Mind-boggling toggling

Jessica Striker
North Dakota State University

October 28, 2019
Mind boggling toggling

1. Posets and order ideals
2. Toggles
3. Rowmotion
4. Promotion
5. Gyration
Mind-boggling toggling

1. Posets and order ideals
2. Toggles
3. Rowmotion
4. Promotion
5. Gyration
Posets

A **poset** is a **partially ordered set**.

**Definition**

A *poset* is a set with a partial order “≤” that is reflexive, antisymmetric, and transitive.
Order ideals

Definition

An order ideal of a poset $P$ is a subset $X \subseteq P$ such that if $y \in X$ and $z \leq y$, then $z \in X$. 
The set of order ideals
Slight digression

How many order ideals are there in this example?  

42 - a Catalan number!
Slight digression

How many order ideals are there in this example?

The Hitchhiker’s Guide to the Galaxy

By Douglas Adams

A masterpiece of sci-fi satire.
Plus, its pages contain the answer to
the question of life, the universe,
and everything!

Schyler
Slight digression

How many order ideals are there in this example?

42 - a Catalan number!
Catalan Numbers

Definition

A **Catalan object** is any collection of sets such that the \(n\)th set is counted by the \(n\)th **Catalan number**:

\[
C_n = \frac{1}{n + 1} \binom{2n}{n} = \frac{(2n)!}{(n + 1)!n!}
\]

The Catalan numbers for \(n = 0, 1, 2, \ldots, 10\) are:

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796

See “Catalan Numbers” by Richard Stanley to learn about hundreds of Catalan objects.
There are beautiful bijections among all these Catalan objects!

\[
\begin{array}{cccccccc}
1 & 2 & 4 & 7 & 8 & 9 & 13 \\
3 & 5 & 6 & 10 & 11 & 12 & 14 \\
\end{array}
\]

1 1 2 4 4 4 7

(0,0) (14,0)
Bijections among Dyck paths, order ideals of the triangular poset, standard Young tableaux, noncrossing matchings

\[(0, 0) \rightarrow (14, 0)\]

\[
\begin{array}{cccccccc}
1 & 2 & 4 & 7 & 8 & 9 & 13 \\
3 & 5 & 6 & 10 & 11 & 12 & 14 \\
\end{array}
\]
Bijections among Dyck paths, order ideals of the triangular poset, standard Young tableaux, noncrossing matchings
Bijections among Dyck paths, order ideals of the triangular poset, standard Young tableaux, noncrossing matchings
Mind-boggling toggling

1. Posets and order ideals
2. Toggles
3. Rowmotion
4. Promotion
5. Gyration
What is a toggle?

www.amazon.com/Shoreline-Marine-Toggle-Switch-Brass/dp/B004LR5N3C
Toggles act on order ideals

Define a toggle, $t_e$, for each $e \in P$. 
Toggles act on order ideals

Toggles $t_e$ add $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ add $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ remove $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ remove $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ do nothing otherwise.
Toggles act on order ideals

Toggles $t_e$ do nothing otherwise.
Toggles act on order ideals

Toggle group actions are compositions of toggles that act on order ideals.
Mind-boggling toggling

1. Posets and order ideals
2. Toggles
3. Rowmotion
4. Promotion
5. Gyration
Rowmotion definition

Definition

The **rowmotion** of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.
The **rowmotion** of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Find the **minimal** elements of $P$ not in $X$. 
Rowmotion definition

Definition

The rowmotion of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Use them to generate a new order ideal $\text{Row}(X)$. 

\begin{center}
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{center}
Rowmotion definition

Definition

The **rowmotion** of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Find the **minimal** elements of $P$ not in $\text{Row}(X)$. 

![Diagram of a poset with red and black elements, representing the rowmotion of an order ideal.]

Jessica Striker (NDSU)
The rowmotion of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Use them to generate a new order ideal $\text{Row} (\text{Row}(X))$. 

[Diagram of a Hasse diagram with white and black nodes, illustrating the rowmotion process.]

Jessica Striker (NDSU)  
Mind-boggling toggling  
October 28, 2019
The **rowmotion** of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Find the **minimal** elements of $P$ not in $\text{Row}(\text{Row}(X))$. 

---

Jessica Striker  (NDSU)
The **rowmotion** of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Use them to generate $\text{Row}(\text{Row}(\text{Row}(X)))$. 

![Diagram](image)
The action of rowmotion on these order ideals
The action of rowmotion on these order ideals

Why is this orbit structure so nice?
Rowmotion as a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Rowmotion as a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Rowmotion as a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Rowmotion as a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Rowmotion as a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Rowmotion as a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Mind-boggling toggling

1. Posets and order ideals
2. Toggles
3. Rowmotion
4. Promotion
5. Gyration
Question
In how many ways can you fill in this shape with the numbers 1 through 10 such that the rows and columns are increasing?
42 - a Catalan number!
Definition

A **standard Young tableau** is a collection of $n$ boxes justified up and to the left filled with positive integers \{1, 2, \ldots, n\} such that the rows are increasing from left to right and columns are increasing from top to bottom.

\[
\begin{array}{cccc}
1 & 2 & 4 & 9 \\
3 & 5 & 6 & 8 \\
7 & 11 \\
\end{array}
\]
Standard Young Tableaux

Definition

A standard Young tableau is a collection of \( n \) boxes justified up and to the left filled with positive integers \( \{1, 2, \ldots, n\} \) such that the rows are increasing from left to right and columns are increasing from top to bottom.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10
\end{array}
\]
Defining Promotion

To compute the **promotion** of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with $n + 1$. Then subtract 1 from all entries.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 8 & 9 \\
\end{array}
\]
Defining Promotion

To compute the promotion of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with $n + 1$. Then subtract 1 from all entries.

\[
\begin{array}{cccc}
2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10
\end{array}
\]
Defining Promotion

To compute the promotion of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with $n + 1$. Then subtract 1 from all entries.

\[
\begin{array}{cccc}
2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10 \\
\end{array}
\]
Defining Promotion

To compute the promotion of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with $n + 1$. Then subtract 1 from all entries.

\[
\begin{array}{cccc}
2 & 3 & & \\
5 & 6 & 8 & 9 & 10 \\
\end{array}
\]
Defining Promotion

To compute the promotion of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with \( n + 1 \). Then subtract 1 from all entries.
Defining Promotion

To compute the promotion of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with \( n + 1 \). Then subtract 1 from all entries.

\[
\begin{array}{cccc}
2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10 \\
\end{array}
\]
Defining Promotion

To compute the **promotion** of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with \( n + 1 \). Then subtract 1 from all entries.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Jessica Striker (NDSU) Mind-boggling toggling  October 28, 2019
Defining Promotion

To compute the promotion of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with $n + 1$. Then subtract 1 from all entries.
Defining Promotion

To compute the promotion of a standard Young tableau, first delete the 1. Then slide the smaller of the numbers in the boxes to the right or below into the empty box. Continue in this manner until you reach an outside corner. Fill this box with $n + 1$. Then subtract 1 from all entries.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 6 & 9 \\
4 & 5 & 7 & 8 & 10 \\
\end{array}
\]
Computing Promotion

**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_i$ swaps $i$ and $i + 1$ when possible.

\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 8 & 9 \\
\hline
7 & 10 & & \\
\hline
\end{array}
Promotion is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_1$ swaps 1 and 2 when possible.
Promotion is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_2$ swaps 2 and 3 when possible.
Computing Promotion

**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_3$ swaps 3 and 4 when possible.

```
1 2 3 4 7
5 6 8 9 10
```
**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_4$ swaps 4 and 5 when possible.
Promotion is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_4$ swaps 4 and 5 when possible.
**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_5$ swaps 5 and 6 when possible.

\[
\begin{array}{cccc}
1 & 2 & 3 & 5 \\
4 & 6 & 8 & 9 \\
5 & 7 & 9 & 10 \\
\end{array}
\]
Promotion is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_5$ swaps 5 and 6 when possible.
**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_6$ swaps 6 and 7 when possible.
Computing Promotion

**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_7$ swaps 7 and 8 when possible.

\[ \begin{array}{ccccc}
1 & 2 & 3 & 6 & 7 \\
4 & 5 & 8 & 9 & 10 \\
\end{array} \]
**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_7$ swaps 7 and 8 when possible.
Promotion is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_8$ swaps 8 and 9 when possible.
**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$, where $\rho_9$ swaps 9 and 10 when possible.

\[
\begin{array}{ccccc}
1 & 2 & 3 & 6 & 9 \\
4 & 5 & 7 & 8 & 10 \\
\end{array}
\]
**Promotion** is the product $\prod_i \rho_i$ of the Bender-Knuth involutions $\rho_i$.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>
The action of promotion on these tableaux
The action of promotion on these tableaux

We’ve seen this before...
The action of rowmotion on these order ideals
The action of rowmotion on these order ideals

Why is this orbit structure so nice?
The action of promotion on these tableaux

Why are these orbit structures the same?
The action of promotion on these tableaux

Why is this orbit structure so nice?
Theorem (D. White)

There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and rotation on non-crossing matchings of $2n$. So promotion has order $2n$. 

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10 \\
\end{array}
\quad \xrightarrow{\text{Promotion}} \quad 
\begin{array}{cccccc}
1 & 2 & 3 & 6 & 9 \\
4 & 5 & 7 & 8 & 10 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10 \\
\end{array}
\quad \xrightarrow{\text{Rotation}} \quad 
\begin{array}{cccccc}
1 & 2 & 3 & 6 & 9 \\
4 & 5 & 7 & 8 & 10 \\
\end{array}
\]
The action of promotion on these tableaux

This explains why these orbits are so nice; promotion on $2 \times n$ tableaux is really a rotation!
Another slight digression

**Theorem (Haiman 1992)**

*Promotion* on $m \times n$ rectangular standard Young tableaux is of order $mn$.

- There are $278,607,172,289,160$ tableaux in a $5 \times 7$ rectangle, but promotion is of order $7 \cdot 5 = 35$.
- Promotion has order $7,554,844,752$ on tableaux of shape $\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 \\
31 & 32 & 33 & 34 & 35 & 36 \\
\end{array}$. 
The action of rowmotion on these order ideals

But why is this orbit structure so nice?

[Diagram showing orbit structure]
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

---

**Diagram:**

```
    1 2 3 4 7
  5 6 8 9 10
```
Proposition (S., N. Williams 2012)

*There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and toggling left to right on order ideals of the triangular poset $\Phi^+(A_{n-1})$.***
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between **promotion** on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10 \\
\end{array}$
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 7 \\
5 & 6 & 8 & 9 & 10 \\
\end{array}
\]
Proposition (S., N. Williams 2012)

*There is an equivariant bijection between promotion on 2 \times n standard Young tableaux and toggling left to right on order ideals of the triangular poset Φ^+(A_{n-1}).*
There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

\[
\begin{array}{cccccc}
1 & 2 & 3 & 5 & 7 \\
4 & 6 & 8 & 9 & 10 \\
\end{array}
\]
Proposition (S., N. Williams 2012)

There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and toggling left to right on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

![Diagram]

Jessica Striker (NDSU)  
Mind-boggling toggling  
October 28, 2019
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

\[
\begin{array}{cccccc}
1 & 2 & 3 & 6 & 7 \\
4 & 5 & 8 & 9 & 10 \\
\end{array}
\]
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between **promotion** on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

- Mind-boggling toggling
- October 28, 2019
Proposition (S., N. Williams 2012)

There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and toggling left to right on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

1 2 3 6 8
4 5 7 9 10
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

\[
\begin{array}{cccccc}
1 & 2 & 3 & 6 & 8 \\
4 & 5 & 7 & 9 & 10 \\
\end{array}
\]
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

\[ \begin{array}{cccccc}
1 & 2 & 3 & 6 & 9 \\
4 & 5 & 7 & 8 & 10 \\
\end{array} \]
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between **promotion** on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

![Diagram]

Jessica Striker (NDSU)  
Mind-boggling toggling  
October 28, 2019
Proposition (S., N. Williams 2012)

There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and toggling left to right on order ideals of the triangular poset $\Phi^+(A_{n-1})$. 

\[ \begin{array}{cccccc}
1 & 2 & 3 & 6 & 9 \\
4 & 5 & 7 & 8 & 10 \\
\end{array} \]
Proposition (S., N. Williams 2012)

There is an **equivariant bijection** between promotion on $2 \times n$ standard Young tableaux and **toggling left to right** on order ideals of the triangular poset $\Phi^+(A_{n-1})$. So toggling left to right has order $2n$. 

\[
\begin{array}{cccccc}
1 & 2 & 3 & 6 & 9 \\
4 & 5 & 7 & 8 & 10 \\
\end{array}
\]
The action of rowmotion on these order ideals

But why is the orbit structure of rowmotion so nice?
Promotion and rowmotion are conjugate actions

Theorem (N. Williams and S. 2012)

In any ranked poset, there is an equivariant bijection between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).

In an equivariant bijection
the orbit structure is preserved.
The action of promotion on $2 \times 5$ tableaux

Promotion and rowmotion have the same orbit structure!
The action of rowmotion on these order ideals

Promotion and rowmotion have the same orbit structure!
Rowmotion in triangular posets $\Phi^+(A_{n-1})$

Corollary (S., N. Williams 2012)

There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and rowmotion on order ideals of $\Phi^+(A_{n-1})$. So rowmotion has order $2n$.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

![Diagram of a circular and triangular graph with labeled nodes](image)
Corollary (N. Williams and S. 2012)

There is an equivariant bijection between rowmotion on order ideals of $a \times b$ and rotation on binary words of length $a + b$ with $b$ ones. So rowmotion has order $a + b$. 

Jessica Striker (NDSU)
Mind-boggling toggling
October 28, 2019
Rowmotion in box posets $a \times b \times 2$

Theorem (N. Williams and S. 2012)

There is an equivariant bijection between rowmotion on order ideals of $a \times b \times 2$ and rotation on noncrossing partitions of $a + b + 1$ into $b + 1$ blocks. So rowmotion has order $a + b + 1$. 

Promotion $\rightarrow$

Rotation $\rightarrow$
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$. 
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$. 
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$. 

\begin{tabular}{ccc}
1 & 3 & 4 \\
4 & 5 & 6 \\
\end{tabular}
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between \( K \)-promotion on \( a \times b \) increasing tableaux with entries at most \( a + b + c - 1 \) and toggling back to front on \( a \times b \times c \).
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$.
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$. 
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$. 

$$
\begin{array}{ccc}
1 & 3 & 5 \\
3 & 4 & 6 \\
\end{array}
$$
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$. 

\[
\begin{array}{ccc}
1 & 3 & 5 \\
3 & 4 & 6 \\
\end{array}
\]
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$.
Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between $K$-promotion on $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ and toggling back to front on $a \times b \times c$. 

\[
\begin{array}{ccc}
1 & 3 & 5 \\
3 & 4 & 7 \\
\end{array}
\]
Promotion on increasing labelings of a poset

An increasing tableau is an **increasing labeling** of a partition shaped poset.

\[
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 5 \\
\hline
2 & 4 & 5 & \quad \\
\hline
3 & & & \\
\hline
\end{array}
\]

We may define generalized promotion, Inc-promotion, on increasing labelings \( \text{Inc}^q(P) \) of any poset \( P \) with labels in \( \{1, 2, \ldots, q\} \).
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under $\text{Inc}$-promotion and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_r}$. 

\[
\begin{align*}
4 & \quad 6 \\
2 & \quad 4 \\
1 & \quad 4 \\
\end{align*}
\]
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under $\text{Inc}$-promotion and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{\mathcal{H}_r}$.
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under Inc-promotion and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_r}$. 

[Diagram of two sets of nodes]
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under $\text{Inc-promotion}$ and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_\Gamma}$.
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under Inc-promotion and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_r}$. 
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between \( \text{Inc}^q(P) \) under Inc-promotion and order ideals of \( \Gamma(P, q) \) under TogPro\(_{H_r}\).
There is an equivariant bijection between $\text{Inc}^q(P)$ under \textit{Inc-promotion} and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{\mathcal{H}_r}$.
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under Inc-promotion and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_\Gamma}$. 
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under \textit{Inc-promotion} and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_r}$. 

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
6
\end{array}
\begin{array}{c}
5
\end{array}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array} \]
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under $\text{Inc}$-promotion and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_r}$.
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under \text{Inc-promotion} and order ideals of $\Gamma(P, q)$ under $\text{TogPro}_{H_r}$.
Theorem (K. Dilks, S., C. Vorland 2019)

There is an equivariant bijection between $\text{Inc}^q(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, q)$ under rowmotion.
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_r}$.
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_F}$.
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{Hr}$.

\[
\begin{align*}
3 & \leq 5 \leq 6 \\
2 & \leq 2 \leq 4 \\
1 & \leq 1 \leq 2
\end{align*}
\]
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under Inc-pro and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_r}$. 

\[
\begin{align*}
3 & \leq 5 \\
2 & \leq 4 \\
1 & \leq 2
\end{align*}
\]
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

*There is an equivariant bijection between $\text{Inc}^R(P)$ under Inc-pro and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_r}$.***
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

*There is an equivariant bijection between $\text{Inc}^R(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_r}$.***
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_r}$.

\[ 3 \leq 5 \leq 6 \]
\[ 2 \leq 4 \leq 4 \]
\[ 1 \leq 1 \leq 2 \]
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under Inc-pro and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{\mathcal{H}_r}$.
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under Inc-pro and order ideals of $\Gamma(P, R)$ under TogPro$_{H_r}$. 

\[
\begin{align*}
3 &\leq 5 \leq 6 \\
2 &\leq 4 \leq 4 \\
1 &\leq 1 \leq 2
\end{align*}
\]
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

*There is an equivariant bijection between $\text{Inc}^R(P)$ under Inc-pro and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_r}$.*

\[
\begin{align*}
3 & \leq 5 & \leq 6 \\
2 & \leq 4 & \leq 4 \\
1 & \leq 1 & \leq 2
\end{align*}
\]
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

There is an equivariant bijection between $\text{Inc}^R(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_r}$.

\[
\begin{align*}
3 & \leq 6 & \leq 6 \\
2 & \leq 4 & \leq 4 \\
1 & \leq 1 & \leq 2
\end{align*}
\]
We generalize this slightly to incorporate an arbitrary restriction function $R$ on the labels of $P$.

**Theorem (K. Dilks, S., C. Vorland 2019)**

*There is an equivariant bijection between $\text{Inc}^R(P)$ under $\text{Inc-pro}$ and order ideals of $\Gamma(P, R)$ under rowmotion.*

\[
\begin{align*}
3 & \leq 6 & \leq 6 \\
2 & \leq 4 & \leq 4 \\
1 & \leq 1 & \leq 2
\end{align*}
\]
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

*There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear toggling left to right on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+(A_{n-1})$.***
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

There is an equivariant bijection between promotion on \( n \times \ell \) flagged tableaux with flag \((2, 4, \ldots, 2n)\) and piecewise-linear toggling left to right on weakly increasing labelings with labels at most \(\ell\) on the triangular poset \(\Phi^+(A_{n-1})\).
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear toggling left to right on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+(A_{n-1})$.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear toggling left to right on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+(A_{n-1})$. 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

There is an equivariant bijection between promotion on \( n \times \ell \) flagged tableaux with flag \((2, 4, \ldots, 2n)\) and piecewise-linear toggling left to right on weakly increasing labelings with labels at most \( \ell \) on the triangular poset \( \Phi^+(A_{n-1}) \).

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 3 & 3 \\
4 & 5 & 6 \\
\end{array}
\]

Mind-boggling toggling

Jessica Striker (NDSU)

October 28, 2019
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

*There is an equivariant bijection between promotion on \( n \times \ell \) flagged tableaux with flag \((2, 4, \ldots, 2n)\) and piecewise-linear toggling left to right on weakly increasing labelings with labels at most \( \ell \) on the triangular poset \( \Phi^+(A_{n-1}) \).*
We apply this to obtain the following new result.

Theorem (J. Bernstein, S., C. Vorland, 2020+)

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear toggling left to right on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+(A_{n-1})$. 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 4 & 4 \\
3 & 5 & 6 \\
\end{array}
\]
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear toggling left to right on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+(A_{n-1})$.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 4 & 4 \\
3 & 5 & 6 \\
\end{array}
\]
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear toggling left to right on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+(A_{n-1})$. 

```
1  1  1
2  4  4
3  5  6
```
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

*There is an equivariant bijection between promotion on \( n \times \ell \) flagged tableaux with flag \((2, 4, \ldots, 2n)\) and piecewise-linear toggling left to right on weakly increasing labelings with labels at most \( \ell \) on the triangular poset \( \Phi^+(A_{n-1}) \).*
We apply this to obtain the following new result.

Theorem (J. Bernstein, S., C. Vorland, 2020+)

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear toggling left to right on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+ (A_{n-1})$. 

\[
\begin{array}{c|c|c}
1 & 1 & 1 \\
\hline
2 & 4 & 4 \\
\hline
3 & 5 & 6 \\
\end{array}
\]
We apply this to obtain the following new result.

**Theorem (J. Bernstein, S., C. Vorland, 2020+)**

There is an equivariant bijection between *promotion* on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear *rowmotion* on weakly increasing labelings with labels at most $\ell$ on the triangular poset $\Phi^+(A_{n-1})$.
Theorem (J. Bernstein, S., C. Vorland, 2020+)

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear rowmotion on weakly increasing labelings with labels at most $\ell$ on $\Phi^+(A_{n-1})$.

Theorem (D. Grinberg, T. Roby 2015)

Piecewise-linear rowmotion on $\Phi^+(A_{n-1})$ is of order $2n$. 

Jessica Striker (NDSU)

Mind-boggling toggling

October 28, 2019
Theorem (J. Bernstein, S., C. Vorland, 2020+)

There is an equivariant bijection between promotion on $n \times \ell$ flagged tableaux with flag $(2, 4, \ldots, 2n)$ and piecewise-linear rowmotion on weakly increasing labelings with labels at most $\ell$ on $\Phi^+(A_{n-1})$.

Corollary (J. Bernstein, S., C. Vorland, 2020+)

Promotion on these flagged tableaux is of order $2n$.
Mind-boggling toggling

1. Posets and order ideals
2. Toggles
3. Rowmotion
4. Promotion
5. Gyration
Physics connection - Square ice

Consider ice frozen in a square two-dimensional grid as below. Such configurations are in bijection with *alternating sign matrices*.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]
Alternating sign matrix definition

Definition

Alternating sign matrices (ASMs) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$
- each row and each column sums to 1
- nonzero entries alternate in sign along a row/column

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$
Examples of alternating sign matrices

- There are seven $3 \times 3$ ASMs.
  
  \[
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
  \end{pmatrix}
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0
  \end{pmatrix}
  \begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1
  \end{pmatrix}
  \begin{pmatrix}
  0 & 1 & 0 \\
  1 & -1 & 1 \\
  0 & 1 & 0
  \end{pmatrix}
  \begin{pmatrix}
  0 & 1 & 0 \\
  1 & -1 & 1 \\
  0 & 0 & 1
  \end{pmatrix}
  \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 1 & -1 \\
  0 & 0 & 1
  \end{pmatrix}
  \begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 1 & 0
  \end{pmatrix}
  \]

- How many $4 \times 4$ ASMs are there?
  
  \[
  \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  1 & -1 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0
  \end{pmatrix}
  \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  1 & -1 & 1 & 0 \\
  0 & 1 & -1 & 1 \\
  0 & 0 & 1 & 0
  \end{pmatrix}
  \]
Examples of alternating sign matrices

- There are seven $3 \times 3$ ASMs.
  \[
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
\end{pmatrix},
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0 \\
\end{pmatrix},
  \begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1 \\
\end{pmatrix},
  \begin{pmatrix}
  0 & 1 & 0 \\
  1 & -1 & 1 \\
  0 & 1 & 0 \\
\end{pmatrix},
  \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0 \\
\end{pmatrix},
  \begin{pmatrix}
  0 & 1 & 0 \\
  1 & -1 & 1 \\
  0 & 1 & 0 \\
\end{pmatrix},
  \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}
  \]

- How many $4 \times 4$ ASMs are there? 42
  \[
  \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  1 & -1 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
\end{pmatrix},
  \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  1 & -1 & 1 & 0 \\
  0 & 1 & -1 & 1 \\
  0 & 0 & 1 & 0 \\
\end{pmatrix}
  \]
Alternating sign matrices

Theorem (D. Zeilberger 1996; G. Kuperberg 1996)

\( n \times n \) alternating sign matrices are counted by:

\[
\prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \frac{1!4!7! \cdots (3n - 2)!}{n!(n + 1)! \cdots (2n - 1)!}.
\]

1, 2, 7, 42, 429, 7436, 218348, 10850216, . . .

This was conjectured by Mills, Robbins, and Rumsey in 1983 and proved in different ways by Zeilberger and Kuperberg. Kuperberg’s proof relied on a beautiful bijection with square ice configurations.
Alternating sign matrix

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}
\]
Alternating sign matrix $\leftrightarrow$ fully-packed loop
Fully-packed loop
Fully-packed loops

Start with an $n \times n$ grid.

\[ \begin{array}{cccccc}
    . & . & . & . & . & . \\
    . & . & . & . & . & . \\
    . & . & . & . & . & . \\
    . & . & . & . & . & . \\
    . & . & . & . & . & . \\
\end{array} \]
Fully-packed loops

Add boundary conditions.
Fully-packed loops

Interior vertices adjacent to 2 edges.
Gyration on fully-packed loops

The local move.

\[\begin{array}{ccc}
& \text{\textbullet} & \\
\text{\textbullet} & \rightarrow & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \\
\end{array}\]
Gyration on fully-packed loops

[Diagram of gyration on fully-packed loops]
Gyration on fully-packed loops

Start with the even squares.
Gyration on fully-packed loops

Apply the local move.
Gyration on fully-packed loops

Apply the local move.

- Diagram showing the gyration on fully-packed loops.
Apply the local move.
Now consider the odd squares.
Gyration on fully-packed loops

Apply the local move.
Gyration on fully-packed loops

Apply the local move.
Gyration on fully-packed loops

Apply the local move.
Gyration on fully-packed loops

Jessica Striker (NDSU)
Mind-boggling toggling
October 28, 2019
Gyration on fully-packed loops

Gyration

Rotation

Jessica Striker (NDSU)
Mind-boggling toggling
October 28, 2019
The square is a circle

**Theorem (B. Wieland 2000)**

*Gyration on an order n fully-packed loop rotates the link pattern by an angle of $\frac{2\pi}{2n}$.***

![Diagram showing the rotation and gyration of a fully-packed loop](image)
How does this relate to toggles?
Gyration on the alternating sign matrix poset
Gyration on the alternating sign matrix poset
Gyration on the alternating sign matrix poset
Gyration on the alternating sign matrix poset
Gyration on the alternating sign matrix poset

Mind-boggling toggling
October 28, 2019
Gyration on the alternating sign matrix poset

\[\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 0 \\
2 & 3 & 4 & 5 & 0 & 1 \\
3 & 4 & 5 & 0 & 1 & 2 \\
4 & 5 & 0 & 1 & 2 & 3 \\
5 & 0 & 1 & 2 & 3 & 4 \\
\end{array}\]
Gyration on the alternating sign matrix poset
Gyration on the alternating sign matrix poset
Gyration on the alternating sign matrix poset
Gyration on the alternating sign matrix poset
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

**Gyration on fully-packed loops is equivalent to toggling even then odd ranks in the alternating sign matrix poset.**
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

**Gyration** on fully-packed loops is equivalent to **toggling even then odd ranks** in the alternating sign matrix poset.
Gyration as a toggle group action

**Theorem (N. Williams and S. 2012)**

*Gyration on fully-packed loops is equivalent to toggling even then odd ranks in the alternating sign matrix poset.*
The conjugacy of rowmotion, promotion, and gyration

Theorem (N. Williams and S. 2012)

In any ranked poset, there are equivariant bijections between the order ideals under rowmotion (toggle top to bottom), promotion (toggle left to right), and gyration (toggle evens then odds).

Rowmotion, promotion, and gyration all have the same orbit structure!
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

**Gyration** on fully-packed loops has the same orbit structure as **rowmotion** on order ideals of the alternating sign matrix poset.
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

*Gyration* on fully-packed loops has the same orbit structure as *rowmotion* on order ideals of the alternating sign matrix poset.
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

*Gyration* on fully-packed loops has the same orbit structure as *rowmotion* on order ideals of the alternating sign matrix poset.
Theorem (N. Williams and S. 2012)

Gyration on fully-packed loops has the same orbit structure as rowmotion on order ideals of the alternating sign matrix poset.
Fully-packed loop orbits under gyration
Order ideals in the ASM poset under rowmotion
Mind Boggling Toggling

1. Posets and order ideals
2. Toggles
3. Rowmotion
4. Promotion
5. Gyration
THANKS