Semidualizing Modules and Bass Numbers

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Bass Numbers

Assumption $(R, \mathfrak{m}, k)$ is a local ring.

Definition. The ith Bass number of R is the integer $\mu^i = \mu^i_R(R) = \dim_k(\operatorname{Ext}^i_R(k, R))$.

Fact. If $\mu^i = 0$ for some $i \geq \operatorname{depth}(R)$, then R is Gorenstein and hence $\mu^j = 0$ for all $j \geq \operatorname{depth}(R)$.

Questions. (Huneke)

(a) If the sequence $\{\mu^i\}_i$ is bounded, must R be Gorenstein?

(b) If the sequence $\{\mu^i\}_i$ is bounded by a polynomial in i, must R be Gorenstein?

Recent progress by Jorgensen and Leuschke; Christensen, Striuli and Veliche; Lorestani, Sather-Wagstaff and Yassemi.

Dualizing Modules

Definition. The homothety morphism for an R-module C is the map $\chi_C^R \colon R \to \mathrm{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$.

Definition. (Grothendieck '61) An R-module D is dualizing if:

(1) D is finitely generated,
(2) $\chi_D^R \colon R \to \mathrm{Hom}_R(D, D)$ is an isomorphism;
(3) $\mathrm{Ext}_R^i(D, D) = 0$ for all $i \geq 1$; and
(4) $\mathrm{id}_R(D) < \infty$.

Fact. R is Gorenstein if and only if R is a dualizing R-module.

Fact. If R is artinian, then $E_R(k)$ is a dualizing R-module.

Fact. (Foxby '72, Reiten '72, Sharp '71) The ring R admits a dualizing module if and only if it is Cohen-Macaulay and a homomorphic image of a Gorenstein ring.

Fact. If R has a dualizing module D and $d = \dim(R)$, then $\beta_i^R(D) = \mu^{i-d}$ for all $i \geq 0$.

Semidualizing Modules

Definition. (Foxby '72, Golod '84, Vasconcelos '74, Wakamatsu '88) An R-module C is semidualizing if:

(1) C is finitely generated,
(2) $\chi^R_C\colon R \to \mathrm{Hom}_R(C, C)$ is an isomorphism; and
(3) $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geq 1$.

Example. D is dualizing iff D is semidualizing and $\mathrm{id}_R(D) < \infty$.

$C \cong R$ iff C is semidualizing and $\mathrm{pd}_R(D) < \infty$.

Fact. If R has a dualizing module D, then the following conditions are equivalent:

(i) R is Gorenstein;
(ii) $R \cong D$; and
(iii) R has only one semidualizing module, up to isomorphism.

Fact. If R is Cohen-Macaulay, then every semidualizing R-module is a maximal Cohen-Macaulay R-module (MCM).

Semidualizing Modules, Examples

Definition. An R-module C is semidualizing if:

(1) C is finitely generated,
(2) $\chi_C^R\colon R \to \mathrm{Hom}_R(C,C)$ is an isomorphism; and
(3) $\mathrm{Ext}_R^i(C,C) = 0$ for all $i \geq 1$.

Example. The ring $R = k[X,Y]/(X^2, XY, Y^2)$ has exactly two non-isomorphic semidualizing modules: R and $E_R(k)$.

The ring $S = k[X,Y,Z,W]/(X^2, XY, Y^2, Z^2, ZW, W^2)$ has exactly four non-isomorphic semidualizing modules.

Fact. If D is dualizing and C is semidualizing, then $\mathrm{Hom}_R(C,D)$ is semidulazing and $\mathrm{Ext}_R^i(C,D) = 0$ for all $i \geq 1$.

Question. Is the number of isomorphism classes of semidualizing R-modules finite? Is it always a power of 2?

Problem. Characterize the local artinian rings with exactly two non-isomorphic semidualizing modules.

Theorem. (SSW 2008) *If R has a dualizing module D and a semidualizing module C such that $C \not\cong R$ and $C \not\cong D$, then the sequence $\{\mu^i\}_i$ is unbounded.*

Proof. Consider minimal free resolutions

$$\begin{array}{l} \cdots \to R^{a_2} \to R^{a_1} \to R^{a_0} \longrightarrow C \longrightarrow 0 \\ \cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to \operatorname{Hom}_R(C,D) \to 0. \end{array}$$

Since $C \not\cong R$ and $C \not\cong D$, we have $a_i, b_i \geq 1$ for all $i \geq 0$.

Since C is semidualizing, we have $D \cong C \otimes_R \operatorname{Hom}_R(C,D)$ and $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C,D)) = 0$ for $i \geq 1$.

Hence, the minimal free resolution of D is the tensor product of the minimal free resolutions of C and $\operatorname{Hom}_R(C,D)$.

It follows that $\mu^{i-d} = \beta_i^R(D) = \sum_{j=0}^{i} a_i b_i \geq i$. □

Definition. Let C be a semidualizing R-module. A finitely generated R-module M is totally C-reflexive if:

(1) The natural biduality map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism; and
(2) $\mathrm{Ext}^i_R(M, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C)$ for all $i \geq 1$.

Example. Every finite rank free module R^n is totally C-reflexive.

If D is dualizing, every MCM R-module is totally D-reflexive.

Definition. Given semidualizing R-modules B and C, write $C \trianglelefteq B$ whenever B is totally C-reflexive.

Example. If C is semidualizing, then $C \trianglelefteq R$.

If C is semidualizing and D is dualizing, then $D \trianglelefteq C$.

Theorem. (SSW 2008) *If R has a chain of semidualizing modules $C_0 \lhd C_1 \lhd \cdots \lhd C_{d+1}$, then the sequence $\{\mu^i\}_i$ is bounded below by a polynomial in i of degree d.*

These ideas generalize to semidualizing complexes.

The paper `arXiv:0812.0643` contains these results, with a survey of the derived category notions needed for the proofs of the results.