Increasing Labelings, Generalized Promotion, and Rowmotion

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Outline

1. Background
2. Previous Results
3. Generalized Increasing Labelings
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3. Generalized Increasing Labelings
Increasing Tableaux

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>4</th>
<th>5</th>
<th>8</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
<td>9</td>
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<td>6</td>
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<td>8</td>
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**Figure:** An increasing tableau $T$ of shape $\lambda = (4, 4, 4, 2)$.

Arise in equivariant $K$-theory of the Grassmanian.
$K$-Promotion

$T = \begin{array}{cccc}
1 & 2 & 4 & 6 \\
4 & 5 & 6 & 7 \\
\end{array}$

Delete 1’s

$\begin{array}{cccc}
2 & 4 & 6 \\
4 & 5 & 6 & 7 \\
\end{array}$

$\begin{array}{cccc}
2 & 4 & 6 \\
4 & 5 & 6 & 7 \\
\end{array}$

Fill and decrement

$\begin{array}{cccc}
1 & 3 & 5 & 6 \\
3 & 4 & 6 & 7 \\
\end{array}$

$= K\text{-Pro}(T)$
Order of $K$-promotion

- Order of $K$-promotion on rectangles $a \times b$ is always small multiple of largest possible label $q$.
- Hits it on the nose for special cases ($a = 2$, $q = a + b$, standard tableaux).
Rowmotion

**Definition**

Let *rowmotion* be the action on $J(P)$ (set of order ideals of $P$) that takes in an order ideal $I$, and returns the order ideal generated by the minimal elements of $J(P)$.
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Toggle Group

Definition
For each element $e \in P$ define its **toggle** $t_e : J(P) \rightarrow J(P)$ as follows.

$$t_e(X) = \begin{cases} 
X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in J(P) \\
X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in J(P) \\
X & \text{otherwise}
\end{cases}$$

Note: $t_e, t_f$ commute whenever neither $e$ nor $f$ covers the other.
Theorem (Cameron–Fon-der-Flaass)

Rowmotion is the same as toggling each element once, in the order of a linear extension.
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Order of Rowmotion

- Rowmotion on order ideals in $a \times b \times c$.
- When $c = 1, 2(, 3?)$, order is exactly $a + b + c - 1$.
- In general, orbit sizes appear to be small multiples of $a + b + c - 1$. 
Outline

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Bijection Between Increasing Tableaux and Order Ideals

Theorem (Dilks, Pechenik, Striker)

*Bijective between order ideals in* $[a] \times [b] \times [c]$ *and increasing labelings of* $a \times b$ *with max entry* $a + b + c - 1$. 
Bijection Between Increasing Tableaux and Order Ideals

$I = \begin{array}{cccc}
0 & 0 & 1 & 3 \\
1 & 2 & 2 & 3 \\
2 & 3 & 3 & 4 \\
3 & 4 & 4 & 4 \\
\end{array}$

Project to bottom face

Rotate $180^\circ$

Add $1+\text{rank}$

$\begin{array}{cccc}
4 & 4 & 4 & 3 \\
4 & 3 & 3 & 2 \\
3 & 2 & 2 & 1 \\
3 & 1 & 0 & 0 \\
\end{array}$

$= \psi_3(I)$
Theorem (Dilks, Pechenik, Striker)

\( K \)-promotion is a series of local involutions.
**Theorem (Dilks, Pechenik, Striker)**

*K-promotion is a series of local involutions.*
Equivariant Bijection

\[ x + y - z = 3 \]
\[ x + y - z = 2 \]
\[ x + y - z = 1 \]
\[ x + y - z = 0 \]
\[ x + y - z = -1 \]
\[ x + y - z = -2 \]
Hyperplane Promotion is Conjugate to Rowmotion

Theorem (Dilks, Pechenik, Striker)

Let $P$ be a poset with an order and rank preserving map $\pi : P \rightarrow \mathbb{Z}^n$, and let $v = (v_1, v_2, v_3, \ldots, v_n)$, where $v_j \in \{ \pm 1 \}$.

Let $T^i_{\pi, v}$ be the product of toggles $t_x$ for all elements $x$ of $P$ that lie on the affine hyperplane $\langle \pi(x), v \rangle = i$.

Then $\text{Pro}_{\pi, v} = \ldots T^{-2}_{\pi, v} T^{-1}_{\pi, v} T^0_{\pi, v} T^1_{\pi, v} T^2_{\pi, v} \ldots$ is conjugate to rowmotion.
Punchline

\( K \)-promotion on increasing labelings of \( a \times b \) with max entry \( a + b + c - 1 \)

has the same orbit structure as

rowmotion on order ideals in \( a \times b \times c \).
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Motivation

How much do these methods generalize.

- Non-square increasing tableaux?
- Non-ranked posets?
Increasing Labeling

**Definition**

An *increasing labeling* of $P$ is a function $f : P \rightarrow \mathbb{Z}$ such that $p_1 <_P p_2$ implies $f(p_1) < f(p_2)$.

**Definition**

For labeling function $R : P \mapsto \mathcal{P}(\mathbb{Z})$, let $\text{Inc}^R(P)$ be the set of increasing labelings of $P$ such that for all $p \in P$, $f(p) \in R(p)$. 
Example

Generalized Increasing Labelings

- **a** \{1,4\}
- **b** \{2,3,5\}
- **c** \{2,4,5\}
- **d** \{3,4,5,6\}
- **e** \{4,6,7,9\}

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Key Idea

- \( \text{Inc}^R(P) \) can be partially ordered by element-wise comparison.
- \( \text{Inc}^R(P) \) is a distributive lattice (meet and join are taking element-wise min/max)
- Birkhoff FTFDL (Fundamental Theorem of Finite Distributive Lattices)
- Find join irreducibles and their relative order (\( \Gamma(P, R) \)).
Generalized Increasing Labelings

{1,4}  {2,3,5}  {4,6,7,9}

{2,4,5}  {3,4,5,6}  

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Special Case: Bounded max entry

Largest entry of 6

a
{1,2,3}

b
{2,3,4,5}

c
{2,3,4}

d
{3,4,5}

e
{4,5,6}

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Generalized Bender-Knuth involutions for arbitrary $R$:

- If a label is currently $i$, and you can increment it to next allowable label (and stay in $\text{Inc}^R(P)$), then do it.
- If you can decrement a label to become $i$ (and stay in $\text{Inc}^R(P)$), then do it.
- Otherwise, do nothing.

Increasing tableaux promotion:
$\text{IncPro} = \ldots \circ \rho_2 \circ \rho_1$. 

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- Generalizes promotion on linear extensions.
- Generalizes $K$-promotion on increasing tableaux.
- In case with largest global entry, can equivalently be described in terms of box sliding.
**Toggle Promotion**

**Definition**

*Toggle order* a function $H : P \to \mathbb{Z}$ where $p_1 \prec p_2 \implies H(p_1) \neq H(p_2)$.

**Definition**

$T_H^i$ is the product of all $t_p$ for $p \in P$ such that $H(p) = i$.

*Toggle-promotion* (wrt $H$), called TogPro$_H$, is the toggle group action given by

$$\ldots T_H^2 T_H^1 T_H^0 T_H^{-1} T_H^{-2} \ldots$$
Bijection is equivariant

For $\Gamma(P, R)$, a natural toggle order is given by $H((p, k)) = k$.

**Theorem (Dilks, Striker, Vorland)**

*The map between $\text{Inc}^R(P)$ and $J(\Gamma(P, R))$ equivariantly takes $\text{IncPro}$ to $\text{TogPro}_H$.*
\[ \rho_i \text{ is } T^i_H \]
\( \rho_i \) is \( T^i_H \)

Graph showing increasing labelings and relations.
\[ \rho_i \text{ is } T_H^i \]
\( \rho_i \) is \( T^i_H \)
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Column Toggle Order

Definition

We say that a function $H : P \rightarrow \mathbb{Z}$ is a column toggle order if whenever $p_1 \preceq p_2$ in $P$, then $H(p_1) = H(p_2) \pm 1$.

Theorem (Dilks, Striker, Vorland)

When $H$ is a column toggle order, then $\text{TogPro}_H$ is conjugate to rowmotion.
Theorem (Dilks, Striker, Vorland)

If $R$ is a restriction function for $P$ that consists of intervals (including global max entry), then the map $H : \Gamma(P, R) \to \mathbb{Z}$ given by $(p, k) \mapsto k$ is a column toggle order.

Therefore, rowmotion on $\Gamma(P, R)$ is conjugate to the corresponding toggle promotion.

Theorem

If $P_1$ and $P_2$ are ranked posets, then $H : P_1 \times P_2 : \to \mathbb{Z}$ given by $H((p_1, p_2)) = \text{rk}_{P_1}(p_1) - \text{rk}_{P_2}(p_2)$ is a column toggle order.
Big result

Theorem (Dilks, Striker, Vorland)

*Increasing promotion on* $\text{Inc}^R(P)$ *is conjugate to rowmotion on* $\Gamma(P, R)$ *when* $R$ *consists of intervals.*
Thanks!