Symbolic Powers of Prime Ideals

Sean Sather-Wagstaff

13 October 2001

Hilbert's Fourteenth Problem and a Question of Cowsik

Hilbert (1902): Let $X_1 = f_1(x_1, \ldots, x_n), \ldots, X_m = f_m(x_1, \ldots, x_n)$ be m polynomials in n variables. Is the ring consisting of all rational functions in $X_1, \ldots, X_m$ which are polynomials in $x_1, \ldots, x_n$ a finitely generated algebra?

On other words, if k is a field of characteristic 0, $A = k[x_1, \ldots, x_n]$ and K is a subfield of the field of fractions $Q(A)$, is the ring $A \cap K$ finitely generated over k?

Zariski (1954): Same question if A is a normal domain, finitely generated over k.

Rees (1958): If A is the coordinate ring of the cone over an elliptic curve, P is a prime corresponding to a point of infinite order, then the symbolic Rees algebra $\oplus_n P^{(n)}$ is a counterexample to Zariski's question. (Nonregular ring)

Nagata (1960): Counterexample to Hilbert's question. (Similar construction to Rees' with a non-prime ideal P.)

Cowsik (1981): If P is a prime ideal in a regular local ring A, is the symbolic Rees algebra finitely generated over A?

Roberts (1985, 1990): Examples of nonfinitely generated symbolic Rees algebras over noncomplete and complete regular rings.

A Question of Eisenbud-Mazur

Eisenbud-Mazur (1997): The relation between evolutions and symbolic powers of prime ideals is formulated and studied.

Definition. Let λ be a ring and and T a local Λ-algebra essentially of finite type. An *evolution* of T over Λ is a local Λ-algebra R essentially of finite type and a surjection $R \to T$ of Λ-algebras inducing an isomorphism $\Omega_{R/\Lambda} \otimes_R T \to \Omega_{T/\lambda}$. The evolution is *trivial* if $R \to T$ is an isomorphism.

Theorem. (Eisenbud-Mazur) Let Λ be a Noetherian ring, $(S, \mathfrak{n})$ a localization of a polynomial ring in finitely many variables over Λ, and I an ideal of S. If $T = S/I$ is reduced and generically separable over Λ, then every evolution of T is trivial iff $I^{(2)} \subseteq \mathfrak{n}I$.

Eisenbud-Mazur construct a prime ideal P in $k[x_1, x_2, x_3, x_4]_{(x_1,x_2,x_3,x_4)}$ such that $P^{(2)} \not\subseteq \mathfrak{m}P$, when char$(k) > 0$.
Kurano-Roberts construct an analagous example in $V[\![S, T, U, V, X, Y]\!]$ where V is a DVR of mixed characteristic 2. In equal characteristic 0, however, no such example is known.

Question. If $(S, \mathfrak{n})$ is a regular local ring containing a field of characteristic 0, and P is a prime ideal, is it true that $P^{(2)} \subseteq \mathfrak{n}P$?

This question is still open for
$S = \mathbb{C}[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ and
$S = \mathbb{C}[\![X_1, \ldots, X_n]\!]$.

Serre's Positivity Conjecture and a Question of Kurano-Roberts

Serre (1965): Theorem and definition. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d with prime ideals $\mathfrak{p}, \mathfrak{q}$ such that $\sqrt{\mathfrak{p} + q} = \mathfrak{m}$. Then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R) = d$. Define the intersection multiplicity of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ as
$\chi(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{d} \text{len}(\text{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q}))$. If R is unramified, then

(Nonnegativity) $\chi(R/\mathfrak{p}, R/\mathfrak{q}) \geq 0$.

(Vanishing) If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) < d$, then $\chi(R/\mathfrak{p}, R/\mathfrak{q}) = 0$.

(Positivity) If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = d$, then $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

Conjecture. The above properties hold when R is ramified.

Roberts, Gillet-Soulé (1985): Theorem. The Vanishing Conjecture is verified.

Gabber ($\approx$ 1995): Theorem. The Nonnegativity Conjecture is verified.

The Positivity Conjecture is still open.

Kurano-Roberts (2000): Theorem. Let $(R, \mathfrak{m})$ be a regular local ring that either contains a field or is ramified, and let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. If $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$, then $\mathfrak{p}^{(m)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{m+1}$ for all $m \geq 1$.

Conjecture. If $(R, \mathfrak{m})$ is a regular local ring with prime ideals $\mathfrak{p}, \mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$, then $\mathfrak{p}^{(m)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{m+1}$ for all $m \geq 1$.

The conjecture follows from the theorem when R contains a field. It is still open in mixed characteristic.

___ (2001): Theorem. Let $(R, \mathfrak{m})$ be a regular local ring containing a field. Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$, then $\mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)} \subseteq \mathfrak{m}^{m+n}$ for all $m, n \geq 1$.

Conjecture. The conclusion of the previous theorem holds when R is any regular local ring.

Theorem. In order to verify either of the previous conjectures in general, it suffices to verify each when R is unramified.

A number of questions generalizing the above conjectures have been posed.

1. Let $(R, \mathfrak{m})$ be a regular local ring with prime ideals $\mathfrak{p}, \mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$.

(a) Does $\mathfrak{p}^{(m)} \cap \mathfrak{q} \subseteq \mathfrak{m}\mathfrak{p}^{(m)}$ for all $m \geq 1$?

(b) Does $\mathfrak{p}^{(m)} \cap \mathfrak{q} \subseteq \mathfrak{m}^m\mathfrak{q}$ for all $m \geq 1$?

2. Let $\text{char}(k) = 0$, and $A = k[\![X_1, \ldots, X_n]\!]$ or $A = k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$. Let I be an ideal of A and let J be the Jacobian ideal of $R = A/I$. Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}_R$.

(a) If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \geq \dim(R)$, does there exist fixed $N \geq 1$ such that $J^N(\mathfrak{p}^{(m)} \cap \mathfrak{q}) \subseteq \mathfrak{m}^{m+1}$ for all $m \geq 1$?

(b) If I is generated by an A-sequence of length c, and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R) + c$, does there exist fixed $N \geq 1$ such that $J^N(\mathfrak{p}^{(m)} \cap \mathfrak{q}) \subseteq \mathfrak{m}^{n+1}$ for all $m \geq 1$?

One can ask similar questions for $\mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)}$.

Example 1(a).

Example 1(b).

Example 2(a). Let $A = k[\![X, Y, Z]\!]$, $s \geq 2$ and $R = A/(X(Y+Z) - Y^s Z) = k[\![x, y, z]\!]$. Then $J = (y+z, x - sy^{s-1}z, x - y^s)$. Let $\mathfrak{p} = (x, y)R$ and $\mathfrak{q} = (x, z)R$. Then $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}_R$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 = \dim(R)$. If $N \geq 1$ and $m \geq N/(s-1)$, then $(x - y^s)^N x^m \in J^N(\mathfrak{p}^{(ms)} \cap \mathfrak{q}) \setminus \mathfrak{m}^{ms+1}$.

Example 2(b). Let $B = k[\![W, X, Y, Z]\!]$ and $S = B/(XY(Z+W) - W^s Z) = k[\![w, x, y, z]\!]$,so $J = (y(w+z), x(w+z), xy - sw^{s-1}z, xy - w^s)$. Let $\mathfrak{p} = (x, w)$ and $\mathfrak{q} = (y, z)$. Then $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}_S$ and $\dim(S/\mathfrak{p}) + \dim(S/\mathfrak{q}) = 4 = \dim(S) + 1$. If $N \geq 2$, $s \geq 3$ and $m > 2N/(s-1)$ then $(xy - w^s)^N x^m y \in J^N(\mathfrak{p}^{(ms)} \cap \mathfrak{q}) \setminus \mathfrak{m}^{ms+1}$.

Results of Ein-Lazarsfeld-Smith and Hochster-Huneke

E-L-S, H-H (2001): Theorem. Let $(R, \mathfrak{m})$ be a regular local ring containing a field, and P a prime ideal of height h. Then, for all $n > 0$ and $k \geq 0$, $P^{(hn+kn)} \subseteq (P^{(k+1)})^n$. In particular, $P^{(hn)} \subseteq P^n$ for $n > 0$.

Ein-Lazarsfeld-Smith prove the second containment for affine regular rings containing a field of characteristic zero, using the theory of multiplier ideals. Hochster-Huneke prove the more general statement (in fact, more general statements) using tight closure in positive characteristic, and reduction to positive characteristic in characteristic 0.

Question. What is the smallest h' such that $P^{(h'n)} \subseteq P^n$ for $n > 0$.

Question. Does the conclusion of the theorem hold in mixed characteristic?

A question of Cutkosky

Question. Let P be a homogeneous prime ideal of $k[X_1, \ldots, X_d]$. Does there exist $e \geq 1$ such that $\text{reg}(P^{(n)}) \leq en$ for all $n \geq 1$? (Here, $\text{reg}(I)$ is the Castelnuovo-Mumford regularity of the homogeneous ideal I.)