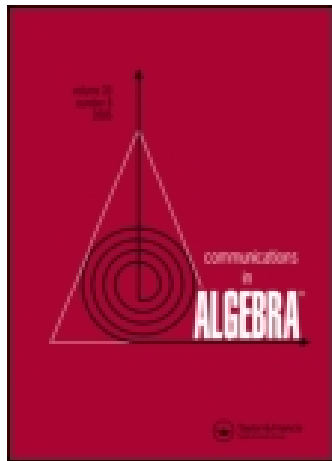


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KRULL DIMENSION OF MONOMIAL IDEALS IN POLYNOMIAL RINGS WITH REAL EXPONENTS

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We develop a new technique for studying monomial ideals in the standard polynomial rings $A[X_1, \dots, X_d]$ where A is a commutative ring with identity. The main idea is to consider induced ideals in the semigroup ring $R = A[\mathbb{M}_{\geq 0}^1 \times \dots \times \mathbb{M}_{\geq 0}^d]$ where $\mathbb{M}^1, \dots, \mathbb{M}^d$ are nonzero additive subgroups of \mathbb{R} . We prove that the set of nonzero finitely generated monomial ideals in R has the structure of a metric space, and we prove that a version of Krull dimension for this setting is lower semicontinuous with respect to this metric space structure. We also show how to use discrete techniques to study certain monomial ideals in this context.

Key Words: Edge ideals; Krull dimension; m -irreducible decompositions; Monomial ideals; Semicontinuous.

2010 Mathematics Subject Classification: Primary: 13C05, 13F20; Secondary: 05C22, 05E40, 05C69.

1. INTRODUCTION

Assumption 1.1. *Throughout this paper, A is a nonzero commutative ring with identity, and $\mathbb{M}^1, \dots, \mathbb{M}^d$ are nonzero additive subgroups of \mathbb{R} . For $i = 1, \dots, d$ set $\mathbb{M}_{\geq 0}^i = \{m \in \mathbb{M}^i \mid m \geq 0\}$.*

We are interested in properties of monomial ideals in the polynomial ring $S = A[X_1, \dots, X_d]$, that is, ideals generated by monomials. These ideals have deep applications to combinatorics; for instance, building from work of Hochster [3] and Reisner [7], Stanley uses monomial ideals to prove his upper bound theorem for simplicial spheres [9]. On the other hand, one can use the combinatorial aspects of these ideals to construct interesting examples and verify their properties. See, e.g., [2, 5, 8, 10, 11] for some aspects of this.

For small values of d , one can study a given monomial ideal $I \subseteq S$ visually. For instance, when $d = 2$, one considers the set of points $(a_1, a_2) \in \mathbb{Z}^2$ such that

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To our teacher and friend, Jim Coykendall.

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$X_1^{a_1} X_2^{a_2} \in I$. This “graph of I ” contains nontrivial information about I ; for example, one can read certain decompositions of I from the graph.

Given the fact that these graphs are (in general) subsets of \mathbb{R}^d , one should be able to study these ideals using geometric techniques, as follows. To prove a result about a given monomial ideal I , prove it for monomial ideals J that are “close” to I in some suitable sense, and then prove that the closeness of I to J forces the conclusion to transfer from J to I . The problem with this idea is the following one: the ideal I is defined by discrete data (e.g., the lattice points (a_1, a_2) described above). Thus, a reasonable notion of “closeness” for classical monomial ideals based on the euclidean metric in \mathbb{R}^n will likely be trivial. As a possible remedy for this defect, we switch perspectives from the discrete setting to a continuous one.

We consider the semigroup ring $R = A[\mathbb{M}_{\geq 0}^1 \times \cdots \times \mathbb{M}_{> 0}^d]$, which is the set of all (finite) A -linear combinations of *monomials* $X^{\underline{m}} = X_1^{m_1} X_2^{m_2} \cdots X_d^{m_d}$ where $\underline{m} = (m_1, m_2, \dots, m_d) \in \mathbb{M}_{\geq 0}^1 \times \cdots \times \mathbb{M}_{> 0}^d$. A *monomial ideal* of R is an ideal of R that is generated by a set of monomials. (This includes the ideal $0 = (\emptyset)R$.) For instance, in the case $\mathbb{M}^i = \mathbb{R}$, the monomials of R correspond to the points of the non-negative orthant $\mathbb{R}_{\geq 0}^d$, and any monomial ideal $I \subseteq S$ induces a monomial ideal $IR \subseteq R$ since $S \subseteq R$.¹ On the other hand, the case $\mathbb{M}^i = \mathbb{Z}$ recovers the monomial ideals of S .

The main results of this paper are in Section 4, where we study a version of the Krull dimension for this setting. We prove that the set of nonzero finitely generated monomial ideals in R has the structure of a metric space in Theorem 4.13. (For example, in the case $\mathbb{M}^i = \mathbb{R}$, this applies to the ideals of the form IR where I a nonzero monomial ideal of S .) Then in Corollary 4.15, we prove that our Krull dimension is lower semicontinuous with respect to this metric space structure. This suggests that one may be able to apply the geometric techniques described above in this setting, even to monomial ideals in S .

In Section 3, we run this idea in reverse, in some sense, by showing how to use discrete techniques from S to study monomial ideals in R for the case $\mathbb{M}^i = \mathbb{R}$. Specifically, we apply techniques from [6] to a special class of monomial ideals of R that behave like edge ideals of weighted graphs. The main result of this section is Theorem 3.18, which provides nontrivial decompositions of these ideals determined by objects that we call “interval vertex covers.”

In a sense, Section 2 consists of background material and examples. On the other hand, many of the results in this section are technically new, being versions of results from [4] for our more general context.

2. MONOMIAL IDEALS AND THEIR DECOMPOSITIONS

This section contains foundational material for use in the rest of the paper. Most of the ideas are standard for the case $\mathbb{M}^i = \mathbb{Z}$, and the case $\mathbb{M}^i = \mathbb{R}$ is developed in [4].

Assumption 2.1. *Throughout this section, set $R = A[\mathbb{M}_{\geq 0}^1 \times \cdots \times \mathbb{M}_{\geq 0}^d]$.*

¹It is worth noting that the ring $A[\mathbb{M}_{\geq 0}^1 \times \cdots \times \mathbb{M}_{\geq 0}^d]$ has been studied previously, for instance, to construct interesting counterexamples to questions about non-noetherian rings; see, e.g., [1]. We are grateful to Jim Coykendall for teaching us about these constructions.

Monomial Basics

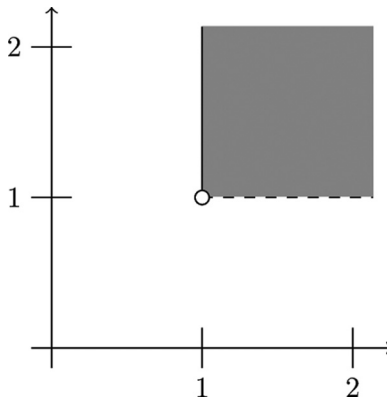
Definition 2.2. For $i = 1, \dots, d$, set $\mathbb{M}_{>0}^i = \{m \in \mathbb{M}^i \mid m > 0\}$ and $\mathbb{M}_{\infty \geq 0}^i = \mathbb{M}_{\geq 0}^i \cup \{\infty\}$. A *pure power* in R is a monomial of the form X_i^r . For any subset $G \subseteq R$, we let $[G]$ denote the set of all monomials in G , so $[G] = G \cap [R]$. For $i = 1, \dots, d$, we define² $X_i^\infty = 0$.

The following is a straightforward consequence of our definitions.

Fact 2.3. Fix a set $\{I_\lambda\}_{\lambda \in \Lambda}$ of monomial ideals of R .

- (a) The monomial ideal I_λ is generated by $[I_\lambda]$, so we have $I_\lambda \subseteq I_\mu$ if and only if $[I_\lambda] \subseteq [I_\mu]$, and hence $I_\lambda = I_\mu$ if and only if $[I_\lambda] = [I_\mu]$.
- (b) Given monomials $f = \underline{X}^t$ and $g = \underline{X}^s$ in R , we have $g \in (f)R$ if and only if for all i there exists $t_i \in \mathbb{M}_{\geq 0}^i$ such that $r_i + t_i = s_i$. When these conditions are satisfied, we have $g = fh$ where $h = \underline{X}^t$.
- (c) Given a monomial $f \in [R]$ and a subset $S \subseteq [R]$, we have $f \in (S)R$ if and only if $f \in (s)R$ for some $s \in S$.
- (d) The sum $\sum_{\lambda \in \Lambda} I_\lambda$ and intersection $\bigcap_{\lambda \in \Lambda} I_\lambda$ are monomial ideals such that $[\sum_{\lambda \in \Lambda} I_\lambda] = \bigcup_{\lambda \in \Lambda} [I_\lambda]$ and $[\bigcap_{\lambda \in \Lambda} I_\lambda] = \bigcap_{\lambda \in \Lambda} [I_\lambda]$.

Example 2.4. As with monomial ideals in the polynomial ring $A[X_1, X_2]$, we can visualize a monomial ideal I in $R = A[\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}]$ via $[I]$. For instance, here is $[I]$ where $I = (X_1 X_2^a \mid a > 1)R$:



Definition 2.5. Let I be a monomial ideal of R , and suppose that $I = (\underline{X}^{z_1}, \dots, \underline{X}^{z_n})R$. We say that the list $\underline{X}^{z_1}, \dots, \underline{X}^{z_n}$ is an *irredundant generating sequence* for I if for all $i \neq j$ we have $\underline{X}^{z_i} \notin (\underline{X}^{z_j})R$.

Fact 2.6. As a consequence of Fact 2.3, one checks readily that every finitely generated monomial ideal in R has a unique irredundant monomial generating sequence. (Note, however, that R may have monomial ideals that are not finitely generated.)

²Despite this notation, note that 0 is not a monomial according to our definition.

The next result is proved as in [4, Lemma 2.7], using Fact 2.3.

Lemma 2.7. For $t = 1, \dots, l$, let $\{K_{t,i}\}_{i=1}^{m_t}$ be a collection of monomial ideals of R . Then the following equalities hold:

$$\bigcap_{t=1}^l \sum_{i=1}^{m_t} K_{t,i} = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_l=1}^{m_l} \bigcap_{t=1}^l K_{t,i_t},$$

$$\sum_{t=1}^l \bigcap_{i=1}^{m_t} K_{t,i} = \bigcap_{i_1=1}^{m_1} \bigcap_{i_2=1}^{m_2} \cdots \bigcap_{i_l=1}^{m_l} \sum_{t=1}^l K_{t,i_t}.$$

Generators of Intersections of Monomial Ideals

Fact 2.3(d) shows that an intersection of monomial ideals is again a monomial ideal. In this subsection, we explicitly describe a monomial generating sequence for any finite intersection of monomial ideals; see Proposition 2.9. This is key for many of our results, and it strongly relies on the assumption that each \mathbb{M}^i is closed under subtraction.

Definition 2.8. Let $\underline{X}^{L_1}, \dots, \underline{X}^{L_k} \in [R]$ with $\underline{r}_i = (r_{i,1}, \dots, r_{i,d})$. The least common multiple of the \underline{X}^{L_i} is $\text{lcm}_{1 \leq i \leq k}(\underline{X}^{L_i}) = \underline{X}^{\underline{p}}$ where \underline{p} is defined as $p_j = \max_{1 \leq i \leq k} \{r_{i,j}\}$.

The next result is proved like [4, Theorem 2.5].

Proposition 2.9. Given subsets $S_1, \dots, S_k \subseteq [R]$, we have

$$\bigcap_{i=1}^k (S_i)R = (\{\text{lcm}_{1 \leq i \leq k}(f_i) \mid f_i \in S_i \text{ for } i = 1, \dots, k\})R.$$

M-irreducible Monomial Ideals

Here we characterize the monomial ideals that cannot be decomposed as a nontrivial, finite intersection of monomial ideals; see Proposition 2.13.

Notation 2.10. Let $\varepsilon \in \{0, 1\}$. Given $r \in \mathbb{M}^i$ and $\alpha \in \mathbb{R}$, we define

$$r \geq_{\varepsilon} \alpha \text{ provided that } \begin{cases} r \geq \alpha & \text{if } \varepsilon = 0 \\ r > \alpha & \text{if } \varepsilon = 1. \end{cases}$$

Given $s \in \mathbb{M}_{\infty \geq 0}^i$, we define

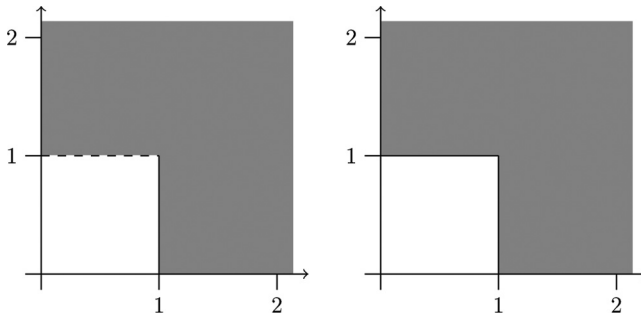
$$s \geq_{\varepsilon} \infty \text{ provided that } s = \infty.$$

Given $\underline{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_{\infty \geq 0}^d$ and $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_d) \in \{0, 1\}^d$, we set

$$J_{\underline{\alpha}, \underline{\varepsilon}} = (\{X_i^{r_i} \mid r_i \in \mathbb{M}_{\infty \geq 0}^i \text{ and } r_i \geq_{\varepsilon_i} \alpha_i \text{ for } i = 1, \dots, d\})R.$$

Note that $J_{\underline{\alpha}, \underline{\varepsilon}}$ is generated by pure powers in R .

Example 2.11. In $R = A[\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}]$, we illustrate the ideals $J_{(1,1),(0,1)}$ and $J_{(1,1),(0,0)} = (X_1, X_2)R$:



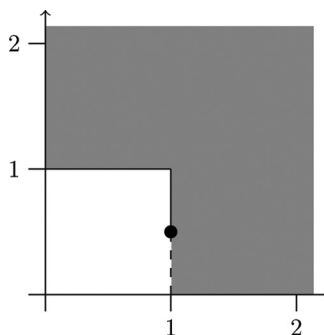
Definition 2.12. A monomial ideal $I \subseteq R$ is *m-irreducible* (short for *mono-mial-irreducible*) provided that for all monomial ideals J and K of R such that $I = J \cap K$, either $I = J$ or $I = K$.

The following characterization of m-irreducible monomial ideals is proved as in [4, Theorem 3.9] using Fact 2.3 and Proposition 2.9.

Proposition 2.13. For a monomial ideal $I \subseteq R$, the following conditions are equivalent:

- (i) I is generated by pure powers of a subset of the variables X_1, \dots, X_d ;
- (ii) there exist $\underline{\alpha} \in \mathbb{R}_{\infty \geq 0}^d$ and $\underline{\varepsilon} \in \{0, 1\}^d$ such that $I = J_{\underline{\alpha}, \underline{\varepsilon}}$;
- (iii) I is m-irreducible.

Example 2.14. Proposition 2.13 implies that the ideals in Example 2.11 are m-irreducible. It is worth noting that, even though the following graph has roughly the same shape as those in Example 2.11,



the ideal $I = (X_1^a, X_2, X_1 X_2^{1/2} \mid a > 1)R$ it represents in $R = A[\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}]$ is not m-irreducible. This follows from Proposition 2.13 because I cannot be generated by pure powers of the variables. One also deduces this by definition using the

decomposition

$$I = (X_1, X_2)R \cap (X_1^a, X_2^{1/2} \mid a > 1)R = J_{(1,1),(0,0)} \cap J_{(1,1/2),(1,0)}.$$

This decomposition is nontrivial, since Fact 2.3 implies that we have $X_1 \in J_{(1,1),(0,0)} \setminus J_{(1,1/2),(1,0)}$ and $X_2^{1/2} \in J_{(1,1/2),(1,0)} \setminus J_{(1,1),(0,0)}$.

M-prime Monomial Ideals

Here we characterize the ideals of R that are prime with respect to monomials, for use in Section 4.

Definition 2.15. A monomial ideal $P \subsetneq R$ is *m-prime* (short for *monomial-prime*) provided that for all $f, g \in [R]$, if $f \cdot g \in P$, then $f \in P$ or $g \in P$. Given a subset $T \subseteq \{1, \dots, d\}$, set

$$Q_T = (X_i^n \mid i \in T \text{ and } n \in \mathbb{M}_{>0}^i)R.$$

Proposition 2.16. For a monomial ideal $I \subseteq R$, the following conditions are equivalent:

- (i) I is *m-prime*;
- (ii) There exists $T \subseteq \{1, \dots, d\}$ such that $I = Q_T$;
- (iii) $I = J_{\underline{\alpha}, \underline{1}}$ where $\alpha_i \in \{0, \infty\}$ for all i and $\underline{1} = (1, \dots, 1)$.

Proof. (i) \Rightarrow (ii): Assume that I is *m-prime*, and set

$$T = \{i \mid X_i^n \in I \text{ for some } n \in \mathbb{M}_{<0}^i\}.$$

First, observe that, if $i \in T$, then $X_i^a \in I$ for all $a \in \mathbb{M}_{>0}^i$. Indeed, by definition of T , there is an element $n \in \mathbb{M}_{>0}^i$ such that $X_i^n \in I$. Fix a positive integer k such that $ak > n$. Since \mathbb{M}^i is closed under subtraction, we have $ak - n \in \mathbb{M}_{>0}^i$. It follows that $X_i^{ak} = X_i^{ak-n} X_i^n \in I$, so the fact that I is *m-prime* implies that $X_i^a \in I$, as claimed.

Now we show that $I = Q_T$. For the containment $I \supseteq Q_T$, it suffices to show that each generator X_i^a of Q_T is in I ; here we have $i \in T$ and $a \in \mathbb{M}_{>0}^i$. This follows from the above observation. For the reverse containment $I \subseteq Q_T$, let $X_1^{\alpha_1} \dots X_d^{\alpha_d} \in [I]$. Since I is *m-prime*, there is an index i such that $\alpha_i > 0$ and $X_i^{\alpha_i} \in I$. It follows that $i \in T$, so $X_1^{\alpha_1} \dots X_d^{\alpha_d} \in (X_i^{\alpha_i})R \subseteq Q_T$.

(ii) \Rightarrow (iii): Let $T \subseteq \{1, \dots, d\}$, and let $I = Q_T$. For $i = 1, \dots, d$, set

$$\alpha_i = \begin{cases} 0 & \text{if } i \in T \\ \infty & \text{if } i \notin T. \end{cases}$$

It is straightforward to show that $I = Q_T = J_{\underline{\alpha}, \underline{1}}$.

(iii) \Rightarrow (i): Let $I = J_{\underline{\alpha}, \underline{1}}$ such that $\alpha_i \in \{0, \infty\}$ for $i = 1, \dots, d$. To show that I is *m-prime*, let $\gamma, \beta \in \mathbb{M}_{\geq 0}^1 \times \dots \times \mathbb{M}_{\geq 0}^d$ such that $\underline{X}^\gamma \underline{X}^\beta \in I$. Since $\underline{X}^\gamma \underline{X}^\beta$ must be a multiple of one of the generators of $J_{\underline{\alpha}, \underline{1}}$, there exists an i such that $\alpha_i = 0$ and $\gamma_i + \beta_i > 0$. Hence, either $\gamma_i > 0$ or $\beta_i > 0$. Suppose without loss of generality that $\gamma_i > 0$. Then $X_i^{\gamma_i} \in I$, which implies that $\underline{X}^\gamma \in I$. Hence, I is *m-prime*. \square

M-irreducible Decompositions

Here we characterize the monomial ideals that can be expressed as finite intersections of monomial irreducible ideals; see Proposition 2.20.

Definition 2.17. Let $I \subseteq R$ be a monomial ideal. An *m-irreducible decomposition* of I is a decomposition $I = \bigcap_{\lambda \in \Lambda} I_\lambda$ where each I_λ is an m-irreducible monomial ideal of R . If the index set Λ is finite, we say that $I = \bigcap_{\lambda \in \Lambda} I_\lambda$ is a *finite m-irreducible decomposition*. An m-irreducible decomposition is *irredundant* if for all distinct $\lambda, \mu \in \Lambda$ we have $I_\lambda \not\subseteq I_\mu$.

Notation 2.18. Given $\underline{\alpha} \in \mathbb{R}_{\geq 0}^d$ and $\underline{\varepsilon} \in \{0, 1\}^d$, we set

$$I_{\underline{\alpha}, \underline{\varepsilon}} = (\{\underline{X}^r \mid i = 1, \dots, d, r_i \geq_{\varepsilon_i} \alpha_i \text{ and } r_i \in \mathbb{M}_{\infty \geq 0}^i\})R.$$

Note that $I_{\underline{\alpha}, \underline{\varepsilon}}$ is not in general generated by pure powers of the variables, so it is different from $J_{\underline{\alpha}, \underline{\varepsilon}}$.

Example 2.19. With the zero-vector $\underline{0} = (0, \dots, 0)$, we have $I_{\underline{\alpha}, \underline{0}} = (\underline{X}^{\underline{\alpha}})R$. From this, it follows that if I is a finitely generated monomial ideal in R , then I is a finite sum of ideals of the form $I_{\underline{\alpha}, \underline{0}}$. Indeed, let $\underline{X}^{z_1}, \dots, \underline{X}^{z_n}$ be a monomial generating sequence for I . Then we have

$$I = (\underline{X}^{z_1}, \dots, \underline{X}^{z_n})R = \sum_{i=1}^n (\underline{X}^{z_i})R = \sum_{i=1}^n I_{z_i, \underline{0}}$$

On the other hand, if $\alpha_i = \infty$ for any i , then $I_{\underline{\alpha}, \underline{\varepsilon}} = 0$.

In $R = A[\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}]$, the ideal $I_{(1,1), (0,1)}$ is graphed in Example 2.4.

We think of the ideal $I_{\underline{\alpha}, \underline{\varepsilon}}$ as “almost principal” since it is very close to the principal ideal $(\underline{X}^{\underline{\alpha}})R$. Hence, a finite sum of ideals of the form $I_{\underline{\alpha}, \underline{\varepsilon}}$ is “almost finitely generated.”

Proposition 2.20. A monomial ideal $I \subseteq R$ has a finite m-irreducible decomposition if and only if it can be expressed as a finite sum of ideals of the form $I_{\underline{\alpha}, \underline{\varepsilon}}$. If I has a finite m-irreducible decomposition, then I has a unique irredundant finite m-irreducible decomposition.

Proof. The first statement is proved as in [4, Theorem 4.12] using Proposition 2.13. For existence in the second statement, let $I = \bigcap_{i=1}^m P_i$ be a finite m-irreducible decomposition. If this decomposition is irredundant, then there is nothing to show. Assume that the decomposition is redundant, so we have $P_k \subseteq P_j$ for some $k \neq j$. It follows that $I = \bigcap_{i \neq j} P_i$ is another finite m-irreducible decomposition. Continue removing redundancies in this way. The process terminates in finitely many steps since the original decomposition is finite.

To show uniqueness, let $I = \bigcap_{i=1}^m P_i = \bigcap_{j=1}^n Q_j$ be two irredundant finite m -irreducible decompositions. For $s = 1, \dots, m$, it follows that $P_s \supseteq I = \bigcap_{j=1}^n Q_j$, so Lemma 2.7 implies that

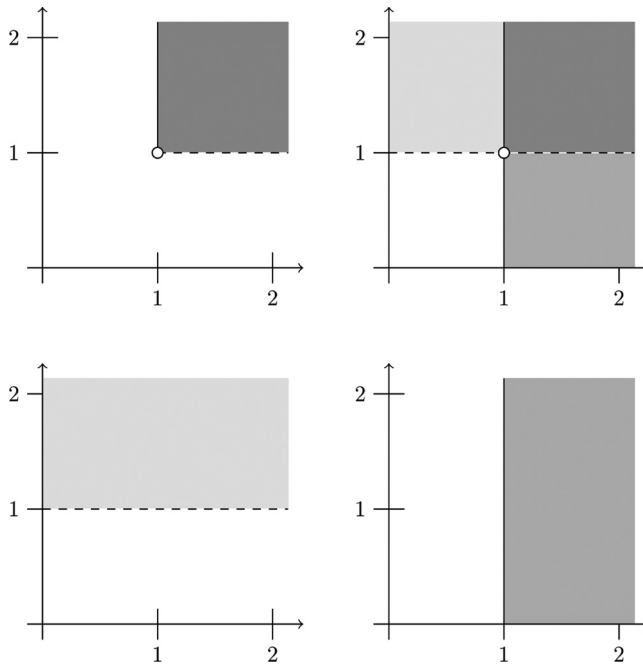
$$P_s = P_s + \bigcap_{j=1}^n Q_j = \bigcap_{j=1}^n (P_s + Q_j).$$

Since P_s is m -irreducible, it follows that $P_s = P_s + Q_t \supseteq Q_t$ for some t . Similarly, there is an index u such that $Q_t \supseteq P_u$. Hence, the irredundancy of the intersection $\bigcap_{i=1}^m P_i$ implies that $P_s = P_u$, and thus $P_s = Q_t$. We conclude that $\{P_1, \dots, P_m\} \subseteq \{Q_1, \dots, Q_n\}$. Symmetrically, we have $\{P_1, \dots, P_m\} \supseteq \{Q_1, \dots, Q_n\}$, so the decompositions are equal. \square

Corollary 2.21. *Every finitely generated monomial ideal $I \subseteq R$ has a finite m -irreducible decomposition.*

Proof. This follows from Example 2.19 and Proposition 2.20. \square

Example 2.22. Here we illustrate an m -irreducible decomposition of the ideal $I = I_{(1,1),(0,1)}$ in $R = A[\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}]$ from Example 2.4. As with such decompositions in the standard polynomial ring $A[X_1, X_2]$, the key is to use the graph of the monomial set $[I]$ to find the decomposition. The first diagram in the following display is the graph of $[I]$. The second one shows how we use the boundary lines from $[I]$ to write $[I]$ as the intersection $[J] \cap [K]$ where J and K are generated by pure powers of X_2 and X_1 , respectively. The third and fourth diagrams show $J = J_{(\infty,1),(0,1)}$ and $K = J_{(1,\infty),(0,1)}$ separately.



One checks readily that $I = J \cap K$, say, using Proposition 2.9.

Note that Example 2.14 provides another m -irreducible decomposition. Moreover, it shows that one needs to be careful when using these diagrams to generate decompositions, as the rough shape of the diagram (ignoring the distinction between dashed and solid lines, etc.) does not contain enough information.

We conclude this section with a discussion of (possibly infinite) irredundant m -irreducible decompositions, beginning with existence.

Proposition 2.23. *Let I be a monomial ideal in R , and let \mathfrak{C}_I denote the set of m -irreducible monomial ideals of R that contain I . Let \mathfrak{C}'_I denote the set of minimal elements of \mathfrak{C}_I with respect to containment.*

- (a) *For every $J \in \mathfrak{C}_I$, there is an ideal $J' \in \mathfrak{C}'_I$ such that $J' \subseteq J$.*
- (b) *With $\underline{1} = (1, \dots, 1)$, we have the following m -irreducible decompositions:*

$$I = \bigcap_{\underline{x} \neq \underline{1}} J_{\underline{x}, \underline{1}} = \bigcap_{J \in \mathfrak{C}_I} J = \bigcap_{J \in \mathfrak{C}'_I} J.$$

The third decomposition is irredundant.

Proof. (a) Let $J \in \mathfrak{C}_I$, and let $\mathfrak{C}_{J,I}$ denote the set of ideals $K \in \mathfrak{C}_I$ contained in J . By Zorn's Lemma, it suffices to show that every chain \mathfrak{T} in $\mathfrak{C}_{J,I}$ has a lower bound in $\mathfrak{C}_{J,I}$. To this end, it suffices to show that the ideal $L := \bigcap_{K \in \mathfrak{T}} K$ is m -irreducible. If the chain \mathfrak{T} has a minimal element, then L is the minimal element, and hence it is m -irreducible. Thus, we assume that \mathfrak{T} does not have a minimal element. Fact 2.3(d) shows that L is a monomial ideal of R , and the containments $L \subseteq J \subsetneq R$ implies that $L \neq R$. Thus, to show that L is m -irreducible, let M and N be monomial ideals of R such that $L = M \cap N$; we need to show that $L = M$ or $L = N$. Since we have $L = M \cap N \subseteq M$, and similarly $L \subseteq N$, it suffices to show that $L \supseteq M$ or $L \supseteq N$.

For each $K \in \mathfrak{T}$, we have $K \supset L = M \cap N$, so Lemma 2.7 implies that

$$K = K + L = K + (M \cap N) = (K + M) \cap (K + N).$$

The fact that K is m -irreducible implies that either $K = K + M \supseteq M$ or $K = K + N \supseteq N$.

Case 1. For every $K \in \mathfrak{T}$, there is a $K' \in \mathfrak{T}$ such that $K \supseteq K' \supseteq M$. In this case, it follows that $L = \bigcap_{K \in \mathfrak{T}} K \supseteq M$, as desired.

Case 2. There is an ideal $K \in \mathfrak{T}$ such that for every $K' \in \mathfrak{T}$ with $K \supseteq K'$, one has $K' \not\supseteq M$. (Note that the fact that \mathfrak{T} does not have a minimal element implies that at least one such K' exists.) From the paragraph before Case 1, we conclude that for every $K' \in \mathfrak{T}$ with $K \supseteq K'$, one has $K' \supseteq N$. It now follows that $L \supseteq N$, as desired.

(b) If $I = R$, then the desired conclusions are trivial since the empty intersection of ideals of R is itself R . Thus, we assume that $I \neq R$. The equality

$I = \bigcap_{\underline{x}^r \notin I} J_{r,1}$ is proved like [4, Proposition 4.14]. For each monomial $\underline{x}^r \notin I$, it follows that $J_{r,1} \in \mathcal{C}_I$, so we have the first containment in the following display:

$$I = \bigcap_{\underline{x}^r \notin I} J_{r,1} \supseteq \bigcap_{J \in \mathcal{C}_I} J \supseteq \bigcap_{J \in \mathcal{C}'_I} J \supseteq I.$$

The second containment follows from part (a), and the third containment follows from the definition of \mathcal{C}'_I . This establishes the desired decompositions. Finally, the decomposition $\bigcap_{J \in \mathcal{C}'_I} J$ is irredundant because there are no proper containments between minimal elements of \mathcal{C}'_I , by definition. \square

The following example shows that infinite irredundant m-irreducible decompositions need not be unique.

Example 2.24. Set $d = 2$ and $I = (\{X^r Y^{1-r} \mid 0 \leq r \leq 1\})R$ with $\mathbb{M}^i = \mathbb{R}$ for $i = 1, 2$. In [4, Example 4.13], it is shown that I does not admit a finite m-irreducible decomposition. However, it is straightforward to show that the following m-irreducible decompositions are irredundant and distinct:

$$I = \bigcap_{0 \leq r \leq 1} J_{(r,1-r),(1,1)} = \bigcap_{0 \leq r \leq 1} J_{(r,1-r),(1,0)}.$$

Moreover, one can use this idea to construct infinitely many distinct irredundant m-irreducible decompositions of I . Indeed, for every subset S of the closed interval $[0, 1]$, we have

$$I = \left(\bigcap_{r \in S} J_{(r,1-r),(1,1)} \right) \cap \left(\bigcap_{r \in [0,1] \setminus S} J_{(r,1-r),(1,0)} \right).$$

3. AN EXTENDED EXAMPLE

Here we show how to use discrete techniques from [6] to compute some decompositions in our setting. This section’s main result is Theorem 3.18.

Assumption 3.1. *Throughout this section, I is an ideal in the ring $R = A[\mathbb{R}_{\geq 0} \times \dots \times \mathbb{R}_{\geq 0}]$ generated by a non-empty set of monomials of the form $X_i^a X_j^a$ with $i \neq j$ and $a \in \mathbb{R}_{>0}$. Also, we consider the standard polynomial ring $S = A[X_1, \dots, X_d]$. Let Ω denote the following set of intervals:*

$$\Omega = \{(a, \infty) \mid a \in \mathbb{R}_{\geq 0}\} \cup \{[b, \infty) \mid b \in \mathbb{R}_{>0}\}.$$

Example 3.2. In the case $d = 3$, we may consider the ideal

$$I = (X_1^a X_2^a, X_2^b X_3^b \mid a \geq 1, b > 2)R = (X_1 X_2, X_2^b X_3^b \mid b > 2)R.$$

By Proposition 2.9, it is routine to verify the irredundant decomposition

$$\begin{aligned} I &= (X_1, X_2^b \mid b > 2)R \cap (X_1, X_3^b \mid b > 2)R \cap (X_2)R \\ &= J_{(1,\infty,2),(0,0,1)} \cap J_{(\infty,1,\infty),(0,0,0)} \cap J_{(1,2,\infty),(0,1,0)}. \end{aligned}$$

Notation 3.3. Define Γ to be the finite simple graph with vertex set $V = \{1, \dots, d\}$ and edge set

$$E = \{ij \mid i \neq j \text{ and } X_i^a X_j^a \in I \text{ for some } a > 0\},$$

where $ij = \{i, j\}$. For each $ij \in E$, set

$$S(ij) = \{a > 0 \mid X_i^a X_j^a \in I\}.$$

$$s(ij) = \inf S(ij)$$

$$\varepsilon(ij) = \begin{cases} 0 & \text{if } s(ij) \in S(ij) \\ 1 & \text{if } s(ij) \notin S(ij). \end{cases}$$

This defines functions $s : E \rightarrow \mathbb{R}_{\geq 0}$ and $\varepsilon : E \rightarrow \{0, 1\}$.

Example 3.4. Continue with the ideal I from Example 3.2. The graph Γ in this case is

$$1 \text{ --- } 2 \text{ --- } 3.$$

The values $S(ij)$ are

$$\begin{aligned} S(12) &= [1, \infty) & S(23) &= (2, \infty) \\ s(12) &= 1 & s(23) &= 2 \\ \varepsilon(12) &= 0 & \varepsilon(23) &= 1. \end{aligned}$$

Fact 3.5. For each $ij \in E$, the set $S(ij)$ is an interval. Indeed, if $a \in S(ij)$, then $X_i^a X_j^a \in I$, so for all $r > 0$, we have $X_i^{r+a} X_j^{r+a} = X_i^r X_j^r X_i^a X_j^a \in I$, implying that $r + a \in S(ij)$. Moreover, it is straightforward to show that

$$S(ij) = \begin{cases} [s(ij), \infty) & \text{if } \varepsilon(ij) = 0 \\ (s(ij), \infty) & \text{if } \varepsilon(ij) = 1. \end{cases}$$

In particular, we have a function $S : E \rightarrow \Omega$.

The ideal I is a finite sum $\sum_{ij \in E} I_{\underline{\alpha}(ij), \underline{\varepsilon}(ij)}$. This essentially follows from the previous paragraph, with the following definitions of $\underline{\alpha}(ij)$ and $\underline{\varepsilon}(ij)$:

$$\alpha(ij)_k = \begin{cases} \infty & \text{if } k \notin \{i, j\} \\ s(ij) & \text{if } k \in \{i, j\} \end{cases} \quad \varepsilon(ij)_k = \begin{cases} 1 & \text{if } k \notin \{i, j\} \\ \varepsilon(ij) & \text{if } k \in \{i, j\}. \end{cases}$$

Proposition 2.20 implies that I has a finite m -irreducible decomposition.

Example 3.6. With the ideal I from Example 3.2, we have

$$I = I_{(1,1,\infty),(0,0,1)} + I_{(\infty,2,2),(1,1,1)}.$$

Definition 3.7. A *vertex cover* of the graph Γ is a subset $W \subseteq V$ such that for all $ij \in E$ we have either $i \in W$ or $j \in W$. A vertex cover is *minimal* if it is minimal in the set of all vertex covers with respect to inclusion.

Example 3.8. Continue with the ideal I from Example 3.2. The graph Γ in this case has two minimal vertex covers, namely $\{1, 3\}$ and $\{2\}$. It has the following nonminimal vertex covers: $\{1, 2\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. We will see below that the irredundant m -irreducible decomposition of I is given by the vertex covers $\{1, 2\}$, $\{1, 3\}$, and $\{2\}$ with some additional data.

The work from [6] takes its cue from the following decomposition result that we know from [11, Proposition 6.1.16].

Fact 3.9. *The edge ideal of the finite simple graph Γ is the ideal $I(\Gamma) = (X_i X_j \mid ij \in E)S$. Then we have the following m -irreducible decompositions*

$$I(\Gamma) = \bigcap_W Q_W = \bigcap_{W_{\min}} Q_W,$$

where the first intersection is taken over all vertex covers of Γ , and the second intersection is taken over all minimal vertex covers of Γ . The second intersection is irredundant.

Example 3.10. Continue with the graph Γ from Example 3.4. Using the minimal vertex covers from Example 3.8, we have

$$I(\Gamma) = Q_{\{1,3\}} \cap Q_{\{2\}} = (X_1, X_3)S \cap (X_2)S.$$

One can verify these equalities using Proposition 2.9, and the irredundancy is straightforward.

To prepare for the decomposition result for the ideal I , we review the decomposition result from [6] for weighted edge ideals.

Definition 3.11. A *weight function* for the graph Γ is a function $\omega : E \rightarrow \mathbb{Z}_{>0}$. For each $ij \in E$, the value $\omega(ij)$ is the *weight* of the edge ij . Write Γ_ω for the ordered pair (Γ, ω) . Fix a weight function ω of Γ .

A *weighted vertex cover* of Γ_ω is a pair (W, δ) such that the following statement hold:

- (1) W is a vertex cover of Γ ;
- (2) $\delta : W \rightarrow \mathbb{Z}_{>0}$ is a function such that for each edge $ij \in E$, either
 - (a) $i \in W$ and $\delta(i) \leq \omega(ij)$, or
 - (b) $j \in W$ and $\delta(j) \leq \omega(ij)$.

The value $\delta(i)$ is the *weight* of the vertex i .

Given two weighted vertex covers (W, δ) and (W', δ') of Γ_ω , we write $(W, \delta) \leq (W', \delta')$ provided that the following statements hold:

- (1) $W \subseteq W'$;
- (2) For all $i \in W$, we have $\delta(i) \geq \delta'(i)$.

A weighted vertex cover is *minimal* if it is minimal in the set of all weighted vertex covers with respect to this ordering.

Given a weighted vertex cover (W, δ) of Γ_ω , set

$$P_{W,\delta} = (X_i^{\delta(i)} \mid i \in W)S.$$

The *weighted edge ideal* of Γ_ω is the ideal

$$I(\Gamma_\omega) = (X_i^{\omega(ij)} X_j^{\omega(ij)} \mid ij \in E)S.$$

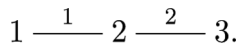
Fact 3.12. *Given a weight function ω of Γ , we have the following m -irreducible decompositions*

$$I(\Gamma_\omega) = \bigcap_{(W,\delta)} P_{W,\delta} = \bigcap_{(W,\delta) \text{ min}} P_{W,\delta},$$

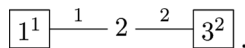
where the first intersection is taken over all weighted vertex covers of Γ_ω , and the second intersection is taken over all minimal weighted vertex covers of Γ_ω . The second intersection is irredundant. See [6, Theorem 3.5].

This technique allows us to find an irredundant m -irreducible decomposition for any ideal J in S generated by monomials of the form $X_i^a X_j^a$ with $i \neq j$ and $a \in \mathbb{Z}_{>0}$. Indeed, such an ideal is of the form $I(\Gamma_\omega)$ where ij is an edge of Γ if and only if the monomial $X_i^a X_j^a$ is in J for some $a \in \mathbb{Z}_{>0}$, and $\omega(ij)$ is the least a such that $X_i^a X_j^a \in J$.

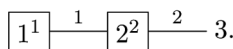
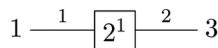
Example 3.13. Continue with the graph Γ from Example 3.4. Consider the weight function ω with $\omega(12) = 1$ and $\omega(23) = 2$. We represent this graphically by labeling each edge ij with the value $\omega(ij)$:



We also represent weighted vertex covers graphically with a box around each vertex in the vertex cover and using a superscript for the weight, as follows:



This weighted graph has three minimal weighted vertex covers, the one represented above, and the following two:



Note that the first two correspond to minimal vertex covers of the unweighted graph Γ , but the third one does not. The irredundant decomposition of $I(\Gamma_\omega)$ coming from Fact 3.12 is

$$I(\Gamma_\omega) = (X_1, X_2^2)S \cap (X_1, X_3^2)S \cap (X_2)S.$$

One can check this equality using Proposition 2.9, and the irredundancy is straightforward.

Now we develop a version of this construction for the ideal I from Assumption 3.1.

Definition 3.14. Let Γ_S denote the ordered pair (Γ, S) where S is from Notation 3.3. An *interval vertex cover* of Γ_S is a pair (W, σ) such that the following statements hold:

- (1) W is a vertex cover of Γ ,
- (2) $\sigma : W \rightarrow \Omega$ is a function such that for each edge $ij \in E$, either
 - (a) $i \in W$ and $S(ij) \subseteq \sigma(i)$, or
 - (b) $j \in W$ and $S(ij) \subseteq \sigma(j)$.

The value $\sigma(i)$ is the “interval weight” of the vertex i .

Given ordered pairs (W, σ) and (W', σ') where $W, W' \subseteq V$ are subsets and $\sigma : W \rightarrow \Omega$ and $\sigma' : W' \rightarrow \Omega$ are functions, write $(W, \sigma) \leq (W', \sigma')$ whenever the following statements hold:

- (1) $W \subseteq W'$;
- (2) For all $i \in W$, we have $\sigma(i) \subseteq \sigma'(i)$.

An interval vertex cover of Γ_S is *minimal* if it is minimal in the set of all interval vertex covers with respect to this ordering.

Given an ordered pair (W, σ) where $W \subseteq V$ and $\sigma : W \rightarrow \Omega$, set

$$Q_{W,\sigma} = (X_i^a \mid i \in W \text{ and } a \in \sigma(i))R.$$

Example 3.15. Continue with the ideal I from Example 3.2. We visualize the associated data from Example 3.4 similarly to the labeled graph from Example 3.13, keeping track of the entire interval $S(ij)$ for each edge ij :

$$1 \xrightarrow{\geq 1} 2 \xrightarrow{> 2} 3.$$

We also represent interval vertex covers graphically with a box around each vertex in the vertex cover and a superscript for the interval weight, as follows:

$$(W_1, \sigma_1) : \quad \boxed{1^{\geq 1}} \xrightarrow{\geq 1} 2 \xrightarrow{> 2} \boxed{3^{> 2}}$$

This weighted graph has three minimal interval vertex covers, the one represented above and the following two:

$$(W_2, \sigma_2) : 1 \xrightarrow{\geq 1} \boxed{2 \geq 1} \xrightarrow{> 2} 3$$

$$(W_3, \sigma_3) : \boxed{1 \geq 1} \xrightarrow{\geq 1} \boxed{2 > 2} \xrightarrow{> 2} 3.$$

Again, the first two correspond to minimal vertex covers of the unweighted graph Γ , but the third one does not.

Fact 3.16. *The ideals in R of the form $Q_{W,\sigma}$ are exactly the ideals of the form $J_{z,\epsilon}$, since they are exactly the ones generated by (intervals of) pure powers of the variables. Thus, the ideals of the form $Q_{W,\sigma}$ are exactly the m -irreducible ideals of R by Proposition 2.13.*

The next lemma is the key to the main result of this section.

Lemma 3.17. *Consider ordered pairs (W, σ) and (W', σ') where $W, W' \subseteq V$ are subsets and $\sigma : W \rightarrow \Omega$ and $\sigma' : W' \rightarrow \Omega$ are functions.*

- (a) *One has $Q_{W,\sigma} \subseteq Q_{W',\sigma'}$ if and only if $(W, \sigma) \leq (W', \sigma')$.*
- (b) *One has $I \subseteq Q_{W,\sigma}$ if and only if (W, σ) is an interval vertex cover of Γ_S .*

Proof. (a) We prove the forward implication; the reverse implication is similar and easier. Assume that $Q_{W,\sigma} \subseteq Q_{W',\sigma'}$. To show that $(W, \sigma) \leq (W', \sigma')$, let $i \in W$; we need to show that $i \in W'$ and that $\sigma(i) \subseteq \sigma'(i)$. Let $a \in \sigma(i)$. By definition, it follows that $X_i^a \in Q_{W,\sigma} \subseteq Q_{W',\sigma'}$. Fact 2.3 implies that X_i^a is a multiple of a monomial generator of $Q_{W',\sigma'}$, so there exist $j \in W'$ and $b \in \sigma'(j)$ such that $X_i^a \in (X_j^b)R$. Note that $a, b > 0$. It follows that $i = j \in W'$ and $b \leq a$. Since $\sigma'(i)$ is an interval of the form $[c, \infty)$ or (c, ∞) , the conditions $b \in \sigma'(i)$ and $b \leq a$ imply that $a \in \sigma'(i)$, as desired.

(b) Again, we prove the forward implication. Assume that $I \subseteq Q_{W,\sigma}$. To show that W is a vertex cover of Γ , let $ij \in E$. By definition of E and $S(ij)$, there is an element $a \in S(ij)$ such that $X_i^a X_j^a \in I \subseteq Q_{W,\sigma}$. By Fact 2.3, the element $X_i^a X_j^a$ is a multiple of a monomial generator of $Q_{W,\sigma}$, so there exist $k \in W$ and $b \in \sigma(k)$ such that $X_i^a X_j^a \in (X_k^b)R$. Since $a, b > 0$, we conclude that either $i = k \in W$ or $j = k \in W$, so W is a vertex cover of Γ .

To show that (W, σ) is an interval vertex cover of Γ , let $ij \in E$. We proceed by cases.

Case 1: $i \notin W$. Since W is a vertex cover, we have $j \in W$. In this case, we need to show that $S(ij) \subseteq \sigma(j)$, so let $a \in S(ij)$. It follows that $X_i^a X_j^a \in I \subseteq Q_{W,\sigma}$. Hence, there is a monomial generator $X_k^b \in Q_{W,\sigma}$ such that $X_i^a X_j^a \in (X_k^b)R$. Since $i \notin W$, the first paragraph of the proof of part (b) shows that $j = k$, and we conclude that $b \leq a$. As in the proof of part (a), we conclude that $a \in \sigma(j)$, as desired.

Case 2: $j \notin W$. This case is handled like Case 1.

Case 3: $i, j \in W$ and $S(ij) \not\subseteq \sigma(i)$. Again, we need to show that $S(ij) \subseteq \sigma(j)$, so let $a \in S(ij)$. The condition $S(ij) \not\subseteq \sigma(i)$ implies that there is an element $a' \in S(ij) \setminus \sigma(i)$. If $a' \leq a$, then it suffices to show that $a' \in \sigma(j)$, as above. If $a \leq a'$, then the assumption $a' \in S(ij) \setminus \sigma(i)$ implies that $a \in S(ij) \setminus \sigma(i)$ because of the shape of the interval $\sigma(i)$. Thus, we may replace a by a' if necessary to assume that $a \in S(ij) \setminus \sigma(i)$.

As above, there is a monomial generator $X_k^b \in \mathcal{Q}_{W,\sigma}$ such that $X_i^a X_j^a \in (X_k^b)R$. It follows that either $i = k$ or $j = k$, and $b \leq a$. If $i = k$, then the condition $X_k^b \in \mathcal{Q}_{W,\sigma}$ implies that $b \in \sigma(i)$; hence the inequality $b \leq a$ implies that $a \in \sigma(i)$ because of the shape of $\sigma(i)$; this is a contradiction. Thus, we have $j = k$ and, as in the previous sentence, $a \in \sigma(j)$. \square

Theorem 3.18. *For the ideal I from Assumption 3.1, we have the m -irreducible decompositions*

$$I = \bigcap_{(W,\sigma)} \mathcal{Q}_{W,\sigma} = \bigcap_{(W,\sigma) \text{ min}} \mathcal{Q}_{W,\sigma}$$

where the first intersection is taken over all interval vertex covers of Γ , and the second intersection is taken over all minimal interval vertex covers of Γ . The second intersection is finite and irredundant, and the graph Γ with data from Notation 3.3 has only finitely many minimal interval vertex covers.

Proof. In the following equation, the containment is from Lemma 3.17(b):

$$I \subseteq \bigcap_{(W,\sigma)} \mathcal{Q}_{W,\sigma} = \bigcap_{(W,\sigma) \text{ min}} \mathcal{Q}_{W,\sigma}.$$

For the equality, we have $\bigcap_{(W,\sigma)} \mathcal{Q}_{W,\sigma} \subseteq \bigcap_{(W,\sigma) \text{ min}} \mathcal{Q}_{W,\sigma}$ by basic properties of intersections, and the reverse containment follows from Lemma 3.17(a). Also, the second intersection is irredundant by Lemma 3.17(a).

By Proposition 2.20 and Fact 3.5, the ideal I has a finite m -irreducible decomposition which is of the form $I = \bigcap_{k=1}^m \mathcal{Q}_{W_k, \sigma_k}$ by Fact 3.16. Note that Lemma 3.17(b) implies that each pair (W_k, σ_k) is an interval vertex cover of Γ . Thus, we have

$$I = \bigcap_{k=1}^m \mathcal{Q}_{W_k, \sigma_k} \supseteq \bigcap_{(W,\sigma)} \mathcal{Q}_{W,\sigma}.$$

With the previous display, this provides the equalities from the statement of the result. Thus, it remains to show that the intersection $\bigcap_{(W,\sigma) \text{ min}} \mathcal{Q}_{W,\sigma}$ is finite. For this, let (W, σ) be a minimal interval vertex cover of Γ . It suffices to show that $(W, \sigma) = (W_k, \sigma_k)$ for some k . From the equalities we have already established, we have

$$\mathcal{Q}_{W,\sigma} \supseteq I = \bigcap_{k=1}^m \mathcal{Q}_{W_k, \sigma_k}.$$

The proof of Proposition 2.20 shows that $\mathcal{Q}_{W,\sigma} \supseteq \mathcal{Q}_{W_k, \sigma_k}$ for some k . So, Lemma 3.17(a) implies that $(W, \sigma) \geq (W_k, \sigma_k)$. As these are both interval vertex covers of Γ , the minimality of (W, σ) yields the desired equality $(W, \sigma) = (W_k, \sigma_k)$. \square

Example 3.19. Continue with the ideal I from Example 3.2. The minimal vertex covers from Example 3.19 provide the following irredundant m -irreducible decomposition

$$I = Q_{(W_1, \sigma_1)} \cap Q_{(W_1, \sigma_2)} \cap Q_{(W_3, \sigma_3)}$$

which is exactly the decomposition computed in Example 3.2.

4. MONOMIAL KRULL DIMENSION

We now introduce and study a notion of Krull dimension for this setting. The main result of this section is Corollary 4.15.

Assumption 4.1. Throughout this section, set $R = A[\mathbb{M}_{\geq 0}^1 \times \cdots \times \mathbb{M}_{\geq 0}^d]$.

Definition 4.2. For an m -prime ideal $P = Q_T$, we employ the notation $v(P)$ to denote the number of variables from which P is generated, i.e.,

$$v(P) = |T| = |\{X_i \mid X_i^k \in P \text{ for some } k \in \mathbb{M}_{>0}^i \text{ and } 1 \leq i \leq d\}|$$

and set $m\text{-dim}^*(R/P) = d - v(P)$.

The *monomial Krull dimension* of an arbitrary monomial ideal I is

$$m\text{-dim}(R/I) = \sup\{m\text{-dim}^*(R/Q) \mid Q \text{ is } m\text{-prime and } Q \supseteq I\}.$$

Fact 4.3. By definition, if I and J are monomial ideals in R such that $I \supseteq J$, then $m\text{-dim}(R/I) \leq m\text{-dim}(R/J)$, and furthermore we have

$$m\text{-dim}(R/I) = \begin{cases} \max\{m\text{-dim}^*(R/Q) \mid Q \text{ is } m\text{-prime and } Q \supseteq I\} & \text{if } I \neq R \\ -\infty & \text{if } I = R \end{cases}$$

Also, given an m -prime ideal P of R , it is straightforward to show that $m\text{-dim}(R/P) = m\text{-dim}^*(R/P)$.

Example 4.4. Let I be a monomial ideal in the ring R . It is straightforward to show that $m\text{-dim}(R/I) = d$ if and only if $I = 0$. Also, if $I \neq R$, then $m\text{-dim}(R/I) = 0$ if and only if for $i = 1, \dots, d$ there is an element $a_i \in \mathbb{M}_{>0}^i$ such that $X_i^{a_i} \in I$.

In the case $d = 2$, this tells us that the ideals from Examples 2.4 and 2.22 have $m\text{-dim}(R/I) = 1$, and the ideals from Examples 2.11 and 2.14 have $m\text{-dim}(R/I) = 0$.

Before proving our main results, we verify some desired properties $m\text{-dim}$.

Proposition 4.5. For a monomial ideal I in R , we have

$$m\text{-dim}(R/I) = \sup\{n \geq 0 \mid \exists m\text{-prime } P_0 \subset P_1 \subset \cdots \subset P_n \text{ and } I \subseteq P_0\}.$$

Proof. Assume without loss of generality that $I \neq R$.

Let $m = \max\{n \geq 0 \mid \exists \text{ m-prime } P_0 \subset P_1 \subset \dots \subset P_n \text{ and } I \subseteq P_0\}$ with corresponding maximal chain $I \subseteq P_0 \subset P_1 \subset \dots \subset P_m$. The maximality of this chain implies that $P_m = Q_{\{1, \dots, d\}}$. Then for $i = 0, \dots, m - 1$ we have $v(P_{i+1}) = v(P_i) + 1$; otherwise, we could find an m-prime P such that $P_i \subset P \subset P_{i+1}$, which would contradict the maximality of m . Hence, we have $d = v(P_m) = v(P_0) + m$, and it follows that $m = \text{m-dim}(R/P_0)$. It remains to show that $\text{m-dim}(R/P_0) = \text{m-dim}(R/I)$.

We have that $\text{m-dim}(R/P_0) \leq \text{m-dim}(R/I)$ by Fact 4.3. Now, suppose that $\text{m-dim}(R/P_0) < \text{m-dim}(R/I)$. That would mean there is an m-prime ideal $P \supseteq I$ such that $v(P) < v(P_0)$. We could then create a chain $P \subset P'_0 \subset \dots \subset P'_m$, contradicting the maximality of m . \square

The next result applies whenever I is finitely generated by Corollary 2.21.

Proposition 4.6. *Given a monomial ideal I in R with a finite m -irreducible decomposition $I = \bigcap_{i=1}^t J_i$, one has*

$$\text{m-dim}(R/I) = \sup_i \{\text{m-dim}(R/J_i)\}.$$

Proof. Assume without loss of generality that $t \geq 1$, i.e., that $I \neq R$. Since $J_i \supseteq I$ for all i , Fact 4.3 implies that $\sup_i \{\text{m-dim}(R/J_i)\} \leq \text{m-dim}(R/I)$.

We now claim that for every m-prime ideal P , if $P \supseteq I$, then there is an index k such that $P \supseteq J_k$. By way of contradiction, suppose that for all k there is a monomial $f_k \in [J_k] \setminus [P]$. Since P is m-prime and $f_k \notin [P]$ for all k , we have that $\prod_{k=1}^t f_k \notin [P]$. However, $f_k \in J_k$ implies that $\prod_{k=1}^t f_k \in \bigcap_{k=1}^t J_k = I \subseteq P$, a contradiction.

Now, let P be m-prime such that $I \subseteq P$ and $\text{m-dim}(R/P) = \text{m-dim}(R/I)$. The claim implies that there is an index k such that $P \supseteq J_k$, so we have

$$\text{m-dim}(R/I) = \text{m-dim}(R/P) \leq \text{m-dim}(R/J_k) \leq \sup_i \{\text{m-dim}(R/J_i)\}$$

as desired. \square

Example 4.7. Consider the case $\mathbb{M}^i = \mathbb{R}$ for all i and the ideal I from Assumption 3.1. Using ideas from [6], one shows that $\text{m-dim}(R/I) = d - \tau(\Gamma)$, where $\tau(\Gamma)$ is the *vertex cover number* of Γ

$$\tau(\Gamma) = \min\{|W| \mid W \text{ is a vertex cover of } \Gamma\}.$$

To consider the semicontinuity of monomial Krull dimension, we introduce the next definition.

Definition 4.8. Let $\varepsilon \in \mathbb{R}_{>0}$. For monomial ideals I, J in R , we say that $\text{dist}(I, J) < \varepsilon$ if the following conditions hold:

- (1) For all $\underline{X}^\gamma \in [I]$, there exists $\underline{X}^\delta \in [J]$ such that $\text{dist}(\underline{\gamma}, \underline{\delta}) < \varepsilon$;
- (2) For all $\underline{X}^\delta \in [J]$, there exists $\underline{X}^\gamma \in [I]$ with $\text{dist}(\underline{\delta}, \underline{\gamma}) < \varepsilon$.

Here $\text{dist}(\underline{\gamma}, \underline{\delta}) = |\underline{\gamma} - \underline{\delta}| = \sqrt{(\gamma_1 - \delta_1)^2 + \dots + (\gamma_d - \delta_d)^2}$.

Example 4.12 shows that the following definition does not define a metric when applied to the set of arbitrary monomial ideals of R . However, it is a metric when restricted to the set of finitely generated monomial ideals, as we show in Theorem 4.13 below.

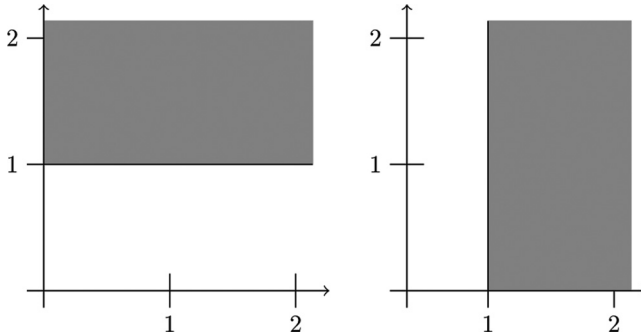
Definition 4.9. The distance between two monomial ideals I, J in R is

$$\text{dist}(I, J) = \inf\{\varepsilon > 0 \mid \text{dist}(I, J) < \varepsilon\}.$$

Example 4.10. Let I be a monomial ideal in R . Since $[0] = \emptyset$, we have

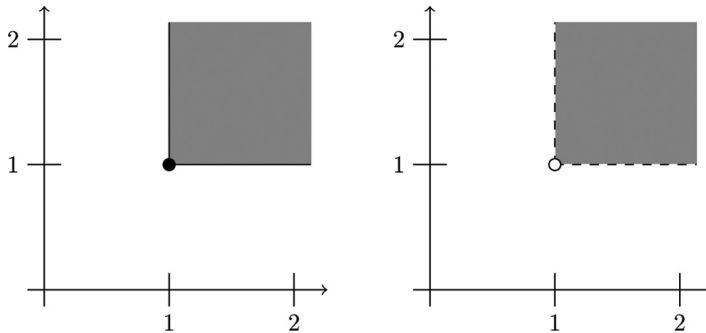
$$\text{dist}(I, 0) = \begin{cases} 0 & \text{if } I = 0 \\ \infty & \text{if } I \neq 0 \end{cases}$$

Example 4.11. The ideals $J_{2,1,0} = (X_2)R$ and $J_{1,1,0} = (X_1)R$ in $R = A[\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}]$ satisfy $\text{dist}((X_2)R, (X_1)R) = 1$.



This is intuitively clear from the above diagrams. To verify this rigorously, first note that, for every monomial $X_1^a X_2^b \in (X_2)R$ one has $X_1^{a+1} X_2^b \in (X_1)R$ and $\text{dist}(X_1^a X_2^b, X_1^{a+1} X_2^b) = 1$. Similarly, every monomial in $(X_1)R$ is distance 1 from a monomial in $(X_2)R$. This implies that $\text{dist}((X_2)R, (X_1)R) < 1 + \epsilon$ for all $\epsilon > 0$, so $\text{dist}((X_2)R, (X_1)R) \leq 1$. Finally, note that the monomial in $(X_2)R$ that is closest to $X_1 \in (X_1)R$ is $X_1 X_2$, which is distance 1 from X_1 , so $\text{dist}((X_2)R, (X_1)R) \geq 1$.

Example 4.12. The ideals $I_{(1,1),(0,0)}$ and $I_{(1,1),(1,1)}$ in $R = A[\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}]$



have $\text{dist}(I_{(1,1),(0,0)}, I_{(1,1),(1,1)}) = 0$ even though $I_{(1,1),(0,0)} \neq I_{(1,1),(1,1)}$. This explains (at least partially) why we restrict to the set of finitely generated monomial ideals in our next result.

Theorem 4.13. *The function dist is a metric on the set of nonzero finitely generated monomial ideals of R .*

Proof. Let I, J, K be nonzero finitely generated monomial ideals of R .

To see that $\text{dist}(I, J) \in \mathbb{R}_{\geq 0}$, let $\underline{X}^\alpha \in [I]$ and $\underline{X}^\beta \in [J]$. Let $\delta > 0$ be given. We claim that $\text{dist}(I, J) < \max\{|\underline{\alpha}|, |\underline{\beta}|\} + \delta$. (a) For every $\underline{X}^\gamma \in [I]$, there is a monomial $\underline{X}^{\gamma+\beta} = \underline{X}^\gamma \underline{X}^\beta \in [J]$ such that $\text{dist}(\gamma + \beta, \gamma) = |\underline{\beta}| < \max\{|\underline{\alpha}|, |\underline{\beta}|\} + \delta$. (b) For every $\underline{X}^\gamma \in [J]$ there is a monomial $\underline{X}^{\gamma+\alpha} = \underline{X}^\gamma \underline{X}^\alpha \in [I]$ such that $\text{dist}(\gamma + \alpha, \gamma) = |\underline{\alpha}| < \max\{|\underline{\alpha}|, |\underline{\beta}|\} + \delta$. This established the claim. Consequently, it follows that $\text{dist}(I, J) \in \mathbb{R}$ such that $0 \leq \text{dist}(I, J) \leq \max\{|\underline{\alpha}|, |\underline{\beta}|\}$, as desired.

The condition $\text{dist}(I, J) = \text{dist}(J, I)$ follows from the symmetry of Definition 4.8. The equality $\text{dist}(I, I) = 0$ is similarly straightforward.

Next, assume that $\text{dist}(I, J) = 0$. We show that $I = J$. Let $\underline{X}^\alpha \in [I]$, and let $J = (\underline{X}^{\beta_1}, \dots, \underline{X}^{\beta_n})R$. For each i, j , we set $\beta'_{i,j} = \max\{\beta_{i,j}, \alpha_j\}$. Then $\beta'_{i,j} \geq \alpha_j$ and $\beta'_{i,j} \geq \beta_{i,j}$. Therefore, for all j , we have $\underline{X}^{\beta'_j} \in (\underline{X}^\alpha)R \subseteq I$ and $\underline{X}^{\beta'_j} \in (\underline{X}^{\beta_j})R \subseteq J$. Since $\text{dist}(I, J) = 0$, we have that $\inf\{\text{dist}(\underline{X}^\alpha, \underline{X}^\gamma) \mid \underline{X}^\gamma \in [J]\} = 0$.

We claim that

$$\inf\{\text{dist}(\underline{X}^\alpha, \underline{X}^\gamma) \mid \underline{X}^\gamma \in [J]\} = \min\{\text{dist}(\underline{X}^\alpha, \underline{X}^{\beta'_i}) \mid 1 \leq i \leq n\}.$$

The condition $\underline{X}^{\beta'_j} \in [J]$ explains the inequality \leq . For the reverse inequality, let $\underline{X}^\gamma \in J$. Then $\underline{X}^\gamma = \underline{X}^{\beta_j} \underline{X}^\delta$ for some j and some $\underline{X}^\delta \in [R]$. We have that $\text{dist}(\underline{X}^\alpha, \underline{X}^\gamma) = \sqrt{(\gamma_1 - \alpha_1)^2 + \dots + (\gamma_n - \alpha_n)^2}$. If $\gamma_i \leq \alpha_i$, then we have $\beta_{j,i} \leq \gamma_i \leq \alpha_i$, so $\beta'_{j,i} = \alpha_i$, which implies that $|\beta'_{j,i} - \alpha_i| = 0 \leq |\gamma_i - \alpha_i|$. If $\gamma_i \geq \alpha_i$, then the condition $\gamma_i \geq \beta_{j,i}$ implies that $\gamma_i \geq \beta'_{j,i} \geq \alpha_j$, so we have that $0 \leq \beta'_{j,i} - \alpha_i \leq \gamma_i - \alpha_i$. Therefore, for all $\underline{X}^\gamma \in J$, we have $\text{dist}(\underline{X}^\alpha, \underline{X}^\gamma) \geq \text{dist}(\underline{X}^\alpha, \underline{X}^{\beta'_j})$ for some j . This proves the claim.

It follows that $\min\{\text{dist}(\underline{X}^\alpha, \underline{X}^{\beta'_i}) \mid 1 \leq i \leq n\} = 0$. Thus, there exists an index i such that $\underline{X}^\alpha = \underline{X}^{\beta'_i} \in J$. As this is so for all $\underline{X}^\alpha \in [I]$, we conclude that $I \subseteq J$. By symmetry, we have $I = J$, as desired.

Now, we check the triangle inequality: $\text{dist}(I, K) \leq \text{dist}(I, J) + \text{dist}(J, K)$. Let $\varepsilon > 0$ be given; we show that $\text{dist}(I, K) < \text{dist}(I, J) + \text{dist}(J, K) + \varepsilon$. Set $d = \text{dist}(I, J)$ and $e = \text{dist}(J, K)$, and let $\underline{X}^\alpha \in [I]$. We need to find a monomial $\underline{X}^\gamma \in [K]$ such that $\text{dist}(\alpha, \gamma) < d + e + \varepsilon$. Since $\text{dist}(I, J) < d + \frac{\varepsilon}{2}$, there is a monomial $\underline{X}^\beta \in [J]$ such that $\text{dist}(\alpha, \beta) < d + \frac{\varepsilon}{2}$. Similarly, there is a monomial $\underline{X}^\gamma \in [K]$ such that $\text{dist}(\beta, \gamma) < e + \frac{\varepsilon}{2}$. From the triangle equality in \mathbb{R}^d , we conclude that

$$\text{dist}(\alpha, \gamma) \leq \text{dist}(\alpha, \beta) + \text{dist}(\beta, \gamma) < d + e + \varepsilon$$

as desired. □

Theorem 4.14. *Given a nonzero finitely generated monomial ideal I in R , there exists $\varepsilon > 0$ such that for all nonzero monomial ideals J , with $\text{dist}(I, J) < \varepsilon$, we have $\text{m-dim}(R/J) \geq \text{m-dim}(R/I)$.*

Proof. Since $\text{m-dim}(R/R) = -\infty$, we assume without loss of generality that $I \neq R$. Let $I = (\underline{X}^{z_1}, \dots, \underline{X}^{z_n})R$, and set $\varepsilon = \min\{\alpha_{ij} \mid \alpha_{ij} > 0\} > 0$. Let J be a nonzero monomial ideal with $\text{dist}(I, J) < \varepsilon$.

Claim: For every m-prime ideal Q of R such that $Q \supseteq I$, we have $Q \supseteq J$. Let $\underline{X}^{\gamma} \in [J]$, and choose $\underline{X}^{\delta} \in I$ such that $\text{dist}(\underline{\gamma}, \underline{\delta}) < \varepsilon$. There exists an index i such that $\underline{X}^{\delta} \in (\underline{X}^{z_i})R \subseteq I \subseteq Q$. Since Q is m-prime, there exists an index j such that $\alpha_{i,j} > 0$ and $X_j^{z_i} \in Q$. Hence, $X_j^t \in Q$ for all $t > 0$. Note that $0 < \varepsilon \leq \alpha_{i,j} \leq \delta_j$ and $|\gamma_j - \delta_j| \leq \text{dist}(\underline{\gamma}, \underline{\delta}) < \varepsilon$. Therefore, we have $-\varepsilon < \gamma_j - \delta_j < \varepsilon$, which implies that $0 \leq \delta_j - \varepsilon < \gamma_j$, so we conclude that $\underline{X}^{\gamma} \in Q$, which establishes the claim.

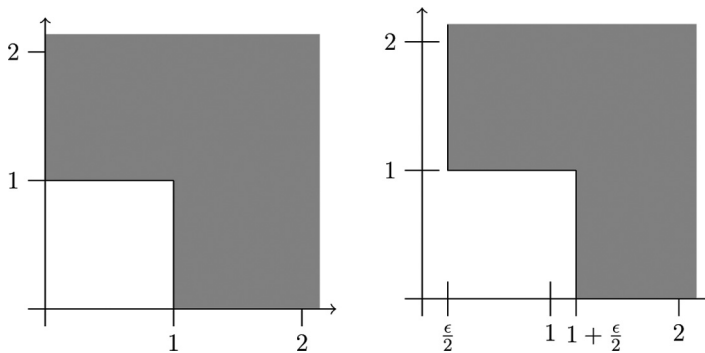
It follows that $\{P \supseteq I \mid P \text{ is m-prime}\} \subseteq \{P' \supseteq J \mid P' \text{ is m-prime}\}$, and hence $\text{m-dim}(R/I) \leq \text{m-dim}(R/J)$. □

Corollary 4.15. *The monomial Krull dimension function is lower semicontinuous on the set of finitely generated monomial ideals of R .*

Example 4.16. Let I be a nonzero finitely generated monomial ideal of R .

If $\text{m-dim}(R/I) = d - 1$, then there is a real number $\epsilon > 0$ such that for all monomial ideals J in R such that $\text{dist}(I, J) < \epsilon$, one has $\text{m-dim}(R/J) = d - 1$. Indeed, Theorem 4.14 provides a real number $\epsilon > 0$ such that for all nonzero monomial ideals J in R such that $\text{dist}(I, J) < \epsilon$, one has $\text{m-dim}(R/J) \geq d - 1$. If $\text{m-dim}(R/J) > d - 1$, then $\text{m-dim}(R/J) = d$, so Example 4.4 implies that $J = 0$, a contradiction.

If $\mathbb{M}^i = \mathbb{R}$ for all i , then (regardless of the value of $\text{m-dim}(R/I)$) for each real number $\epsilon > 0$ there is a nonzero finitely generated monomial ideal $J \subset I$ in R such that $0 < \text{dist}(I, J) < \epsilon$ and $\text{m-dim}(R/J) = d - 1$. Indeed, consider the ideal $X_1^{\epsilon/2}I$, which is nonzero and finitely generated since I is so. As in Example 4.11, it is straightforward to show that $\text{dist}(I, X_1^{\epsilon/2}I) = \epsilon/2 < \epsilon$. However, the ideal $X_1^{\epsilon/2}I$ is contained in $Q_{\{1\}} = (X_1^a \mid a > 0)R$, so we have $\text{m-dim}(X_1^{\epsilon/2}I) \geq d - v(Q_{\{1\}}) = d - 1$. In particular, if $\text{m-dim}(R/I) < d - 1$, then strict inequality can occur in Theorem 4.14. This behavior is depicted in the following two diagrams with $d = 2$: the first diagram represents the ideal $I = J_{(1,1),(0,0)} = (X_1, X_2)R$ and the second one represents $X_1^{\epsilon/2}I = (X_1^{1+(\epsilon/2)}, X_1^{\epsilon/2}X_2)R = (X_1^{\epsilon/2})R \cap (X_1^{1+(\epsilon/2)}, X_2)R$:



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