorname{Hom}_R(C,M))$$

and so $\operatorname{Ext}^{\geqslant 1}_R(M,M) = 0$ if and only if $\operatorname{Ext}^{\geqslant 1}(\operatorname{Hom}_R(C,M),\operatorname{Hom}_R(C,M)) = 0$. Furthermore, the following diagram commutes

$$\begin{tikzcd}
R \arrow[r, "\chi^R_{\operatorname{Hom}_R(C,M)}"] \arrow[d, "\chi^R_M"] & \operatorname{Hom}_R(\operatorname{Hom}_R(C,M),\operatorname{Hom}_R(C,M)) \arrow[d, "\cong"'] \\
\operatorname{Hom}_R(M,M) \arrow[r, "\operatorname{Hom}_R(\xi^C_M,M)"] & \operatorname{Hom}_R(C \otimes_R \operatorname{Hom}_R(C,M),M)
\end{tikzcd}$$

wherein the unspecified isomorphism is Hom-tensor adjointness. The diagram show that $\chi^R_{\operatorname{Hom}_R(C,M)}$ is an isomorphism if and only if ξ^C_M is an isomorphism. This implies that $C \otimes_R M$ is semidualizing if and only if M is semidualizing, as desired.

The proof of part (b) is similar. □

3.3. **Base Change.**

Theorem 3.3.1. *Let C be a semidualizing R-module and let $\varphi\colon R \to S$ be a ring homomorphism. Then $S \in \mathcal{A}_C(R)$ if and only if $\operatorname{Tor}^R_{\geqslant 1}(C,S) = 0$ and $C \otimes_R S$ is a semidualizing S-module.*

Proof. Since the condition $S \in \mathcal{A}_C(R)$ includes the vanishing $\operatorname{Tor}^R_{\geqslant 1}(C,S) = 0$, we assume without loss of generality that $\operatorname{Tor}^R_{\geqslant 1}(C,S) = 0$. Let P be an R-free resolution of C such that each P_i is finitely generated.

In the following sequence, the first isomorphism is Hom-tensor adjointness

$$\begin{aligned}
\operatorname{Hom}_S(P \otimes_R S, C \otimes_R S) &\cong \operatorname{Hom}_R(P,\operatorname{Hom}_R(S, C \otimes_R S)) \\
&\cong \operatorname{Hom}_R(P, C \otimes_R S)
\end{aligned}$$

The second isomorphism is Hom-cancellation. The assumption $\operatorname{Tor}^R_{\geqslant 1}(C,S) = 0$ implies that the complex $P \otimes_R S$ is an S-free resolution of $C \otimes_R S$, and this explains the first isomorphism in the next sequence

$$\begin{aligned}
\operatorname{Ext}^i_S(C \otimes_R S, C \otimes_R S) &\cong \operatorname{H}_{-i}(\operatorname{Hom}_S(P \otimes_R S, C \otimes_R S)) \\
&\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(P, C \otimes_R S)) \\
&\cong \operatorname{Ext}^i_R(C, C \otimes_R S).
\end{aligned}$$

The second isomorphism is from the previous displayed sequence, and the third isomorphism comes from the fact that P is an R-free resolution of C. From this, we see that $\operatorname{Ext}^{\geqslant 1}_S(C \otimes_R S, C \otimes_R S) = 0$ if and only if $\operatorname{Ext}^{\geqslant 1}_R(C, C \otimes_R S) = 0$.

Let $f\colon \operatorname{Hom}_S(S, C \otimes_R S) \to C \otimes_R S$ be the Hom-cancellation isomorphism given by $f(\psi) = \psi(1)$. This fits into a commutative diagram

$$\begin{tikzcd}
S \arrow[r, "\chi^S_{C \otimes_R S}"] \arrow[d, "\gamma^C_S"'] & \operatorname{Hom}_S(C \otimes_R S, C \otimes_R S) \arrow[d, "\cong"] \\
\operatorname{Hom}_R(C, C \otimes_R S) & \operatorname{Hom}_R(C, \operatorname{Hom}_S(S, C \otimes_R S)) \arrow[l, "\operatorname{Hom}_R(C,f)"', "\cong"]
\end{tikzcd}$$

where the unspecified isomorphism is Hom-tensor adjointness. this diagram shows that $\chi^S_{C\otimes_R S}$ is an isomorphism if and only if γ^C_S is an isomorphism. This completes the proof. □

The next result generalizes Proposition 2.2.2.

Corollary 3.3.2. *Let C be a semidualizing R-module and let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension. Then $\operatorname{Tor}^R_{\geqslant 1}(C,S) = 0$, and $C \otimes_R S$ is a semidualizing S-module.*

Proof. Since S has finite flat dimension as an R-module, Proposition 3.1.6 implies that S is in $\mathcal{A}_C(R)$. Hence, the result follows from Theorem 3.3.1. □

Part (a) of the next result generalizes part of Proposition 2.2.1.

Corollary 3.3.3. *Let C be a semidualzing R-module, and let $\mathbf{x} = x_1,\ldots,x_n \in R$.*

(a) *The sequence $\mathbf{x}$ is C-regular if and only if $\mathbf{x}$ is R-regular.*
(b) *If $\mathbf{x}$ is R-regular, then $C/\mathbf{x}C$ is a semidualizing $R/\mathbf{x}R$-module.*
(c) *If R is local, then $\operatorname{depth}_R(C) = \operatorname{depth}(R)$.*

Proof. (a) First note that, as in the proof of Proposition 2.2.1, we see that $(\mathbf{x})C = C$ if and only if $(\mathbf{x})R = R$. so we assume without loss of generality that $(\mathbf{x})R \neq R$. We proceed by induction on n. The base case $n = 1$ follows from Proposition 2.2.1.

For the induction step, assume that $n \geqslant 2$ and that the result holds for sequences of length $n-1$. It follows that the sequence $\mathbf{x}' = x_1,\ldots,x_{n-1}$ is R-regular if and only if it is C-regular. Thus, it remains to assume that $\mathbf{x}'$ is R-regular (and hence C-regular) and show that x_n is $R/\mathbf{x}'R$-regular if and only if it is $C/\mathbf{x}'C$-regular. Since $\mathbf{x}'$ is R-regular, part (b) implies that $C/\mathbf{x}'C$ is a semidualizing $R/\mathbf{x}'R$-module. The base case implies that $\overline{x_n} \in R/\mathbf{x}'R$ is $R/\mathbf{x}'R$-regular if and only if it is $C/\mathbf{x}'C$-regular. It follows that x_n is $R/\mathbf{x}'R$-regular if and only if it is $C/\mathbf{x}'C$-regular.

(b) Assume that $\mathbf{x}$ is R-regular. Then $\operatorname{pd}_R(R/\mathbf{x}R) < \infty$, and so Corollary 3.3.2 implies that $\operatorname{Tor}^R_{\geqslant 1}(C, R/\mathbf{x}R) = 0$ and $C \otimes_R R/\mathbf{x}R \cong C/\mathbf{x}C$ is a semidualizing $R/\mathbf{x}R$-module.

Part (c) follows from part (a). □

Here is a compliment to the previous result.

Proposition 3.3.4. *Let $\mathbf{x} = x_1,\ldots,x_n \in R$ be an R-regular sequence in the Jacobson radical of R. Let A be a finitely generated R-module such that $\mathbf{x}$ is A-regular. If $A/(\mathbf{x})A$ is a semidualizing $R/(\mathbf{x})R$-module, then A is a semidualizing R-module.*

Proof. Arguing by induction on n we may assume that $n = 1$. Set $x = x_1$ and $\overline{R} = R/(x)R$. Note that x is $\operatorname{Hom}_R(A,A)$-regular. Indeed, use the exact sequence

$$0 \to A \xrightarrow{x} A \to A/xA \to 0 \tag{3.3.4.1}$$

and the left-exactness of $\operatorname{Hom}_R(A,-)$ to conclude that the sequence

$$0 \to \operatorname{Hom}_R(A,A) \xrightarrow{x} \operatorname{Hom}_R(A,A)$$

is exact. It follows that $\operatorname{Tor}^R_1(\overline{R}, \operatorname{Hom}_R(A,A)) = 0$.

Consider the following commutative diagram

$$\begin{CD}
\overline{R} @>{\chi^{\overline{R}}_{\overline{R}\otimes_R A}}>{\cong}> \operatorname{Hom}_{\overline{R}}(\overline{R}\otimes_R A,\overline{R}\otimes_R A) @>{\cong}>> \operatorname{Hom}_R(A,\operatorname{Hom}_{\overline{R}}(\overline{R},\overline{R}\otimes_R A))\\
@V{\cong}VV @VV{\cong}V\\
\overline{R}\otimes_R R @>{\overline{R}\otimes_R\chi^R_A}>> \overline{R}\otimes_R\operatorname{Hom}_R(A,A) @>{\omega_{AA\overline{R}}}>{\cong}> \operatorname{Hom}_R(A,\overline{R}\otimes_R A)
\end{CD}$$

wherein the unspecified vertical isomorphisms are induced by Hom- and tensor-cancellation, and the unspecified horizontal isomorphism is Hom-tensor adjointness. The diagram shows that $\overline{R}\otimes_R\chi^R_A$ is an isomorphism.

We claim that χ^R_A is surjective. To see this, consider the exact sequence

$$R\xrightarrow{\chi^R_A}\operatorname{Hom}_R(A,A)\to\operatorname{Coker}(\chi^R_A)\to 0.$$

Use the right-exactness of $\overline{R}\otimes_R -$ to see that the next sequence is exact

$$\overline{R}\otimes_R R\xrightarrow[\cong]{\overline{R}\otimes_R\chi^R_A}\overline{R}\otimes_R\operatorname{Hom}_R(A,A)\to\overline{R}\otimes_R\operatorname{Coker}(\chi^R_A)\to 0.$$

It follows that $\overline{R}\otimes_R\operatorname{Coker}(\chi^R_A)$, and the fact that x is in the Jacobson radical of R implies that $\operatorname{Coker}(\chi^R_A)=0$ by Nakayama's Lemma.

We claim that χ^R_A is injective. To see this, consider the exact sequence

$$0\to\operatorname{Ker}(\chi^R_A)\to R\xrightarrow{\chi^R_A}\operatorname{Hom}_R(A,A)\to 0$$

and take the long exact sequence in $\operatorname{Tor}^R(\overline{R},-)$:

$$\underbrace{\operatorname{Tor}_1(\overline{R},\operatorname{Hom}(A,A))}_{=0}\to\overline{R}\otimes\operatorname{Ker}(\chi^R_A)\to\overline{R}\otimes R\xrightarrow[\cong]{\overline{R}\otimes\chi^R_A}\overline{R}\otimes\operatorname{Hom}(A,A)\to 0.$$

It follows that $\overline{R}\otimes\operatorname{Ker}(\chi^R_A)=0$ and, as above, that $\operatorname{Ker}(\chi^R_A)=0$.

We conclude the proof by showing that $\operatorname{Ext}^{\geqslant 1}_R(A,A)=0$. Since x is A-regular and $x(\overline{R}\otimes_R A)=0$, we have

$$\operatorname{Ext}^i_R(A,A/xA)\cong\operatorname{Ext}^i_R(A,\overline{R}\otimes_R A)\cong\operatorname{Ext}^i_{\overline{R}}(\overline{R}\otimes_R A,\overline{R}\otimes_R A)=0$$

for each $i\geqslant 1$; see, e.g., [17, p. 140]. (The vanishing holds because $\overline{R}\otimes_R A$ is a semidualizing $\overline{R}$-module.) For $i\geqslant 1$, part of the long exact sequence in $\operatorname{Ext}_R(A,-)$ associated to the sequence (3.3.4.1) has the following form:

$$\operatorname{Ext}^i_R(A,A)\xrightarrow{x}\operatorname{Ext}^i_R(A,A)\to\underbrace{\operatorname{Ext}^i_R(A,A/xA)}_{=0}.$$

It follows that $\operatorname{Ext}^i_R(A,A)=x\operatorname{Ext}^i_R(A,A)$. Since $\operatorname{Ext}^i_R(A,A)$ is finitely generated and x is in the Jacobson radical of R, Nakayama's Lemma implies $\operatorname{Ext}^i_R(A,A)=0$, as desired. □

Here is some flat base-change.

Proposition 3.3.5. *Let C be a semidualizing R-module and let $\varphi\colon R\to S$ be a flat ring homomorphism. If $M\in\mathcal{A}_C(R)$, then $M\otimes_R S\in\mathcal{A}_{C\otimes_R S}(S)$. If $M\in\mathcal{B}_C(R)$, then $M\otimes_R S\in\mathcal{B}_{C\otimes_R S}(S)$. The converses hold when φ is faithfully flat.*

Proof. We prove the result for Auslander classes. The proof for Bass classes is similar. Note that $C \otimes_R S$ is a semidulazing S-module by Proposition 2.2.2.

For $i \geqslant 1$, we have

$$\operatorname{Tor}_i^S(C \otimes_R S, M \otimes_R S) \cong \operatorname{Tor}_i^R(C, M) \otimes_R S.$$

If $\operatorname{Tor}_i^R(C, M) = 0$, then this shows that $\operatorname{Tor}_i^S(C \otimes_R S, M \otimes_R S) = 0$. The converse holds when φ is faithfully flat.

For $i \geqslant 1$, we have

$$\begin{aligned}\operatorname{Ext}_S^i(C \otimes_R S, (C \otimes_R S) \otimes_S (M \otimes_R S)) &\cong \operatorname{Ext}_S^i(C \otimes_R S, (C \otimes_R M) \otimes_R S) \\ &\cong \operatorname{Ext}_R^i(C, C \otimes_R M) \otimes_R S.\end{aligned}$$

If $\operatorname{Ext}_R^i(C, C \otimes_R M) = 0$, then this shows that $\operatorname{Ext}_S^i(C \otimes_R S, (C \otimes_R S) \otimes_S (M \otimes_R S)) = 0$. The converse holds when φ is faithfully flat.

Finally, there is a commutative diagram

$$\begin{tikzcd}
M \otimes_R S \arrow[r, "\gamma^{C\otimes_R S}_{M\otimes_R S}"] \arrow[d, "\gamma^C_M \otimes_R S"'] & \operatorname{Hom}_S(C \otimes_R S, (C \otimes_R S) \otimes_S (M \otimes_R S)) \arrow[d, "\cong"] \\
\operatorname{Hom}_R(C, C \otimes_R M) \otimes_R S \arrow[r, "\cong"] & \operatorname{Hom}_S(C \otimes_R S, (C \otimes_R M) \otimes_R S)
\end{tikzcd}$$

where the unspecified maps are the natural isomorphisms. If the natural map γ^C_M is an isomorphism, then so is $\gamma^C_M \otimes_R S$, and so the diagram shows that $\gamma^{C\otimes_R S}_{M\otimes_R S}$ is an isomorphism. Conversely, if $\gamma^{C\otimes_R S}_{M\otimes_R S}$ is an isomorphism, then the diagram shows that $\gamma^C_M \otimes_R S$ is an isomorphism; if we also assume that φ is faithfully flat, then this implies that γ^C_M is an isomorphism. The desired result now follows. □

Here is some other base-change.

Proposition 3.3.6. *Let C be a semidualizing R-module. Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be an R-regular sequence and set $\overline{R} = R/(\mathbf{x})R$. Let M be an R-module such that $\mathbf{x}$ is M-regular. If $M \in \mathcal{A}_C(R)$, then $M \otimes_R \overline{R} \in \mathcal{A}_{C\otimes_R\overline{R}}(\overline{R})$. If $M \in \mathcal{B}_C(R)$, then $M \otimes_R \overline{R} \in \mathcal{B}_{C\otimes_R\overline{R}}(\overline{R})$. The converses hold when $\mathbf{x}$ is in the Jacobson radical of R and M is finitely generated.*

Proof. By induction on n, we assume without loss of generality that $n = 1$. Set $x = x_1$. We prove the result for Auslander classes. The proof for Bass classes is similar. Set $\overline{C} = C \otimes_R \overline{R}$ and $\overline{M} = M \otimes_R \overline{R}$, and note that $\overline{C}$ is a semidulazing $\overline{R}$-module by Corollary 3.3.3(c).

For $i \geqslant 0$, we have

$$\operatorname{Tor}_i^{\overline{R}}(\overline{C}, \overline{M}) \cong \operatorname{Tor}_i^{\overline{R}}(C \otimes_R \overline{R}, \overline{M}) \cong \operatorname{Tor}_i^R(C, \overline{M}) \tag{3.3.6.1}$$

$$\operatorname{Ext}^i_{\overline{R}}(\overline{C}, \overline{C} \otimes_{\overline{R}} \overline{M}) \cong \operatorname{Ext}^i_{\overline{R}}(C \otimes_R \overline{R}, \overline{C} \otimes_{\overline{R}} \overline{M}) \cong \operatorname{Ext}^i_R(C, \overline{C} \otimes_{\overline{R}} \overline{M}) \tag{3.3.6.2}$$

see, e.g., [17, p. 140].

Assume that $M \in \mathcal{A}_C(R)$ and consider the exact sequence

$$0 \to M \xrightarrow{x} M \to \overline{M} \to 0. \tag{3.3.6.3}$$

Since $M \in \mathcal{A}_C(R)$, Proposition 3.1.4(a) implies $\overline{M} \in \mathcal{A}_C(R)$. The isomorphisms in (3.3.6.1) then imply that $\operatorname{Tor}_i^{\overline{R}}(\overline{C}, \overline{M}) \cong \operatorname{Tor}_i^R(C, \overline{M}) = 0$ for $i \geqslant 1$, and so the induced sequence

$$0 \to C \otimes_R M \xrightarrow{x} C \otimes_R M \to C \otimes_R \overline{M} \to 0 \tag{3.3.6.4}$$

is exact. Foxby Equivalence 3.2.1(a) implies that $C \otimes_R M \in \mathcal{B}_C(R)$, and so Proposition 3.1.4(b) implies $C \otimes_R \overline{M} \in \mathcal{A}_C(R)$. The isomorphisms in (3.3.6.1) and (3.3.6.2) then imply that

$$\operatorname{Ext}^i_{\overline{R}}(\overline{C}, \overline{C} \otimes_{\overline{R}} \overline{M}) \cong \operatorname{Ext}^i_R(C, \overline{C} \otimes_{\overline{R}} \overline{M}) \cong \operatorname{Ext}^i_R(C, C \otimes_R \overline{M}) = 0$$

for $i \geqslant 1$. To complete this implication, consider the next commutative diagram

$$\begin{tikzcd}
\overline{M} \arrow[r, "\gamma^{C\otimes_R \overline{R}}_{\overline{M}}"] \arrow[d, "\gamma^{C}_{\overline{M}}"', "\cong"] & \operatorname{Hom}_{\overline{R}}(C \otimes_R \overline{R}, C \otimes_R \overline{R} \otimes_{\overline{R}} \overline{M}) \arrow[d, "\cong"] \\
\operatorname{Hom}_R(C, C \otimes_R \overline{M}) \arrow[r, "\cong"] & \operatorname{Hom}_R(C, C \otimes_R \overline{R} \otimes_{\overline{R}} \overline{M})
\end{tikzcd}$$

where the unspecified isomorphisms are induced by Hom-tensor adjointness and the natural cancellation morphisms. The diagram shows that $\gamma^{C\otimes_R \overline{R}}_{\overline{M}} = \gamma^{\overline{C}}_{\overline{M}}$ is an isomorphism, and so $\overline{M} \in \mathcal{A}_{\overline{C}}(\overline{R})$, as desired.

qqq need to prove the converse.

□

Here are local-global principals for Foxby classes.

Proposition 3.3.7. *Let C be a semidualizing R-module and M an R-module. The following conditions are equivalent:*

(i) $M \in \mathcal{A}_C(R)$;
(ii) $U^{-1}M \in \mathcal{A}_{U^{-1}C}(U^{-1}R)$ *for each multiplicatively closed subset* $U \subset R$;
(iii) $M_{\mathfrak{p}} \in \mathcal{A}_{C_{\mathfrak{p}}}(R_{\mathfrak{p}})$ *for each prime ideal* $\mathfrak{p} \subset R$;
(iv) $M_{\mathfrak{m}} \in \mathcal{A}_{C_{\mathfrak{m}}}(R_{\mathfrak{m}})$ *for each maximal ideal* $\mathfrak{m} \subset R$.

Proof. The implication (i) $\implies$ (ii) is in Proposition 3.3.5, and the implications (ii) $\implies$ (iii) $\implies$ (iv) are straightforward.

(iv) $\implies$ (i). As in the proof of Proposition 3.3.5, for each $i \geqslant 1$ and each each maximal ideal $\mathfrak{m} \subset R$, we have isomorphisms

$$\operatorname{Tor}^R_i(C, M)_{\mathfrak{m}} \cong \operatorname{Tor}^{R_{\mathfrak{m}}}_i(C_{\mathfrak{m}}, M_{\mathfrak{m}}) = 0$$

$$\operatorname{Ext}^i_R(C, C \otimes_R M)_{\mathfrak{m}} \cong \operatorname{Ext}^i_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, C_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}) = 0.$$

and a commutative diagram

$$\begin{tikzcd}
M_{\mathfrak{m}} \arrow[r, "\gamma^{C_{\mathfrak{m}}}_{M_{\mathfrak{m}}}", "\cong"'] \arrow[d, "(\gamma^C_M)_{\mathfrak{m}}"'] & \operatorname{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, C_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}) \arrow[d, "\cong"] \\
\operatorname{Hom}_R(C, C \otimes_R M)_{\mathfrak{m}} \arrow[r, "\cong"] & \operatorname{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, (C \otimes_R M)_{\mathfrak{m}}).
\end{tikzcd}$$

Since this is so for each $\mathfrak{m}$ and each $i \geqslant 1$, we conclude that $\operatorname{Tor}^R_{\geqslant 1}(C, M) = 0 = \operatorname{Ext}^{\geqslant 1}_R(C, C \otimes_R M)$ and that γ^C_M is an isomorphism. Hence $M \in \mathcal{A}_C(R)$ as desired. □

Proposition 3.3.8. *Let C be a semidualizing R-module and M an R-module. The following conditions are equivalent:*

(i) $M \in \mathcal{B}_C(R)$;
(ii) $U^{-1}M \in \mathcal{B}_{U^{-1}C}(U^{-1}R)$ *for each multiplicatively closed subset* $U \subset R$;
(iii) $M_\mathfrak{p} \in \mathcal{B}_{C_\mathfrak{p}}(R_\mathfrak{p})$ *for each prime ideal* $\mathfrak{p} \subset R$;
(iv) $M_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ *for each maximal ideal* $\mathfrak{m} \subset R$.

Proof. Similar to Proposition 3.3.7. □

The following result characterizes the semidualizing modules that are locally isomorphic. The symmetry in conditions (ii) and (iii) implies that condition (i) is also symmetric.

Proposition 3.3.9. *Let B and C be semidualzing R-modules. The following conditions are equivalent:*

(i) $B \cong C \otimes_R P$ *for some finitely generated projective R-module of rank 1;*
(ii) $B_\mathfrak{p} \cong C_\mathfrak{p}$ *for each prime ideal* $\mathfrak{p} \subset R$;
(iii) $B_\mathfrak{m} \cong C_\mathfrak{m}$ *for each maximal ideal* $\mathfrak{m} \subset R$.

When these conditions are satisfied, one has $P \cong \operatorname{Hom}_R(C,B)$ *and* $\operatorname{Hom}_R(P,R) \cong \operatorname{Hom}_R(B,C)$ *(also a rank 1 projective) and* $C \cong B \otimes_R \operatorname{Hom}_R(P,R)$.

Proof. The implication (ii) $\Longrightarrow$ (iii) is straightforward.

(i) $\Longrightarrow$ (ii). The assumptions on P imply that $P_\mathfrak{p} \cong R_\mathfrak{p}$ for each prime ideal $\mathfrak{p} \subset R$, and so there are isomorphisms

$$B_\mathfrak{p} \cong (C \otimes_R P)_\mathfrak{p} \cong C_\mathfrak{p} \otimes_{R_\mathfrak{p}} P_\mathfrak{p} \cong C_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p} \cong C_\mathfrak{p}$$

as desired.

(iii) $\Longrightarrow$ (i). For each maximal ideal $\mathfrak{m} \subset R$, fix an isomorphism $\iota^\mathfrak{m} \colon B_\mathfrak{m} \to C_\mathfrak{m}$. (We use this notation to avoid confusion because $\iota_\mathfrak{m}$ looks too much like the localization of fixed map ι.) For each maximal ideal $\mathfrak{m} \subset R$, this yields isomorphisms

$$\operatorname{Hom}_R(C,B)_\mathfrak{m} \cong \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},B_\mathfrak{m}) \cong \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},C_\mathfrak{m}) \cong R_\mathfrak{m}$$

and so $\operatorname{Hom}_R(C,B)$ is a projective R-module of rank 1.

Consider the natural evaluation map $\xi_B^C \colon C \otimes_R \operatorname{Hom}_R(C,B) \to B$ given by $\xi_B^C(c \otimes \psi) = \psi(c)$. For each maximal ideal $\mathfrak{m} \subset R$ the following diagram commutes

$$\begin{tikzcd}
C_\mathfrak{m} \otimes_{R_\mathfrak{m}} \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},B_\mathfrak{m}) \arrow[r, "(\xi_B^C)_\mathfrak{m}"] \arrow[d, "C_\mathfrak{m}\otimes_{R_\mathfrak{m}}\operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},\iota^\mathfrak{m})"', "\cong"] & B_\mathfrak{m} \arrow[dr, "\iota^\mathfrak{m}", "\cong"'] & \\
C_\mathfrak{m} \otimes_{R_\mathfrak{m}} \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},C_\mathfrak{m}) \arrow[r, "C_\mathfrak{m}\otimes_{R_\mathfrak{m}}\chi_{C_\mathfrak{m}}^{R_\mathfrak{m}}", "\cong"'] & C_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m} \arrow[r, "\cong"'] & C_\mathfrak{m}
\end{tikzcd}$$

where the unspecified isomorphism is given by $c \otimes r \mapsto rc$. It follows that ξ_B^C is locally an isomorphism, and so it is an isomorphism. Hence, the module $P = \operatorname{Hom}_R(C,B)$ satisfies the desired conditions.

For the final conclusions, assume that conditions (i)–(iii) are satisfied. The isomorphism $P \cong \operatorname{Hom}_R(C,B)$ is verified in the proof of the implication (iii) $\Longrightarrow$ (i). By symmetry, we conclude that the module

$$\begin{aligned}
Q &= \operatorname{Hom}_R(B,C) \\
&\cong \operatorname{Hom}_R(C \otimes_R \operatorname{Hom}_R(C,B), C) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(C,B), \operatorname{Hom}_R(C,C)) \\
&\cong \operatorname{Hom}_R(P,R)
\end{aligned}$$

is projective of rank 1 such that $C \cong B \otimes_R Q$, as desired. □

The following result characterizes the semidualizing modules B, C which yield the same Bass class.

Proposition 3.3.10. *Let B and C be semidualzing R-modules. The following conditions are equivalent:*

(i) $B_{\mathfrak{m}} \cong C_{\mathfrak{m}}$ *for each maximal ideal* $\mathfrak{m} \subset R$*;*
(ii) $\mathcal{A}_C(R) = \mathcal{A}_B(R)$*;*
(iii) $\mathcal{B}_C(R) = \mathcal{B}_B(R)$*;*
(iv) $C \in \mathcal{B}_B(R)$ *and* $B \in \mathcal{B}_C(R)$*;*
(v) $\operatorname{Hom}_R(C, B) \in \mathcal{A}_C(R)$ *and* $\operatorname{Hom}_R(B, C) \in \mathcal{A}_B(R)$.

Proof. The implication (iii) $\Longrightarrow$ (iv) follows from Corollary 3.2.2(a) which says that $C \in \mathcal{B}_C(R)$ and $B \in \mathcal{B}_B(R)$. The implications (iv) $\Longleftrightarrow$ (v) follow from Theorem 3.2.1(b).

(i) $\Longrightarrow$ (ii). Let M be an R-module. Proposition 3.3.7 says that $M \in \mathcal{A}_C(R)$ if and only if $M_{\mathfrak{m}} \in \mathcal{A}_{C_{\mathfrak{m}}}(R_{\mathfrak{m}}) = \mathcal{A}_{B_{\mathfrak{m}}}(R_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m} \subset R$, that is, if and only if $M \in \mathcal{A}_B(R)$.

The implication (i) $\Longrightarrow$ (iii) is proved similarly.

(ii) $\Longrightarrow$ (iv). Assume that $\mathcal{A}_C(R) = \mathcal{A}_B(R)$, and let E be a faithfully injective R-module; for instance, one can use the module $E = \oplus_{\mathfrak{m} \in \text{m-Spec}(R)} E_R(R/\mathfrak{m})$. Corollary 3.2.2(a) says that $C \in \mathcal{B}_C(R)$, and so Proposition 3.2.3(a) implies that $\operatorname{Hom}_R(C, E) \in \mathcal{A}_C(R) = \mathcal{A}_B(R)$. Another application of Proposition 3.2.3(a) implies that $C \in \mathcal{B}_B(R)$. Similarly, we have $B \in \mathcal{B}_C(R)$.

(iv) $\Longrightarrow$ (i). Since inclusion in a Bass class is a local condition, we assume within this implication that R is local and show that $B \cong C$. Let F be a minimal free resolution of B. Let G be a minimal free resolution of $\operatorname{Hom}_R(C, B)$. Let H be a minimal free resolution of $\operatorname{Hom}_R(B, C)$. The condition $B \in \mathcal{B}_C(R)$ implies that $F \otimes_R G$ is a minimal free resolution of $C \otimes_R \operatorname{Hom}_R(C, B) \cong B$. The condtion $C \in \mathcal{B}_B(R)$ then implies that $F \otimes_R G \otimes_R H$ is a minimal free resolution of the module $B \otimes_R \operatorname{Hom}_R(B, C) \cong C$. The uniqueness of minimal free resolutions implies $F \otimes_R G \otimes_R H \cong F$

We complete the argument using Poincaré series, that is, the generating function for the ranks of the free modules in the given resolutions. Specifically, we consider the following formal power series with nonnegative integer coefficients:

$$P_F(t) = \sum_{i \geqslant 0} \operatorname{rank}_R(F_i) t^i \qquad P_G(t) = \sum_{i \geqslant 0} \operatorname{rank}_R(G_i) t^i$$
$$P_H(t) = \sum_{i \geqslant 0} \operatorname{rank}_R(H_i) t^i.$$

The final isomorphism from the previous paragraph yields an equality of formal power series

$$P_F(t) = P_F(t) P_G(t) P_H(t).$$

Since the coefficients of these series are nonnegative integers, it follows that $P_G(t) = 1 = P_H(t)$. This implies $\operatorname{Hom}_R(C, B) \cong R \cong \operatorname{Hom}_R(B, C)$, and hence

$$C \cong B \otimes_R \operatorname{Hom}_R(B, C) \cong B \otimes_R R \cong B$$

as desired. □

For clarity, we single out the local case of the previous result explicitly.

Corollary 3.3.11. *Assume that R is local, and let B and C be semidualzing R-modules. The following conditions are equivalent:*

(i) $B \cong C$;
(ii) $\mathcal{A}_C(R) = \mathcal{A}_B(R)$;
(iii) $\mathcal{B}_C(R) = \mathcal{B}_B(R)$;
(iv) $C \in \mathcal{B}_B(R)$ *and* $B \in \mathcal{B}_C(R)$;
(v) $\operatorname{Hom}_R(C, B) \in \mathcal{A}_C(R)$ *and* $\operatorname{Hom}_R(B, C) \in \mathcal{A}_B(R)$. □

Here is a description of the semidualizing modules of finite projective dimension.

Corollary 3.3.12. *Let C be a semidualizing R-module of finite projective dimension. Then C is a rank 1 projective R-module. If R is local, then $C \cong R$.*

Proof. Assume first that R is local. The Auslander-Buchsbaum formula explains the first equality in the next sequence

$$\operatorname{pd}_R(C) = \operatorname{depth}(R) - \operatorname{depth}_R(C) = 0$$

and the second equality is from Corollary 3.3.3(c). This shows that C is projective, and, since R is local, that C is free. Hence $C \cong R^n$ for some integer $n \geqslant 1$. From the isomorphisms

$$R \cong \operatorname{Hom}_R(C, C) \cong \operatorname{Hom}_R(R^n, R^n) \cong R^{n^2}$$

it follows that $n = 1$ and so $C \cong R$.

Assume now that R is not necessarily local. For each maximal ideal $\mathfrak{m} \subset R$, the $R_\mathfrak{m}$-module $C_\mathfrak{m}$ is semidualizing and has finite projective dimension, and so we have $C_\mathfrak{m} \cong R_\mathfrak{m}$ for each $\mathfrak{m}$. Proposition 3.3.9 implies that there is a rank 1 projective R-module P such that $C \cong R \otimes_R P \cong P$, an so C is a rank 1 projective R-module. □

We next show that semidualizing modules over Gorenstein rings are trivial.

Corollary 3.3.13. *Assume that R is Gorenstein, and let C be a semidualizing R-module. Then C is a rank 1 projective R-module. If R is local, then $C \cong R$.*

Proof. The fact that R is Gorenstein says that R has finite injective dimension as an R-module, and so $R \in \mathcal{B}_C(R)$. On the other hand, every R-module is in $catb_R(R)$, and so $C \in \mathcal{B}_R(R)$. Propositions 3.3.9 and 3.3.10 show that there is a rank 1 projective R-module such that $C \cong R \otimes_R P \cong P$, and so C is a rank 1 projective R-module. When R is local, Corollary 3.3.11 implies directly that $C \cong R$. □

4. Totally C-Reflexive Modules

This section is about duality.

4.1. **Basic Properties.** Recall that the terms "semidualizing" and "totally C-reflexive" are defined in 1.4.3 and 1.4.3, respectively.

Proposition 4.1.1. *Let C be a semidualizing R-module, and let M and N be finitely generated R-modules. Then $M \oplus N$ is totally C-reflexive if and only if M and N are both totally C-reflexive.*

Proof. It is straightforward to show that $M \oplus N$ is finitely generated if and only if M and N are both finitely generated. The isomorphism

$$\operatorname{Ext}^i_R(M \oplus N, C) \cong \operatorname{Ext}^i_R(M, C) \oplus \operatorname{Ext}^i_R(N, C)$$

shows that $\operatorname{Ext}^{\geqslant 1}_R(M \oplus N, C) = 0$ if and only if $\operatorname{Ext}^{\geqslant 1}_R(M, C) = 0 = \operatorname{Ext}^{\geqslant 1}_R(N, C)$. The isomorphisms

$$\begin{aligned}\operatorname{Ext}^i_R(\operatorname{Hom}_R(M \oplus N, C), C) &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(M, C) \oplus \operatorname{Hom}_R(N, C), C)\\ &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(M, C), C) \oplus \operatorname{Ext}^i_R(\operatorname{Hom}_R(N, C), C)\end{aligned}$$

shows that $\operatorname{Ext}^{\geqslant 1}_R(\operatorname{Hom}_R(M \oplus N, C), C) = 0$ if and only if $\operatorname{Ext}^{\geqslant 1}_R(\operatorname{Hom}_R(M, C), C) = 0 = \operatorname{Ext}^{\geqslant 1}_R(\operatorname{Hom}_R(N, C), C)$. There is a commutative diagram

$$\begin{tikzcd} M \oplus N \arrow[r, "\delta^C_{M\oplus N}"] \arrow[dr, "\delta^C_M \oplus \delta^C_N"'] & \operatorname{Hom}_R(\operatorname{Hom}_R(M \oplus N, C), C) \arrow[d, "\cong"] \\ & \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C) \oplus \operatorname{Hom}_R(\operatorname{Hom}_R(N, C), C) \end{tikzcd}$$

and so $\delta^C_{M\oplus N}$ is an isomorphism if and only if $\delta^C_M \oplus \delta^C_N$ is an isomorphism, that is, if and only if δ^C_M and δ^C_N are both isomorphisms. The desired result now follows. □

Proposition 4.1.2. *Let C be a semidualizing R-module. If P is a finitely generated projective R-module, then P and $C \otimes_R P$ are totally C-reflexive. In particular, for each integer $n \geqslant 0$, the modules R^n and C^n are totally C-reflexive.*

Proof. Proposition 1.4.4 shows that R and C are totally C-reflexive. Since P is a finitely generated projective R-module, there is a second finitely generated projective R-module Q such that $P \oplus Q \cong R^n$ for some integer $n \geqslant 0$. Since R is totally C-reflexive we conclude from Proposition 4.1.1 that $R^n \cong P \oplus Q$ is totally C-reflexive and then that P and Q are totally C-reflexive. Since C is totally C-reflexive we conclude from Proposition 4.1.1 that C^n is totally C-reflexive and then, using the isomorphisms

$$C^n \cong C \otimes_R R^n \cong (C \otimes_R P) \oplus (C \otimes_R Q).$$

that $C \otimes_R P$ and $C \otimes_R Q$ are totally C-reflexive. □

Proposition 4.1.3. *Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

such that M'' is totally C-reflexive. Then M' is totally C-reflexive if and only if M is totally C-reflexive.

Proof. Since M'' is totally C-reflexive, we have $\operatorname{Ext}^{\geqslant 1}_R(M'', C) = 0$. Using the long exact sequence in $\operatorname{Ext}^i_R(-, C)$ associated to the given sequence, we conclude that $\operatorname{Ext}^{\geqslant 1}_R(M', C) = 0$ if and only if $\operatorname{Ext}^{\geqslant 1}_R(M, C) = 0$. Furthermore, the condition $\operatorname{Ext}^1_R(M'', C) = 0$ yields a second exact sequence

$$(*) \quad 0 \to \operatorname{Hom}_R(M'', C) \xrightarrow{\operatorname{Hom}_R(g,C)} \operatorname{Hom}_R(M, C) \xrightarrow{\operatorname{Hom}_R(f,C)} \operatorname{Hom}_R(M', C) \to 0.$$

Since M'' is totally C-reflexive, we have $\operatorname{Ext}^{\geqslant 1}_R(\operatorname{Hom}_R(M'', C), C) = 0$. Using the long exact sequence in $\operatorname{Ext}^i_R(-, C)$ associated to the sequence $(*)$, we see that

$\operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(M',C),C)=0$ if and only if $\operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(M,C),C)=0$. For convenience, set $(-)^{\dagger\dagger}=\operatorname{Hom}_R(\operatorname{Hom}_R(-,C),C)$. The naturality of the biduality maps yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\delta^C_{M'}"'] & M \arrow[r, "g"] \arrow[d, "\delta^C_M"'] & M'' \arrow[r] \arrow[d, "\delta^C_{M''}"', "\cong"] & 0 \\
0 \arrow[r] & (M')^{\dagger\dagger} \arrow[r, "f^{\dagger\dagger}"] & M^{\dagger\dagger} \arrow[r, "g^{\dagger\dagger}"] & (M'')^{\dagger\dagger} \arrow[r] & 0.
\end{tikzcd}$$

The top row is exact by assumption. The top row is exact because $(*)$ is exact and $\operatorname{Ext}_R^1(\operatorname{Hom}_R(M',C),C)=0$. The Snake Lemma shows that $\delta^C_{M'}$ is an isomorphism if and only if δ^C_M is an isomorphism. This completes the proof. □

The next example shows that, given an exact sequence $0\to M'\to M\to M''\to 0$, if M' and M are totally C-reflexive, then M'' need not be totally C-reflexive. See however Proposition 4.1.5 below.

Example 4.1.4. Let k be a field and $R=k[\![X]\!]$ a formal power series ring in one variable. Consider the exact sequence

$$0\to R\xrightarrow{X}R\to k\to 0.$$

The module R is totally reflexive, but k is not because $\operatorname{Ext}_R^1(k,R)\cong k\neq 0$.

For the next result, argue as in the proof of Proposition 3.1.4 or 4.1.5.

Proposition 4.1.5. *Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules*

$$0\to M'\xrightarrow{f}M\xrightarrow{g}M''\to 0$$

such that M' and M are totally C-reflexive. Then M'' is totally C-reflexive if and only if $\operatorname{Ext}_R^1(M,C)=0$. □

For the next result, argue as in the proof of Proposition 3.2.3.

Proposition 4.1.6. *Let C be a semidualizing R-module, let P be a finitely generated projective R-module, and let M be a finitely generated R-module.*

(a) *If M is is totally C-reflexive, then so are $M\otimes_R P$ and $\operatorname{Hom}_R(P,M)$.*
(b) *If P is faithfully projective and either $M\otimes_R P$ or $\operatorname{Hom}_R(P,M)$ is totally C-reflexive, then M is is totally C-reflexive.* □

Proposition 4.1.7. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then M is totally C-reflexive if and only if $\operatorname{Hom}_R(M,C)$ is totally C-reflexive and $\operatorname{Ext}_R^{\geqslant 1}(M,C)=0$.*

Proof. □

Proposition 4.1.8. *Let B and C be a semidualizing R-modules. The following conditions are equivalent:*

(i) *B is totally C-reflexive;*
(ii) *$C\in\mathcal{B}_B(R)$;*
(iii) *$\operatorname{Hom}_R(B,C)$ is a semidualizing R-module and $\operatorname{Ext}_R^{\geqslant 1}(B,C)=0$.*

When these conditions are satisfied, the module $\operatorname{Hom}_R(B,C)$ is totally C-reflexive, and $B\in\mathcal{A}_{\operatorname{Hom}_R(B,C)}(R)$ and $\operatorname{Hom}_R(B,C)\in\mathcal{A}_B(R)$ and $C\cong B\otimes_R\operatorname{Hom}_R(B,C)$.

Proof. □

4.2. **Base Change.** This subsection is similar to subsection 3.3

Proposition 4.2.1. *Let C be a semidualizing R-module, let $\varphi\colon R \to S$ be a flat ring homomorphism, and Let M be a finitely generated R-module. If M is is totally C-reflexive, then the S-module $S \otimes_R M$ is totally $S \otimes_R C$-reflexive; the converse holds when φ is faithfully flat.*

Proof. Because φ is flat, Corollary 3.3.2 implies that $S \otimes_R C$ is a semidualizing S-module. Since M is finitely generated and φ is flat, we have isomorphisms

$$\mathrm{Ext}^i_S(S \otimes_R M, S \otimes_R C) \cong S \otimes_R \mathrm{Ext}^i_R(M, C)$$

$$\mathrm{Ext}^i_S(\mathrm{Hom}_S(S \otimes_R M, S \otimes_R C), S \otimes_R C) \cong S \otimes_R \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C).$$

Hence, if $\mathrm{Ext}^{\geqslant 1}_R(M, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C)$, then we conclude that $\mathrm{Ext}^{\geqslant 1}_S(S \otimes_R M, S \otimes_R C) = 0 = \mathrm{Ext}^{\geqslant 1}_S(\mathrm{Hom}_S(S \otimes_R M, S \otimes_R C), S \otimes_R C)$; and the converse holds when φ is faithfully flat. We also have a commutative diagram

$$\begin{tikzcd}
S \otimes_R M \arrow[r, "\delta^{S\otimes_R C}_{S\otimes_R}"] \arrow[d, "S\otimes_R \delta^C_M"'] & \mathrm{Hom}_S(\mathrm{Hom}_S(S \otimes_R M, S \otimes_R C), S \otimes_R C) \arrow[d, "\cong"] \\
S \otimes_R \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C) \arrow[r, "\cong"] & \mathrm{Hom}_S(S \otimes_R \mathrm{Hom}_S(M, C), S \otimes_R C)
\end{tikzcd}$$

where the unspecified isomorphisms are the natural ones. Hence, if the biduality map δ^C_M is an isomorphism, then so is $\delta^{S\otimes_R C}_{S\otimes_R}$; and the converse holds when φ is faithfully flat. This justifies the desired implications. □

Here is a local global principal for totally reflexive modules. Its proof is similar to that of Proposition 3.3.7.

Proposition 4.2.2. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. The following conditions are equivalent.*

(i) *M is a totally C-reflexive R-module;*
(ii) *$U^{-1}M$ is a totally $U^{-1}C$-reflexive $U^{-1}R$-module for each multiplicatively closed subset $U \subset R$;*
(iii) *$M_{\mathfrak{p}}$ is a totally $C_{\mathfrak{p}}$-reflexive $R_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p} \subset R$;*
(iv) *$M_{\mathfrak{m}}$ is a totally $C_{\mathfrak{m}}$-reflexive $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subset R$.* □

Here is a companion for Proposition 3.3.9.

Proposition 4.2.3. *Let B and C be semidualizing R-modules The following conditions are equivalent.*

(i) *$B_{\mathfrak{m}} \cong C_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$;*
(ii) *$\mathcal{G}_B(R) = \mathcal{G}_C(R)$;*
(iii) *B is totally C-reflexive and C is totally B-reflexive.*

Proof. □

Proposition 4.2.4. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Let $\mathbf{x} = x_1, \ldots, x_d \in R$ be a sequence that is R-regular and M-regular, and set $S = R/(\mathbf{x})R$. If M is is totally C-reflexive, then the S-module $S \otimes_R M$ is totally $S \otimes_R C$-reflexive. The converse holds when $\mathbf{x}$ is in the Jacobson radical of R.*

Proof. □

5. G_C-Dimension

5.1. **Basic Properties.**

Definition 5.1.1. Let C be a semidualizing R-module and M a finitely generated R-module. An *augmented G_C-resolution* of M is an exact sequence

$$G^+ = \cdots \xrightarrow{\partial^G_{i+1}} G_i \xrightarrow{\partial^G_i} G_{i-1} \xrightarrow{\partial^G_{i-1}} \cdots G_1 \xrightarrow{\partial^G_1} G_0 \xrightarrow{\tau} M \to 0$$

wherein each G_i is totally C-reflexive. The *G_C-resolution* of M associated to G^+ is the sequence obtained by truncating:

$$G^+ = \cdots \xrightarrow{\partial^G_{i+1}} G_i \xrightarrow{\partial^G_i} G_{i-1} \xrightarrow{\partial^G_{i-1}} \cdots G_1 \xrightarrow{\partial^G_1} G_0 \to 0$$

Definition 5.1.2. Let C be a semidualizing R-module and M a finitely generated R-module. If M admits a G_C-resolution G such that $G_i = 0$ for $i \gg 0$, then we say that M has finite G_C-dimension. More specifically, the *G_C-dimension* of M is the shortest such resolution:

$$\mathrm{G}_C\text{-}\dim_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_n \neq 0\} \mid G \text{ is a } \mathrm{G}_C\text{-resolution of } M\}.$$

6. Application: Complete Intersection Dimension and Quasi-deformations

The goal of this section is to prove the following generalization of Proposition 1.2.6. It is another application of the theory of semidualizing modules that does not refer to semidualizing modules in the statement.

Theorem 6.0.3. *Let $(R, \mathfrak{m})$ be a local ring and M an R-module of finite complete intersection dimension. Then there exists a quasi-deformation $R \xrightarrow{\rho} R' \leftarrow Q$ such that $\mathrm{pd}_Q(R' \otimes_R M) < \infty$ and such that Q is complete with algebraically closed residue field and such that the closed fibre $R'/\mathfrak{m}R'$ is artinian and Gorenstein.*

7. Application: Composition of Local Ring Homomorphisms

The goal of this section is to prove the following result which answers a special case of Question 1.3.10. It is an application of the theory of semidualizing modules that does not refer to semidualizing modules in the statement.

Theorem 7.0.4. *Let $\varphi\colon R \to S$ and $\psi\colon S \to T$ be local ring homomorphisms. If φ has finite Gorenstein dimension and ψ has finite complete intersection dimension, then the composition $\psi \circ \varphi$ has finite Gorenstein dimension.*

Appendix A. Some aspects of homological algebra

A.1. **Natural Maps.** Hom and tensor-evaluation. amplitude inequality. operations on complexes. Künneth formula. flat dimension, projective dimension, injective dimension. Poincarśeries

Definition A.1.1. Let L, M, N be R-modules.
The *tensor evaluation* homomorphism

$$\omega_{LMN}\colon \mathrm{Hom}_R(L, M) \otimes_R N \to \mathrm{Hom}_R(L, M \otimes_R N)$$

is given by $\omega_{LMN}(\psi \otimes n)(l) = \psi(l) \otimes n$.

The *Hom evaluation* homomorphism

$$\theta_{LMN}\colon L \otimes_R \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(L,M),N)$$

is given by $\theta_{LMN}(l \otimes \psi)(\phi) = \psi(\phi(l))$.

The next two lemmata are from Ishikawa [ref].

Lemma A.1.2. *Let L, M, N be R-modules. The tensor evaluation homomorphism $\omega_{LMN}\colon \operatorname{Hom}_R(L,M) \otimes_R N \to \operatorname{Hom}_R(L, M \otimes_R N)$ is an isomorphism under either of the following conditions:*

(1) *L is finitely generated and projective; or*
(2) *L is finitely generated and N is flat.*

Proof. First observe that, for R-modules L', L'', the following commutative diagram shows that the map $\omega_{(L'\oplus L'')MN}$ is an isomorphism if and only if $\omega_{L'MN}$ and $\omega_{L''MN}$ are isomorphisms:

$$\begin{tikzcd}
\operatorname{Hom}_R(L' \oplus L'', M) \otimes_R N \arrow[r, "\cong"] \arrow[d, "\omega_{(L'\oplus L'')MN}"'] & (\operatorname{Hom}_R(L', M) \otimes_R N) \oplus (\operatorname{Hom}_R(L'', M) \otimes_R N) \arrow[d, "\omega_{L'MN} \oplus \omega_{L''MN}"] \\
\operatorname{Hom}_R(L' \oplus L'', M \otimes_R N) \arrow[r, "\cong"] & \operatorname{Hom}_R(L', M \otimes_R N) \oplus \operatorname{Hom}_R(L'', M \otimes_R N).
\end{tikzcd}$$

(1) It is straightforward to show that ω_{LMN} is an isomorphism when $L = R$. Hence, an induction argument using the above observation shows that ω_{LMN} is an isomorphism when $L = R^n$ for some n. When L is finitely generated and projective, there is an R-module L'' such that $L \oplus L'' \cong R^n$ for some n, so again the previous paragraph implies that ω_{LMN} is an isomorphism in this case.

(2) Assume that L is finitely generated and M is flat. Since R is noetherian, there exists an exact sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} L \to 0.$$

The left-exactness of $\operatorname{Hom}_R(-,M)$ implies that the next sequence is exact

$$0 \to \operatorname{Hom}_R(L,M) \xrightarrow{g^M} \operatorname{Hom}_R(R^n,M) \xrightarrow{f^M} \operatorname{Hom}_R(R^m,M)$$

where $f^M = \operatorname{Hom}_R(f,M)$ and $g^M = \operatorname{Hom}_R(g,M)$. Since N is flat, the top row of the next commutative diagram is also exact

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(L,M) \otimes N \arrow[r, "g^M \otimes N"] \arrow[d, "\omega_{LMN}"'] & \operatorname{Hom}(R^n,M) \otimes N \arrow[r, "f^M \otimes N"] \arrow[d, "\omega_{R^nMN}"', "\cong"] & \operatorname{Hom}(R^m,M) \otimes N \arrow[d, "\omega_{R^mMN}"', "\cong"] \\
0 \arrow[r] & \operatorname{Hom}(L,M \otimes N) \arrow[r, "g^{M\otimes N}"] & \operatorname{Hom}(R^n,M \otimes N) \arrow[r, "f^{M\otimes N}"] & \operatorname{Hom}(R^m,M \otimes N).
\end{tikzcd}$$

The bottom row is exact because $\operatorname{Hom}_R(-,M\otimes N)$ is left-exact. The maps ω_{R^mMN} and ω_{R^nMN} are isomorphisms by case (1), so a diagram chase shows that ω_{LMN} is an isomorphism as well. □

Lemma A.1.3. *Let L, M, N be R-modules. The Hom evaluation homomorphism $\theta_{LMN}\colon L \otimes_R \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(L,M),N)$ is an isomorphism under either of the following conditions:*

(1) *L is finitely generated and projective; or*
(2) *L is finitely generated and N is injective.*

Proof. Similar to the proof of Lemma A.1.2. □

A.2. **Fidelity.**

Lemma A.2.1. *Let M and N be nonzero R-modules. If M is finitely generated and* $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$*, then* $\operatorname{Hom}_R(M,N) \neq 0$ *and* $M \otimes_R N \neq 0$.

Proof. If R is local with maximal ideal $\mathfrak{m}$, then $\operatorname{Hom}_R(M, R/\mathfrak{m}) \neq 0$. Indeed, Nakayama's Lemma implies that $M/\mathfrak{m}M$ is a nonzero vector space over $R/\mathfrak{m}$ and so any composition $M \twoheadrightarrow M/\mathfrak{m}M \twoheadrightarrow R/\mathfrak{m}$ gives a nonzero element of $\operatorname{Hom}_R(M, R/\mathfrak{m})$. It follows that, for each $\mathfrak{p} \in \operatorname{Spec}(R)$, we have

$$\operatorname{Hom}_R(M, R/\mathfrak{p})_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \neq 0$$

and so $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$.

Use the fact that R is noetherian to conclude that N has an associated prime $\mathfrak{p}$, and hence a monomorphism $R/\mathfrak{p} \hookrightarrow M$. Apply $\operatorname{Hom}_R(M, -)$ to find a monomorphism $0 \neq \operatorname{Hom}_R(M, R/\mathfrak{p}) \hookrightarrow \operatorname{Hom}_R(M,N)$. It follows that $\operatorname{Hom}_R(M,N) \neq 0$.

For the tensor product, note that the identity $N \to N$ is a nonzero element of $\operatorname{Hom}_R(N,N)$. Therefore, the previous paragraph provides the nonvanishing in the next sequence while the isomorphism is by Hom-tensor adjointness:

$$\operatorname{Hom}_R(M \otimes N, N) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(N,N)) \neq 0.$$

It follows that $M \otimes N \neq 0$. □

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