MATH 720, Algebra I
Exam 1
Due Fri 30 Sep

Let $R$ be a commutative ring with identity.

**Exercise 1.** Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be a non-empty set of commutative rings with identity. For each $\mu \in \Lambda$ let $p_\mu: \prod_{\lambda \in \Lambda} R_\lambda \to R_\mu$ be defined as $p_\mu((r_\lambda)) := r_\mu$.

(a) Prove that for each $\mu \in \Lambda$ the function $p_\mu: \prod_{\lambda \in \Lambda} R_\lambda \to R_\mu$ is an epimorphism of commutative rings with identity.

(b) Let $\{f_\lambda: R \to R_\lambda\}_{\lambda \in \Lambda}$ be a set of homomorphisms of commutative rings with identity. Prove that there is a unique homomorphism $F: R \to \prod_{\lambda \in \Lambda} R_\lambda$ such that for each $\mu \in \Lambda$ the composition $p_\mu \circ F$ is $f_\mu$.

**Exercise 2.** Let $I$ be an ideal of $R$. Let $R \ltimes I$ be the additive abelian group $R \oplus I$ with the following multiplication: for all $(r, i), (r', i') \in R \ltimes I$ we set $(r, i)(r', i') := (rr', ri' + r'i)$. 

(a) Prove that $R \ltimes I$ is a commutative ring with identity under these operations.

(b) Prove that the map $g: R \to R \ltimes I$ given by $g(r) = (r, 0)$ is a monomorphism of rings with identity.

(c) Prove that the subset $0 \oplus I \subseteq R \ltimes I$ is an ideal of $R$ such that $(R \ltimes I)/(0 \oplus I) \cong R$.

(d) Prove that $(0 \oplus I)^2 = 0$.

(e) Prove that if $I$ is generated (as an ideal of $R$) by a set $S \subseteq R$, then $0 \oplus I$ is generated (as an ideal of $R \ltimes I$) by the set $\{(0, s) \in R \ltimes I \mid s \in S\}$.

**Definition 1.** Let $I$ be an ideal of $R$. The *radical* of $I$ is the set $\text{rad}(I) = \{x \in R \mid$ there is an integer $n \geq 1$ such that $x^n \in I\}$.

**Exercise 3.** Let $I$ and $J$ be ideals of $R$.

(a) Prove that $\text{rad}(I)$ is an ideal of $R$ such that $I \subseteq \text{rad}(I) = \text{rad}(\text{rad}(I))$.

(b) Prove that if $I \subseteq J$, then $\text{rad}(I) \subseteq \text{rad}(J)$.

(c) Prove that $\text{rad}(I) = R$ if and only if $I = R$.

(d) Prove that if $I$ is finitely generated and $I \subseteq \text{rad}(J)$, then there is an integer $q \geq 1$ such that $I^q \subseteq J$.

(e) Assume that $R$ is a unique factorization domain, and let $u$ be a unit of $R$. Let $p_1, \ldots, p_n$ be primes of $R$ such that for all $i, j$ such that $1 \leq i < j \leq n$ the elements $p_i$ and $p_j$ are not associates. Given integers $e_1, \ldots, e_n \geq 1$ prove that $\text{rad}((up_1^{e_1} \cdots p_n^{e_n})R) = (p_1 \cdots p_n)R$.

(f) Find an example of a commutative ring $R$ with identity and two ideals $I$ and $J$ such that $\text{rad}(I) = \text{rad}(J)$ but $I \not\subseteq J$ and $J \not\subseteq I$. 

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