Semidualizing Modules for Rings of Codimension Two

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**Definition.** (Foxby, Golod, Vasconcelos, Wakamatsu)
A finitely generated $R$-module $C$ is **semidualizing** if

1. the natural map $\chi^R_C: R \to \text{Hom}_R(C, C)$ is bijective; and
2. $\text{Ext}^i_R(C, C) = 0$ for all $i \geq 1$.

An $R$-module $D$ is **dualizing** if it is semidualizing and $\text{id}(D) < \infty$. 

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An \(R\)-module \(D\) is **dualizing** if it is semidualizing and \(\text{id}(D) < \infty\).

Fact. The following conditions are equivalent:

1. \(R\) is Gorenstein;
2. \(D \cong R\);
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(ii) \(D \cong R\);
(iii) \(\beta^R_i(D) = 0\) for some (equivalently, all) \(i \geq 1\); and
(iv) \(R\) has a unique semidualizing module.
Question. (Huneke) If the sequence \( \{ \beta^R_i(D) \}_i \) is bounded, must \( R \) be Gorenstein?

Theorem. (SSW, 2008) If \( C \) is a semidualizing \( R \)-module such that \( D \not\sim = C \not\sim = R \), then the sequence \( \{ \beta^R_i(D) \}_i \) is unbounded.

Tools for proof.

(a) The following conditions are equivalent:

(i) \( C \sim = R \);

(ii) \( C \) is cyclic;

(iii) \( \text{pd}_R(C) < \infty \).

(b) \( C^\dagger = \text{Hom}_R(C, D) \) is semidualizing, and \( C^{\dagger\dagger} \sim = C \).

(c) \( C \sim = D \) if and only if \( \text{pd}_R(C^\dagger) < \infty \).

(d) \( D \sim = C \otimes_R C^\dagger \) and \( \text{Tor}_R^1(C, C^\dagger) = 0 \).

(e) \( \beta^R_i(D) = \sum_{j=0}^{\infty} \beta^R_j(C) \beta^R_{i-j}(C^\dagger) \).
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(a) The following conditions are equivalent:

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**Proof.** Since $C \not\cong R$, we have $\beta_i^R(C) \geq 1$ for all $i \geq 0$.

Since $C \not\cong D$, we have $\beta_i^R(C^\perp) \geq 1$ for all $i \geq 0$.

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**Strategy.** Identify all $R$ for which the only two semidualizing modules (up to isomorphism) are $R$ and $D$. These are the only rings where Huneke’s question is still open.
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Strategy. Identify all $R$ for which the only two semidualizing modules (up to isomorphism) are $R$ and $D$. These are the only rings where Huneke’s question is still open.

Goal. Given $R$, count the semidualizing $R$-modules and describe them explicitly.
**Theorem.** (SSW, 2008) If $C$ is a semidualizing $R$-module such that $D \not\cong C \not\cong R$, then the sequence $\{\beta^R_i(D)\}_i$ is unbounded.

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**Goal.** Given $R$, count the semidualizing $R$-modules and describe them explicitly.

**Focus.** Cohen-Macaulay local rings of codimension 2.
Note. If codim$(R) \leq 1$, then $R$ is Gorenstein, so $R$ has a single semidualizing module.
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Note. Ghione and Gulliksen show that the sequence $\{\beta_i^R(D)\}_i$ is unbounded when $R$ is not Gorenstein and $\operatorname{codim}(R) = 2$. See also Christensen, Striuli, and Veliche.
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2. Is the number of isomorphism classes of semidualizing $R$-modules equal to $2^n$ for some integer $n$?
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4. If $P \in \text{Ass}(R)$, then $\text{length}_{R_P}(C_P) = \text{length}_{R_P}(R_P)$?
Lemma. (SMC-SSW, 2009) Let $R$ be a Cohen-Macaulay local ring of codimension 2. Let $C$ be a semidualizing $R$-module such that $\beta^R_0(C) \leq \beta^R_1(C)$. Then $C$ is dualizing for $R$. 

Proof. Jorgensen and Leuschke show that $\beta^R_1(D) = \beta^R_0(D) + 1$. Suppose that $C$ is not dualizing. The assumption $\beta^R_0(C) \leq \beta^R_1(C)$ implies that $C \not\simeq R$. Thus $\beta^R_0(D) + 1 = \beta^R_1(D) = \beta^R_1(C) \beta^R_0(C^\ast) + \beta^R_0(C) \geq \beta^R_0(D) + 2$. This is an egregious contradiction.
Lemma. (SMC-SSW, 2009) Let $R$ be a Cohen-Macaulay local ring of codimension 2. Let $C$ be a semidualizing $R$-module such that $\beta^R_0(C) \leq \beta^R_1(C)$. Then $C$ is dualizing for $R$.

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$$\beta^R_0(D) + 1 = \beta^R_1(D)$$

$$= \beta^R_1(C) \beta^R_0(C^\dagger) + \beta^R_0(C) \beta^R_1(C^\dagger)$$

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$$\geq \beta^R_0(D) + 2.$$ 

This is an egregious contradiction.
Lemma. (SMC-SSW, 2009) Let $R$ be a Cohen-Macaulay local ring. Let $C$ be an $R$-module with $\text{pd}_R(C) = \infty$. Assume that for every $P \in \text{Ass}(R)$ one has $\text{length}_{R_P}(C_P) = \text{length}_{R_P}(R_P)$. Then $\beta_0^R(C) \leq \beta_1^R(C)$. 
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Proof. Count lengths in an exact sequence

$$0 \rightarrow L \rightarrow R^{\beta_1^R(C)} \rightarrow R^{\beta_0^R(C)} \rightarrow C \rightarrow 0$$

to conclude that $\beta_0^R(C) - 1 \leq \beta_1^R(C)$. 

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Suppose that $\beta_0^R(C) > \beta_1^R(C)$; then $\beta_0^R(C) - 1 = \beta_1^R(C)$.

Consider a second exact sequence $0 \to K \to R^{\beta_0^R(C)} \to C \to 0$.

Another length-count shows that $\beta_0^R(K) = \text{rank}_R(K)$. 
Lemma. (SMC-SSW, 2009) Let $R$ be a Cohen-Macaulay local ring. Let $C$ be an $R$-module with $\text{pd}_R(C) = \infty$. Assume that for every $P \in \text{Ass}(R)$ one has $\text{length}_{R_P}(C_P) = \text{length}_{R_P}(R_P)$. Then $\beta_0^R(C) \leq \beta_1^R(C)$.

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Thus, $K$ is free, contradicting the assumption $\text{pd}_R(C) = \infty$. \qed
Proposition. (SMC-SSW, 2009) Let $R$ be a Cohen-Macaulay local ring of codimension 2 where $\text{length}_{R_P}(C_P) = \text{length}_{R_P}(R_P)$ for each $P \in \text{Ass}(R)$ and each semidualizing $R$-module $C$. The only two semidualizing $R$-modules (up to isomorphism) are $R$ and $D$.

Proof. Combine the previous two lemmas.
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Proof. Combine the previous two lemmas.

Question. If $R$ is a Cohen-Macaulay local ring of codimension 2, are the only two semidualizing modules (up to isomorphism) $R$ and $D$?
Special Cases

Theorem. (SMC-SSW, 2009) Let $R$ be a Cohen-Macaulay local ring of codimension 2. Assume that one of the following holds:

1. $R$ is generically Gorenstein, e.g., $R$ is reduced;
2. $R_P = 0$ for each $P \in \text{Ass}(R)$;
3. $\hat{R} \sim = S/I$ where $S = k[[X_0, X_1, X_2]]$ and $I \subset k[[X_0, X_1, X_2]]$ is the ideal of vanishing for a fat point scheme in $\mathbb{P}^2_k$; or
4. $\hat{R} \sim = k[[X_1, \ldots, X_n]]/I$ where $I$ is a monomial ideal.

The only two semidualizing $R$-modules (up to isomorphism) are $R$ and $D$.

Proof. In each case, prove that length$_R(C_P) = \text{length}_R(R_P)$ for every $P \in \text{Ass}(R)$ and every semidualizing $R$-module $C$. This is straightforward in cases (1) and (2). It requires more work in cases (3) and (4).
**Theorem.** (SMC-SSW, 2009) Let $R$ be a Cohen-Macaulay local ring of codimension 2. Assume that one of the following holds:

(1) $R$ is generically Gorenstein, e.g., $R$ is reduced;

(2) $P^2 R_{P} = 0$ for each $P \in \text{Ass}(R)$;

(3) $\hat{R} \cong S/I S$ where $S = k[\[X_0, X_1, X_2\]]$ and $I \subset k[\[X_0, X_1, X_2\]]$ is the ideal of vanishing for a fat point scheme in $\mathbb{P}^2_k$; or

(4) $\hat{R} \cong k[\[X_1, \ldots, X_n\]]/I$ where $I$ is a monomial ideal.

The only two semidualizing $R$-modules (up to isomorphism) are $R$ and $D$. Proof. In each case, prove that length $R_{P}(C_{P}) = \text{length } R_{P}(R_{P})$ for every $P \in \text{Ass}(R)$ and every semidualizing $R$-module $C$. This is straightforward in cases (1) and (2). It requires more work in cases (3) and (4).
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The only two semidualizing $R$-modules (up to isomorphism) are $R$ and $D$.

Proof. In each case, prove that $\text{length } R_P(C_P) = \text{length } R_P(R_P)$ for every $P \in \text{Ass}(R)$ and every semidualizing $R$-module $C$. This is straightforward in cases (1) and (2). It requires more work in cases (3) and (4).
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Then $e(I;C_P) = e(I;R_P)$ for each $m$-primary ideal $I$.

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