Dynamical algebraic combinatorics: Resonance

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Dynamical algebraic combinatorics: Resonance

1. Promotion and rowmotion
2. Resonance defined
3. Multidimensional promotion and rowmotion
4. Increasing labeling promotion and rowmotion
Dynamical algebraic combinatorics: Resonance

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Toggles act on order ideals

Define a toggle, $t_e$, for each $e \in P$. 
Toggles act on order ideals

Toggles $t_e$ add $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ add $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ remove $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ remove $e$ when possible.
Toggles act on order ideals

Toggles $t_e$ do nothing otherwise.
Toggles act on order ideals

Toggles $t_e$ do nothing otherwise.
Rowmotion

Definition

The **rowmotion** of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$. 

An order ideal $X$

![Diagram of an order ideal X](image-url)
The \textbf{rowmotion} of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Find the \textbf{minimal} elements of $P$ not in $X$. 
Rowmotion

Definition

The rowmotion of an order ideal $X$ is the order ideal generated by the minimal elements of $P$ not in $X$.

Use them to generate a new order ideal $\text{Row}(X)$. 
Rowmotion on order ideals is a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Rowmotion on order ideals is a product of toggles

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Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let $P$ be a poset and $X$ an order ideal of $P$. Then rowmotion on $X$ is the order ideal obtained from $X$ by toggling the elements of $P$ from top to bottom.
Promotion and rowmotion are conjugate actions

\textbf{Theorem (N. Williams and S. 2012)}

\textit{In any ranked poset, there are equivariant bijections between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).}

In an \textit{equivariant} bijection, the orbit structure is preserved.
Rowmotion in rectangular posets $a \times b$

**Corollary (N. Williams and S. 2012)**

There is an equivariant bijection between order ideals of $a \times b$ under rowmotion and binary words of length $a + b$ with $b$ ones under a cyclic shift. So rowmotion has order $a + b$ and exhibits the cyclic sieving phenomenon.
Theorem (N. Williams and S. 2012)

There is an equivariant bijection between \( J(a \times b \times 2) \) under rowmotion and noncrossing partitions of \( a + b + 1 \) into \( b + 1 \) blocks under rotation. So rowmotion has order \( a + b + 1 \) and exhibits the cyclic sieving phenomenon.
There is an equivariant bijection between $J(a \times b \times 2)$ under rowmotion and noncrossing partitions of $a + b + 1$ into $b + 1$ blocks under rotation. So rowmotion has order $a + b + 1$ and exhibits the cyclic sieving phenomenon.
What is the order of rowmotion on $a \times b \times c$?

- The order of rowmotion on $a \times b = a \times b \times 1$ is $a + b$.
- The order of rowmotion on $a \times b \times 2$ is $a + b + 1$.
- The order of rowmotion on $a \times b \times 3$ is $a + b + 2$ for as high as we can check.
- This may lead one to conjecture that the order of rowmotion on $a \times b \times c$ is $a + b + c - 1$.
- But the orbits of rowmotion on $4 \times 4 \times 4$ are of size 11 and 33.
Cyclic sieving of rowmotion

Definition (V. Reiner, D. Stanton, D. White 2004)

The triple $(S, f(q), g)$ for a set $S$, a polynomial $f(q)$, and a bijective action $g$ of order $n$, exhibits the cyclic sieving phenomenon when the evaluation of $f$ at $\zeta^d$, where $\zeta = e^{2\pi i/n}$, counts the elements of $S$ fixed under $g^d$.

Cyclic sieving occurs with respect to rowmotion in the following posets:

- Minuscule posets ($a \times b$ is type A) (D. Rush, X. Shi)
- Minuscule poset $\times 2$ (D. Rush, X. Shi 2013)
- Positive root posets ($A_b$ is type A) (D. Armstrong, C. Stump, H. Thomas 2013)

$\rightarrow$ Cyclic sieving does not occur in general for $a \times b \times c$. 
References on cyclic sieving

- Sage code: Computes the unique polynomial of degree less than \( n \) such that the given set and action exhibit the CSP. http://doc.sagemath.org/html/en/reference/combinat/sage/combinat/cyclic_sieving_phenomenon.html
What is the order of rowmotion on $a \times b \times c$?

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Resonance in musical instruments

Definition (Dilks, Pechenik, Striker 2015+)

Let $G = \langle g \rangle$ be a cyclic group acting on a set $X$, $C_\omega = \langle c \rangle$ a cyclic group of order $\omega$ acting nontrivially on a set $Y$, and $f : X \to Y$ a surjection. If $c \cdot f(x) = f(g \cdot x)$ for all $x \in X$, we say the triple $(X, G, f)$ exhibits resonance with frequency $\omega$. 

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow f & & \downarrow f \\
Y & \xleftarrow{C} & Y
\end{array}
\]
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Promotion and rowmotion are conjugate actions

Theorem (N. Williams and S. 2012)

In any ranked poset, there are equivariant bijections between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).

Question: What do we mean by ‘left-to-right’?
A **lattice projection** of a poset $P$ is an order and rank preserving map $\pi : P \rightarrow \mathbb{Z}^n$, where $x \leq y$ in $\mathbb{Z}^n$ if and only if the component-wise difference $y - x$ is in $\mathbb{N}^n$. 
Multidimensional promotion and rowmotion

Definition

A lattice projection of a poset $P$ is an order and rank preserving map $\pi : P \to \mathbb{Z}^n$, where $x \leq y$ in $\mathbb{Z}^n$ if and only if the component-wise difference $y - x$ is in $\mathbb{N}^n$. 
Multidimensional promotion and rowmotion

Definition

A **lattice projection** of a poset $P$ is an order and rank preserving map $\pi : P \rightarrow \mathbb{Z}^n$, where $x \leq y$ in $\mathbb{Z}^n$ if and only if the component-wise difference $y - x$ is in $\mathbb{N}^n$. 

![Diagram showing lattice projection and poset structures]
Multidimensional promotion and rowmotion

Definition

Let $P$ be a poset with an $n$-dimensional lattice projection $\pi$, and let $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_n)$, where $\nu_j \in \{\pm 1\}$. Let $T^i_{\pi,\nu}$ be the product of toggles $t_x$ for all elements $x$ of $P$ that lie on the affine hyperplane $\langle \pi(x), \nu \rangle = i$. Then define **promotion with respect to $\pi$ and $\nu$** as

$$\text{Pro}_{\pi,\nu} = \ldots T^{-2}_{\pi,\nu} T^{-1}_{\pi,\nu} T^0_{\pi,\nu} T^1_{\pi,\nu} T^2_{\pi,\nu} \ldots .$$
\[\text{Pro}_{\text{id},(1,1,-1)} = T_{\pi,v}^{-2} T_{\pi,v}^{-1} T_{\pi,v}^0 T_{\pi,v}^1 T_{\pi,v}^2 T_{\pi,v}^3\]

\[x + y - z = -2\]
\[x + y - z = -1\]
\[x + y - z = 0\]

\[x + y - z = 1\]
\[x + y - z = 2\]
\[x + y - z = 3\]
Multidimensional promotion and rowmotion

**Definition**

Let $P$ be a poset with an $n$-dimensional lattice projection $\pi$, and let $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_n)$, where $\nu_j \in \{\pm 1\}$. Let $T_{\pi,\nu}^i$ be the product of toggles $t_x$ for all elements $x$ of $P$ that lie on the affine hyperplane $\langle \pi(x), \nu \rangle = i$. Then define **promotion with respect to $\pi$ and $\nu$** as

$$\text{Pro}_{\pi,\nu} = \ldots \ T_{\pi,\nu}^{-2} \ T_{\pi,\nu}^{-1} \ T_{\pi,\nu}^0 \ T_{\pi,\nu}^1 \ T_{\pi,\nu}^2 \ \ldots$$

**Proposition**

For any finite ranked poset $P$ and lattice projection $\pi$, 

$$\text{Pro}_{\pi,(1,1,\ldots,1)} = \text{Row}.$$
Multidimensional promotion and rowmotion

Theorem (K. Dilks, O. Pechenik, S. 2017)

Let \( P \) be a finite poset with an \( n \)-dimensional lattice projection \( \pi \). Let \( v = (v_1, v_2, v_3, \ldots, v_n) \) and \( w = (w_1, w_2, w_3, \ldots, w_n) \), where \( v_j, w_j \in \{ \pm 1 \} \). Then there is an equivariant bijection between \( J(P) \) under \( \text{Pro}_{\pi,v} \) and \( J(P) \) under \( \text{Pro}_{\pi,w} \).

Rowmotion and \( 2^n - 1 \) other promotions have the same orbit structure!
Multidimensional promotion and rowmotion

**Theorem (K. Dilks, O. Pechenik, S. 2017)**

Let $P$ be a finite poset with an $n$-dimensional lattice projection $\pi$. Let $v = (v_1, v_2, v_3, \ldots, v_n)$ and $w = (w_1, w_2, w_3, \ldots, w_n)$, where $v_j, w_j \in \{\pm 1\}$. Then there is an equivariant bijection between $J(P)$ under $\text{Pro}_{\pi, v}$ and $J(P)$ under $\text{Pro}_{\pi, w}$.

Rowmotion and $2^n - 1$ other promotions have the same orbit structure!

**Proof.**

Similar argument as in [N. Williams and S. 2012].
Increasing tableaux

Definition

An **increasing tableau** of shape $\lambda$ is a filling of a partition shape $\lambda$ with positive integers so that labels strictly increase from left to right across rows and from top to bottom down columns. Let $\text{Inc}^q(\lambda)$ denote the set of increasing tableaux of shape $\lambda$ with entries at most $q$.

An increasing tableau in $\text{Inc}^{10}(4,4,4,2)$:

$$
\begin{array}{cccc}
1 & 4 & 5 & 8 \\
2 & 5 & 7 & 9 \\
6 & 7 & 9 & 10 \\
8 & 10
\end{array}
$$
### $K$-Promotion on an increasing tableau

<table>
<thead>
<tr>
<th>Original Tableau</th>
<th>Delete 1's</th>
<th>Fill and Decrement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 4 6 4 5 6 7</td>
<td>2 4 6 4 5 6 7</td>
<td>1 3 5 6 3 4 6 7</td>
</tr>
</tbody>
</table>

- **Delete 1's:**
  - Original: 1 2 4 6 4 5 6 7
  - After: 2 4 6 4 5 6 7

- **Fill and Decrement:**
  - Original: 2 4 6 4 5 6 7
  - After: 1 3 5 6 3 4 6 7
Resonance of $K$-promotion

Theorem (K. Dilks, O. Pechenik, S. 2017)

$(\text{Inc}^q(\lambda), \langle K\text{-Pro} \rangle, \text{Con})$ exhibits resonance with frequency $q$, that is, there is a projection from an increasing tableau to its binary content vector such that $K$-promotion maps to a cyclic shift.

\[
\begin{array}{cccc}
1 & 2 & 4 & 7 \\
3 & 5 & 6 & 8 \\
5 & 7 & 8 & 10 \\
7 & 9 & 10 & 12 \\
\end{array}
\quad\quad\quad
\begin{array}{cccc}
1 & 3 & 5 & 6 \\
2 & 4 & 7 & 9 \\
4 & 6 & 9 & 11 \\
6 & 8 & 11 & 12 \\
\end{array}
\]

$K$-Promotion

\[
\begin{array}{cccc}
1, 1, 1, 1, 1, 1, 1, 1, 0, 1 \\
1, 1, 1, 1, 1, 1, 1, 1, 0, 1 \\
1, 1, 1, 1, 1, 1, 1, 0, 1, 1 \\
\end{array}
\quad\quad\quad
\begin{array}{cccc}
1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1 \\
1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1 \\
1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1 \\
\end{array}
\]

Cyclic shift
Proposition (K. Dilks, O. Pechenik, S. 2017)

\[ K\text{-Promotion is the product } \prod_i K\text{-BK}_i, \text{ where } K\text{-BK}_i \text{ increments } i \text{ and/or decrements } i + 1 \text{ when possible.} \]
Proposition (K. Dilks, O. Pechenik, S. 2017)

*K-Promotion* is the product $\prod_i K \cdot \text{BK}_i$, where $K \cdot \text{BK}_i$ increments $i$ and/or decrements $i + 1$ when possible.
$K$-Promotion on as a product of involutions

Proposition (K. Dilks, O. Pechenik, S. 2017)

$K$-Promotion is the product $\prod_i K$-$BK_i$, where $K$-$BK_i$ increments $i$ and/or decrements $i + 1$ when possible.
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**Proposition (K. Dilks, O. Pechenik, S. 2017)**

*K-Promotion is the product* \( \prod_i K\text{-}BK_i \), *where* \( K\text{-}BK_i \) *increments* \( i \) *and/or increments* \( i + 1 \) *when possible.*

\[
\begin{array}{cccc}
1 & 3 & 4 & 6 \\
3 & 5 & 6 & 7 \\
\end{array}
\]
K-Promotion on as a product of involutions

Proposition (K. Dilks, O. Pechenik, S. 2017)

\[ K\text{-Promotion is the product } \prod_i K\text{-BK}_i, \text{ where } K\text{-BK}_i \text{ increments } i \text{ and/or decrements } i + 1 \text{ when possible.} \]
**K-Promotion on as a product of involutions**

Proposition (K. Dilks, O. Pechenik, S. 2017)

*K-Promotion is the product* \( \prod_i K \cdot BK_i \), *where* \( K \cdot BK_i \) *increments* \( i \) *and/or decrements* \( i + 1 \) *when possible.*
**Proposition (K. Dilks, O. Pechenik, S. 2017)**

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\[
\begin{array}{cccc}
1 & 3 & 5 & 6 \\
3 & 4 & 6 & 7 \\
\end{array}
\]
A bijection between increasing tableaux $\text{Inc}^{a+b+c-1}(a \times b)$ and plane partitions $J(a \times b \times c)$
An equivariant bijection between $\text{Inc}^{a+b+c-1}(a \times b)$ and $J(a \times b \times c)$

**Theorem (K. Dilks, O. Pechenik, S. 2017)**

$J(a \times b \times c)$ under $\text{Row}$ is in equivariant bijection with $\text{Inc}^{a+b+c-1}(a \times b)$ under $K$-$\text{Pro}$.

**Lemma (K. Dilks, O. Pechenik, S. 2017)**

$\Psi_3$ intertwines $\text{Pro}_{\text{id},(1,1,-1)}$ and $K$-$\text{Pro}$.
Each hyperplane-toggle corresponds to a $K\cdot BK_i$ involution.
Resonance of rowmotion on $a \times b \times c$

**Theorem (K. Dilks, O. Pechenik, S. 2017)**

*Rowmotion on order ideals of $a \times b \times c$ exhibits resonance with frequency $a + b + c - 1$, that is, there is a projection from an order ideal in $a \times b \times c$ through the corresponding increasing tableau to its binary content vector such that rowmotion maps to a cyclic shift.*

So even though rowmotion on $a \times b \times c$ does not have order $a + b + c - 1$, it projects to something that does!
Corollaries of the equivariant bijection

**Theorem (K. Dilks, O. Pechenik, S. 2017)**

\[ J(a \times b \times c) \text{ under } \text{Row} \text{ is in equivariant bijection with } \text{Inc}^{a+b+c-1}(a \times b) \text{ under } K\text{-Pro}. \]

**Corollary (K. Dilks, O. Pechenik, S. 2017)**

There are $K$-Pro-equivariant bijections between $\text{Inc}^{a+b+c-1}(\lambda)$ for $\lambda = a \times b, a \times c, \text{ or } b \times c$. 
Corollaries of the equivariant bijection

Corollary (K. Dilks, O. Pechenik, S. 2017)

There are $K$-Pro-equivariant bijections between the sets
\[ \text{Inc}^{a+b+c-1}(a \times b), \text{Inc}^{a+b+c-1}(a \times c), \text{and} \]
\[ \text{Inc}^{a+b+c-1}(b \times c). \]

Corollary

The order of $K$-Pro on $\text{Inc}^{a+b}(a \times b)$ is $a + b$.

Proof.

Using the tri-fold symmetry, there is a $K$-Pro-equivariant bijection between the sets $\text{Inc}^{a+b}(a \times b)$ and $\text{Inc}^{a+b}(1 \times a)$. The result is then immediate.
Corollaries of the equivariant bijection

Corollary

*The order of $K$-Pro on $\text{Inc}^{a+b+1}(a \times b)$ is $a + b + 1$.*

Proof.

Using the tri-fold symmetry, there is a $K$-Pro-equivariant bijection between the sets $\text{Inc}^{a+b+1}(a \times b)$ and $\text{Inc}^{a+b+1}(2 \times a)$. The result is then immediate by a result of Pechenik on increasing tableaux or by the result of Cameron and Fon-der-Flaass on $J(a \times b \times 2)$. 

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Conjecture

The order of Row on $J(a \times b \times 3)$ is $a + b + 2$.

This is equivalent to the order of $K$-promotion being $a + b + 2$ on either $\text{Inc}^{a+b+2}(a \times b)$ or $\text{Inc}^{a+b+2}(3 \times a)$. We have verified conjecture for $a \leq 7$ and $b$ arbitrary.
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Promotion on increasing labelings of a poset

An increasing tableau is an **increasing labeling** of a partition shaped poset.

We may define generalized promotion on increasing labelings of any poset by either generalized *Bender-Knuth* involutions.
Increasing Labelings

Definition

An *increasing labeling* of $P$ is a function $f : P \rightarrow \mathbb{Z}$ such that $p_1 <_P p_2$ implies $f(p_1) < f(p_2)$.

Definition

For labeling function $R : P \mapsto \mathcal{P}(\mathbb{Z})$, let $\text{Inc}^R(P)$ be the set of increasing labelings of $P$ such that for all $p \in P$, $f(p) \in R(p)$.

\[
\text{Inc}^R(P) : \quad \begin{array}{c}
\text{a} \quad \{1,4\} \\
\text{b} \quad \{2,3,5\} \\
\text{c} \quad \{2,4,5\} \\
\text{d} \quad \{3,4,5,6\} \\
\text{e} \quad \{4,6,7,9\}
\end{array}
\]
Increasing Labelings with entries in \(\{1, \ldots, q\}\)

**Definition**

An *increasing labeling* of \(P\) is a function \(f : P \to \mathbb{Z}\) such that \(p_1 <_P p_2\) implies \(f(p_1) < f(p_2)\).

**Definition**

Let \(\text{Inc}^q(P)\) be the set of increasing labelings of \(P\) with entries in \(\{1, \ldots, q\}\).

\[
\text{Inc}^6(P) : \quad \{1,2,3\} \quad \{2,3,4,5\} \quad \{4,5,6\}
\]
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\text{Inc}^q(P)$ is the product $\text{IncPro} = \prod_{i=1}^{q-1} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_i$ increments $i$ and/or decrements $i+1$ when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_1$ increments 1 and/or decrements 2 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_1$ increments 1 and/or decrements 2 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_2$ increments 2 and/or decrements 3 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_2$ increments 2 and/or decrements 3 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_3$ increments 3 and/or decrements 4 when possible.

![Diagram of promotion on labelings]

Dynamical algebraic combinatorics

Jessica Striker (NDSU)
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_3$ increments 3 and/or decrements 4 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_4$ increments 4 and/or decrements 5 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_4$ increments 4 and/or decrements 5 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_5$ increments 5 and/or decrements 6 when possible.
Promotion on increasing labelings $\text{Inc}^{q}(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^{8}(P)$ is the product $\prod_{i=1}^{7} \rho_{i}$ of the generalized Bender-Knuth involutions $\rho_{i}$, where $\rho_{5}$ increments 5 and/or decrements 6 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^7 \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_6$ increments 6 and/or decrements 7 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_6$ increments 6 and/or decrements 7 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_7$ increments 7 and/or decrements 8 when possible.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized Bender-Knuth involutions

**Promotion** on $\text{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions $\rho_i$, where $\rho_7$ increments 7 and/or decrements 8 when possible.
Promotion on increasing labelings $\text{Inc}^R(P)$ by generalized Bender-Knuth involutions

The generalized Bender-Knuth involution $\rho_i$ for an arbitrary restriction function $R$ acts as follows:

- If a label is currently $i$, and you can increment it to next allowable label (and stay in $\text{Inc}^R(P)$), then do so.
- If you can decrement a label to become $i$ (and stay in $\text{Inc}^R(P)$), then do so.
- Otherwise, do nothing.

Then generalized increasing labeling promotion on $\text{Inc}^R(P)$ is: $\text{IncPro} = \cdots \circ \rho_2 \circ \rho_1 \circ \cdots$. 
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Delete any 1’s.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 2’s. Fill any blank spot that has a 2 covering it by a 2. Simultaneously, replace any 2’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 3’s. Fill any blank spot that has a 3 covering it by a 3. Simultaneously, replace any 3’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 4’s. Fill any blank spot that has a 4 covering it by a 4. Simultaneously, replace any 4’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 4’s. Fill any blank spot that has a 4 covering it by a 4. Simultaneously, replace any 4’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 5’s. Fill any blank spot that has a 5 covering it by a 5. Simultaneously, replace any 5’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 5’s. Fill any blank spot that has a 5 covering it by a 5. Simultaneously, replace any 5’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 6’s. Fill any blank spot that has a 6 covering it by a 6. Simultaneously, replace any 5’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 7’s. Fill any blank spot that has a 7 covering it by a 7. Simultaneously, replace any 7’s covering blank spots by a blank spot.

\[\begin{array}{cccc}
7 & 8 & 6 & 7 \\
5 & 3 & 4 & \\
\end{array}\]
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 7’s. Fill any blank spot that has a 7 covering it by a 7. Simultaneously, replace any 7’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 8’s. Fill any blank spot that has a 8 covering it by a 8. Simultaneously, replace any 8’s covering blank spots by a blank spot.
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

When all blank spots are maximal, fill the spots with $q + 1$. 

\[ \begin{array}{cccccccc}
7 & 9 & 8 & 9 & 7 & 6 & 3 & 4 \\
5 & 7 & 8 & 6 & 4 & & & \\
7 & & & & & & & \\
\end{array} \]
Promotion on increasing labelings $\text{Inc}^q(P)$ by generalized jeu de taquin slides

Subtract 1 from all labels.
Theorem (K. Dilks, S., C. Vorland 2018+)

The two definitions of promotion on increasing labelings $\text{Inc}^q(P)$ coincide.

Corollary (K. Dilks, S., C. Vorland 2018+)

$(\text{Inc}^P(q), \langle \text{IncPro} \rangle, \text{Con})$ exhibits resonance with frequency $q$. 
Goal
Relate increasing labeling promotion to rowmotion on the order ideals of some poset.

Idea:

- $\text{Inc}^R(P)$ can be partially ordered by componentwise comparison.
- $\text{Inc}^R(P)$ is a distributive lattice where the meet and join are given by taking componentwise min and max.
- Use the Fundamental Theorem of Finite Distributive Lattices.
- Determine how to construct $\Gamma(P, R)$, the subposet of join irreducibles.
Theorem (K. Dilks, S., C. Vorland 2018+)

Order ideals of $\Gamma(P, R)$ are in bijection with the increasing labelings $\text{Inc}^R(P)$.
An equivariant bijection between $\text{Inc}^R(P)$ and $J(\Gamma(P, R))$

**Definition**

$H : P \to \mathbb{Z}$ is a **toggle order** if $p_1 < p_2$ implies $H(p_1) \neq H(p_2)$. Given a toggle order $H$, define $T_H^i$ to be the toggle group action that is the product of all $t_p$ for $p \in P$ such that $H(p) = i$.

**Definition**

**Toggle-promotion** with respect to a toggle order $H$, denoted $\text{TogPro}_H$, is ... $T_H^{-2} T_H^{-1} T_H^0 T_H^1 T_H^2$ ...

**Theorem (K. Dilks, S., C. Vorland 2018+)**

Let $H_\Gamma : \Gamma(P, R) \to \mathbb{Z}$ be the toggle order taking $(p, k)$ to $k$. Then $\text{Inc}^R(P)$ under $\text{IncPro}$ is in equivariant bijection with order ideals of $\Gamma(P, R)$ under $\text{TogPro}_{H_\Gamma}$. 

Jessica Striker (NDSU)  
Dynamical algebraic combinatorics  
March 14, 2018
Each $\rho_i$ involution corresponds to a generalized toggle $T^i_H$.
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Each $\rho_i$ involution corresponds to a generalized toggle $T^i_H$. 

\begin{center}
\begin{tikzpicture}
\node[red] at (0,0) (2) {2};
\node[below of=2] (1) {1};
\node[above of=1] (4) {4};
\node[below of=4] (6) {6};
\node[below of=6] (4) {4};
\draw (2) -- (1);
\draw (2) -- (4);
\draw (4) -- (6);
\end{tikzpicture}
\end{center}
Each $\rho_i$ involution corresponds to a generalized toggle $T^i_H$.
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Each $\rho_i$ involution corresponds to a generalized toggle $T^i_H$. 
Each $\rho_i$ involution corresponds to a generalized toggle $T_H^i$.
How does TogPro relate to rowmotion?

**Definition**

\[ H : P \rightarrow \mathbb{Z} \] is a column toggle order if whenever \( p_1 \preceq p_2 \) in \( P \), then \( H(p_1) = H(p_2) \pm 1 \).

**Theorem (K. Dilks, S., C. Vorland 2018+)**

When \( H_{\Gamma} \) is a column toggle order, there is an equivariant bijection between \( \text{Inc}^R(P) \) under \( \text{IncPro} \) and \( J(\Gamma(P, R)) \) under \( \text{Row} \).

**Corollary**

There is an equivariant bijection between \( \text{Inc}^q(P) \) under \( \text{IncPro} \) and order ideals in \( \Gamma(P, q) \) under \( \text{Row} \).
How does TogPro relate to rowmotion?

Corollary

Let $P$ be a ranked poset with a Cartesian embedding into ranked posets $(P_1, P_2)$. Let $H$ map the element of $P$ embedded at coordinate $(p_1, p_2)$ to the difference of ranks $\text{rk}_{P_1}(p_1) - \text{rk}_{P_2}(p_2)$. Then there is an equivariant bijection on $J(P_1 \times P_2)$ between $\text{TogPro}_H$ and $\text{Row}$.

Hyperplane promotion $\text{Pro}_{\pi, \nu}$ with respect to a lattice embedding $\pi$ can be thought of a special case of this.
Conclusion

We have seen examples of higher-dimensional combinatorial objects with natural actions that no longer have a nice order, but rather exhibit resonance.

Question

Can we find an analogues of cyclic sieving and homomesy for combinatorial actions that exhibit resonance?

In tomorrow’s talk, we will see another example of toggling and resonance in the setting of alternating sign matrices, along with some additional open questions.
Dynamical algebraic combinatorics: Resonance

1 Promotion and rowmotion

2 Resonance defined

3 Multidimensional promotion and rowmotion

4 Increasing labeling promotion and rowmotion

