Susan M. Cooper - Research Statement

1. Introduction and Overview

My research is motivated by the links between commutative algebra, algebraic geometry, and combinatorics[1]. My work has involved topics such as the Eisenbud–Green–Harris Conjecture [9, 10], determinantal schemes [11] and enumerating semidualizing modules [13]. I am especially interested in obtaining information about a scheme in projective n-space, $\mathbb{P}^n$, using tools which encode algebraic invariants associated to the homogeneous ideal defining the scheme. Examples of applications come from areas such as coding theory, graph theory, combinatorial commutative algebra, and computational commutative algebra. These connections lead to exciting research and collaborations at all levels.

To introduce one key invariant, consider the polynomial ring $R = k[x_0, \ldots, x_n]$ where k is an algebraically closed field of characteristic zero (e.g., $R = \mathbb{C}[x_0, \ldots, x_n]$). We say that $F \in R$ is **homogeneous** if every term of F has the same degree. An ideal $I \subseteq R$ is homogeneous if it can be generated by a set of homogeneous polynomials. We group the elements of I by degree which results in a collection of finite dimensional vector spaces. For example, let $I = (F, G) \subseteq k[x_0, x_1, x_2]$ where $F = x_0 - x_2$ and $G = x_1^2 - x_1x_2$. The set $\{x_0F, x_1F, x_2F, G\}$ forms a k-basis for the space of degree 2 homogeneous polynomials in I, and so we write $\dim_k I_2 = 4$. The degree-by-degree dimensions $\dim_k I_d$ of a homogeneous ideal $I \subseteq R$ give a function of d, called the Hilbert function, introduced by David Hilbert [32] in his work in invariant theory.

Definition 1.1. Let $I \subseteq R$ be a homogeneous ideal. The **Hilbert function** of R/I, denoted $H(R/I)$, is the function where $H(R/I, d) := \dim_k(R/I)_d = \dim_k R_d - \dim_k I_d = \binom{n+d}{d} - \dim_k I_d$ for $d \geq 0$.

Related to the Hilbert function are invariants obtained by looking at the relations on the generators of I, and the relations on these relations (called the syzygies), etc. When we study these relations we consider invariants called the **graded Betti numbers for** I.

Question 1.2. What does the zero-locus of an ideal tell us about its Hilbert function and vice versa? For the graded Betti numbers?

My research has been driven by this type of question with a focus on ideals of sets of points[2] in $\mathbb{P}^n$ and monomial ideals. On a first encounter, 0-dimensional schemes may seem to be devoid of substance. However, to the contrary, the study of points has intrigued mathematicians for hundreds of years. For example, Castelnuovo showed that non-trivial information about a curve can be retrieved from the points arising as the general hyperplane sections of the curve [20]. Although the study of points has a deep history (touching questions such as Hilbert's 14th Problem), it manages to host many exciting open problems [7]. The study of monomial ideals has a similar deep history with many connections to graph theory and computational commutative algebra. This note focuses on my past, current, and future work in the following three settings:

(a) At the heart of many open problems in commutative algebra and algebraic geometry is the difference between a symbolic power $I^{(m)}$ (defined below) and a regular power I^r of a homogeneous ideal $0 \neq I \subseteq R$. In 2013 B. Harbourne and C. Huneke [30] announced a number of conjectures relating these powers for ideals of point sets in $\mathbb{P}^n$. However, the conjectures have also received much attention in the non-geometric case. Indeed, the computation of symbolic powers of monomial ideals can be used to find symbolic powers of other arbitrary ideals (such as Pfaffian ideals) via Gröbner bases. In two joint projects [4, 14], we study **symbolic powers of monomial ideals** and compare them with regular powers, recovering all but one

[1] MSC2010: 13A02, 13C05, 13C14, 13C40, 13D02, 13D40, 13F20, 13H15, 13P99, 14M06, 14M12, 14N05.

[2] A point in $\mathbb{P}^n$ is the set of non-zero $(n+1)$-tuples of elements of the field k modulo the relation $(s_0, \ldots, s_n) \sim (ts_0, \ldots, ts_n)$ for $0 \neq t \in k$.

of the conjectures of Harbourne and Huneke for square-free monomial ideals. We also consider the **Waldschmidt constant** (which is a limit related to failure of containments for these powers) for square-free monomial ideals and relate it to a simple **linear program**, allowing us to connect the Waldschmidt constant to properties of graphs for special families of graphs. Moreover, in order to work with a non-square-free monomial ideal I, we introduce the **symbolic polyhedron** which is a convex polyhedron in $\mathbb{R}^{n+1}$ with the property that when scaled by a factor of m it contains the exponent vectors of all monomials in the mth symbolic power $I^{(m)}$.

(b) To many fat point schemes (defined below) we can associate a linear code. Much work has been done on **relating the minimum Hamming distance of the code and homological data of the ideal defining the fat point scheme**. Our goal is to further this investigation for non-homogeneous fat point schemes [15].

(c) **Hilbert functions of fat point schemes** are not well-understood. However, in joint work with B. Harbourne and Z. Teitler [12], we produce easy to compute upper and lower bounds for Hilbert functions of fat points in $\mathbb{P}^2$ and give an explicit criterion for when these bounds coincide. The result is then applied to obtain upper and lower bounds for graded Betti numbers. Although still largely unexplored, the underlying approach can be applied to give bounds for fat points in any projective space. These results are proving to be fruitful in considering conjectures about the **initial degree** of a fat point scheme [5, 16].

2. The Ideal Containment Problem for Monomial Ideals

Understanding the difference between a symbolic power $I^{(m)}$ and a regular power I^r of a homogeneous ideal $0 \neq I \subseteq R = k[x_0, \ldots, x_n] = k[\mathbb{P}^n]$ is key to many open problems in commutative algebra. However, despite much effort, symbolic powers and their invariants are very difficult to compute.

Definition 2.1. Let $0 \neq I$ be a homogeneous ideal in $R = k[x_0, \ldots, x_n]$. The **m-th symbolic power** of I, denoted $I^{(m)}$, is the ideal

$$I^{(m)} = \bigcap_{\wp \in \mathrm{Ass}(I)} (I^m R_\wp \cap R),$$

where $\mathrm{Ass}(I)$ denotes the set of associated prime ideals of I and $R_\wp$ denotes the localization of R at the prime ideal $\wp$.

We know that $I^r \subseteq I^{(m)}$ if and only if $r \geq m$ (by Lemma 8.1.4 of [1]). However, determining which symbolic powers are contained in which regular powers is quite challenging. One celebrated containment of Ein–Lazarsfeld–Smith [19] and Hochster–Huneke [33] is $I^{(rn)} \subseteq I^r$ for $r > 0$. In search of tighter containments, Harbourne and Huneke [30] are motivated to ask the following question:

Question 2.2. [30, Questions 1.3, 1.4 and Conjecture 4.1.5] For which m, i and j do we have $I^{(m)} \subseteq M^j I^i$, where $M = (x_0, \ldots, x_n)$?

In [30], Harbourne and Huneke state a number of conjectures of this flavour comparing regular and symbolic powers of homogeneous ideals. Since the degree to which symbolic and regular powers of homogeneous ideals differ plays a key role in many open problems, it is not surprising that these conjectures have gained much attention. A summary of some known cases along with an additional conjecture can be found in joint work of C. Bocci, B. Harbourne and myself [5]. Although the majority of the conjectures are stated for radical ideals of finite sets of points in $\mathbb{P}^n$, it is natural and useful to ask similar questions in a non-geometric setting such as for monomial ideals. Indeed, as demonstrated in [41], the computation of symbolic powers of monomial ideals can be used to find symbolic powers of other arbitrary ideals via Gröbner bases. In [14], R. Embree, A. Hoefel, T. Hà, and I, show the general containment:

Theorem 2.3. [14] *Suppose $I \subseteq R = k[x_0, \ldots, x_n]$ is a square-free monomial ideal. Then for all positive integers m, t and r we have the containment*

$$I^{(t(m+e-1)-e+r)} \subseteq M^{(t-1)(e-1)+r-1}(I^{(m)})^t$$

where e is the big-height of I and $M = (x_0, \ldots, x_n)$.

Here, the big-height of an ideal $I \subseteq R$ is the maximum of the heights of the associated primes of I.

Theorem 2.3 recovers all but one of the conjectures of Harbourne–Huneke for square-free monomial ideals. The remaining conjecture deals with the initial degree of the ideal.

Definition 2.4. Let $0 \neq I \subseteq R = k[x_0, \ldots, x_n]$ be a homogeneous ideal. The **initial degree** of I is

$$\alpha(I) = \min\{t \geq 0 \mid I_t \neq 0\}.$$

Question for Current & Future Research. [30] For a monomial ideal $I \subseteq R = k[x_0, \ldots, x_n]$, is $I^{(m)} \subseteq I^r$ whenever $m/r \geq n\alpha(I)/(\alpha(I)+1)$?

In order to work with a non-square-free monomial ideal I, we also introduce the **symbolic polyhedron**, a convex polyhedron in $\mathbb{R}^{n+1}$. When scaled by a factor of m, the symbolic polyhedron contains the exponent vectors of all monomials in the mth symbolic power $I^{(m)}$ [14, Theorem 5.4]. In the definition below, $conv\{L\}$ denotes the convex hull of a set L and $\operatorname{maxAss}(I)$ denotes the set of maximal associated prime ideals of I.

Definition 2.5. Let $I \subseteq R = k[x_0, \ldots, x_n]$ be a monomial ideal with primary decomposition $I = Q_1 \cap \cdots \cap Q_s$. The **symbolic polyhedron** of I is

$$\mathcal{Q} = \bigcap_{\wp \in \operatorname{maxAss}(I)} conv\{(a_0, \ldots, a_n) \mid x_0^{a_0} \cdots x_n^{a_n} \in Q_{\subseteq \wp}\} \subseteq \mathbb{R}^{n+1}$$

where

$$Q_{\subseteq \wp} = R \cap IR_{\wp} = \bigcap_{\sqrt{Q_i} \subseteq \wp} Q_i.$$

Using the symbolic polyhedron we have the following containment for *any* monomial ideal.

Theorem 2.6. [14] *Suppose $\mathcal{Q}$ is the symbolic polyhedron of a monomial ideal $I \subseteq k[x_0, \ldots, x_n]$ with big-height e. For all integers $r \geq 0$ and $m \geq \max(er, \beta(I^r)/\alpha(\mathcal{Q}))$,*

$$I^{(m)} \subseteq M^{\lceil \alpha(\mathcal{Q})m \rceil - \beta(I^r)} I^r,$$

where $M = (x_0, \ldots, x_n)$, $\alpha(\mathcal{Q}) = \min\{a_0 + \cdots + a_n \mid (a_0, \ldots, a_n) \in \mathcal{Q}\}$ and $\beta(I^r)$ is the maximum of the degrees of the minimal generators of I^r.

The ideal containment problems of Harbourne and Huneke were a main topic at the recent Mathematisches Forschungsinstitut Oberwolfach (MFO) mini-workshop *Ideals of Linear Subspaces, Their Symbolic Powers and Waring Problems* in February 2015. A subset of the participants focused their attention on the Waldschmidt constant which bounds the resurgence $\rho(I)$ of a homogeneous ideal I in R.

Definition 2.7. Let $I \subseteq R = k[x_0, \ldots, x_n]$ be a homogeneous ideal.

(1) The **resurgence** of I is

$$\rho(I) = \sup\{m/r \mid I^{(m)} \not\subseteq I^r\}.$$

(2) The **Waldschmidt constant** of I is

$$\widehat{\alpha}(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

It is known that $\widehat{\alpha}(I)$ does exist (see [45]). Despite the Waldschmidt constant appearing in many other areas of mathematics such as complex analysis [39] and number theory [8, 45, 46], it is very difficult to compute this limit. However, when you can find $\widehat{\alpha}(I)$, you can obtain information about the symbolic powers of I with the following bound:

$$\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho(I).$$

In joint work [4] stemming from the MFO mini-workshop, we show that if I is a square-free monomial ideal then $\widehat{\alpha}(I)$ can be expressed as the optimal solution to a linear program arising from the primary decomposition of the ideal I. More precisely,

Theorem 2.8. [4] *Let $I \subseteq R = k[x_0, \ldots, x_n]$ be a square-free monomial ideal with minimal primary decomposition $I = Q_1 \cap Q_2 \cap \cdots \cap Q_s$. Let $A = (A_{i,j})$ be the matrix where $A_{i,j} = \begin{cases} 1 & \text{if } x_j \in Q_i \\ 0 & \text{if } x_j \notin Q_i. \end{cases}$
Consider the following linear program:*

$$\text{minimize } \mathbf{1}^T\mathbf{y} \text{ subject to } A\mathbf{y} \geq \mathbf{1} \text{ and } \mathbf{y} \geq \mathbf{0}$$

and suppose that $\mathbf{y}^*$ is a feasible solution that realizes the optimal value. Then

$$\widehat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m} = \mathbf{1}^T\mathbf{y}^*.$$

The applications of this result are wide-ranging. For example,

(1) We can now express $\widehat{\alpha}(I)$ in terms of the fractional chromatic number of a hypergraph. As such we can find $\widehat{\alpha}(I)$ when I is an edge ideal for well-known families of graphs such as bipartite, perfect, and cycles. In addition, we have shown that $\widehat{\alpha}(I)$ has upper and lower bounds involving the chromatic and clique numbers of the graph. (See [4] for details.)

(2) Geramita–Harbourne–Migliore–Nagel [23] recently showed that if $\tilde{I}$ is an ideal obtained from I by replacing each variable with a homogeneous polynomial such that the polynomials form a regular sequence, then $\widehat{\alpha}(\tilde{I})$ can be related to $\widehat{\alpha}(I)$. Thus, the results of [4] can be applied to study ideal containments for a larger family of ideals.

In addition, the linear programming approach allows us to affirmatively answer a conjecture which is in the same spirit as a well-known conjecture of Chudnovsky:

Conjecture 2.9 (Chudnovsky's Conjecture [8])**.** If $I \subseteq k[x_0, \ldots, x_n]$ is the radical ideal of a finite set of points in $\mathbb{P}^n$, then, for all $r > 0$,

$$\frac{\alpha(I) + n - 1}{n} \leq \frac{\alpha(I^{(r)})}{r}.$$

For monomial ideals we have:

Conjecture 2.10. [14] If $I \subseteq k[x_0, \ldots, x_n]$ is a monomial ideal with big-height e and $\mathcal{Q}$ is the symbolic polyhedron of I, then

$$\alpha(\mathcal{Q}) \geq \frac{\alpha(I) + e - 1}{e}.$$

The linear programming machinery can be applied to yield:

Theorem 2.11. [4] *Let $I \subseteq k[x_0, \ldots, x_n]$ be a square-free monomial ideal with big-height e and symbolic polyhedron $\mathcal{Q}$. Then*

$$\alpha(\mathcal{Q}) = \widehat{\alpha}(I) \geq \frac{\alpha(I) + e - 1}{e}.$$

Question for Current & Future Research. For any monomial ideal I, can we determine $\widehat{\alpha}(I)$ and the values of m and r such that $I^{(m)} \subseteq I^r$? If so, how do we best apply the results to non-monomial ideals via Gröbner bases?

3. Fat Points and Coding Theory

Although they seem unrelated, fat point schemes and linear codes have nice connections. Much work has gone into relating properties of linear codes to those of the ideal defining the associated fat points scheme. We will denote by $\mathbf{I}(\mathbb{W}) \subseteq R = k[x_0, \ldots, x_n]$ the ideal consisting of all the homogeneous polynomials vanishing on the point set $\mathbb{W} \subset \mathbb{P}^n$.

Definition 3.1. Let $\mathbb{X} = \{P_1, \ldots, P_r\}$ be a finite set of distinct points in $\mathbb{P}^n$ and $m_1, \ldots, m_r$ be positive integers. The **fat point scheme** $\mathbb{Y} = m_1P_1 + \cdots + m_rP_r$ of $\mathbb{P}^n$ is defined by the homogeneous ideal

$$\mathbf{I}(\mathbb{Y}) = \mathbf{I}(P_1)^{m_1} \cap \mathbf{I}(P_2)^{m_2} \cap \cdots \cap \mathbf{I}(P_r)^{m_r} \subseteq R = k[x_0, \ldots, x_n]$$

(i.e., $\mathbf{I}(\mathbb{Y})$ is the ideal generated by all the homogeneous polynomials vanishing at the points P_i to order at least m_i). We call $\mathbb{X} = \{P_1, \ldots, P_r\}$ the **support** of $\mathbb{Y}$ and $m_1, \ldots, m_r$ the **multiplicities** of $\mathbb{Y}$. We also say that $\mathbb{Y}$ is **homogeneous** if $m_1 = m_2 = \cdots = m_r$.

When the points $P_1, \ldots, P_r$ are not all contained in a hyperplane, we can associate a fat point scheme $\mathbb{Y} = m_1P_1 + \cdots + m_rP_r \subset \mathbb{P}^n$ with a linear code with generating matrix

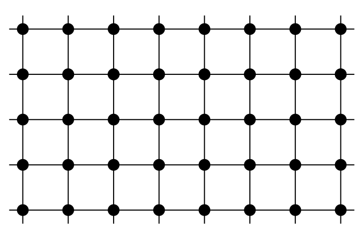
$$A(\mathbb{Y}) = [\underbrace{c_1 \cdots c_1}_{m_1} \quad \cdots \quad \underbrace{c_r \cdots c_r}_{m_r}],$$

where each c_i is a column vector with entries equal to the homogeneous coordinates of the point P_i. The code has parameters $[m_1 + \cdots + m_r, n+1, d]$ where $d = d(\mathbb{Y})$ is the minimum Hamming distance of the code. Tohǎneanu and Van Tuyl [44] showed that

$$d(\mathbb{Y}) = \min\{d \mid \text{there exists } r-d \text{ columns of } A(\mathbb{Y}) \text{ that span an } n\text{-dimensional space}\}.$$

Much work has been done on studying $d(\mathbb{Y})$ using homological data of $R/\mathbf{I}(\mathbb{Y})$. The majority of the program thus far has involved reduced points (i.e., $m_1 = \cdots = m_r = 1$); see [26, 29, 42, 43]) and homogeneous fat point schemes (see [44]). Our goal is to further this investigation for non-homogeneous fat point schemes.

In work in progress, I am investigating fat points and coding theory in the structured setting of complete intersections. Working with complete intersections is a natural starting place since such investigations for reduced points was initiated with complete intersections in [26, 28, 29]. A set $\mathbb{X} = \{P_1, P_2, \ldots, P_{ab}\} \subset \mathbb{P}^2$ of distinct points is called a **complete intersection of type** (a, b) if $\mathbf{I}(\mathbb{X}) = (F, G)$, where F and G share no common factors and $\deg F = a$ and $\deg G = b$. Without loss of generality, we will assume $a \leq b$. A **grid complete intersection** of type (a, b) is a complete intersection of type (a, b) whose ideal is generated by two totally reducible forms. A grid complete intersection of type (a, b) can be visualized as the intersection of two sets of lines on an $a \times b$ grid, where we number the points on the bottom row by $P_1, \ldots, P_b$, the points on the row immediately above the bottom row $P_{b+1}, \ldots, P_{2b}$, and so on. Below is a figure representing a grid complete intersection of type $(5, 8)$.



Theorem 3.2. *If $\mathbb{Y} = mP_1 + \cdots + mP_{ab-1} + (m+c)P_{ab} \subset \mathbb{P}^2$ is a fat point scheme with support a grid complete intersection of type (a, b) with $a \geq 2$, then $d(\mathbb{Y}) = (a-1)mb$.*

This has allowed us to connect Hamming distance to a special graded Betti number via the **minimum socle degree** $s_2(\mathbb{Y})$:

Theorem 3.3. *If $\mathbb{Y} = mP_1 + \cdots + mP_{ab-1} + (m+1)P_{ab} \subset \mathbb{P}^2$ is a fat point scheme with support a grid complete intersection of type (a, b), then $s_2(\mathbb{Y}) = ma$.*

Corollary 3.4. *With $\mathbb{Y}$ as in Theorem 3.3 where $a \geq 2$, we have*

$$s_2(\mathbb{Y}) = \frac{d(\mathbb{Y}) + mb}{b}.$$

The proof of Theorem 3.2 is straightforward. However, Theorem 3.3 is really a corollary of a more complicated result. This is not surprising since the graded Betti numbers of fat point schemes are not easily described. The approach taken was to show that $R/\mathbf{I}(\mathbb{Y})$ and $R/\mathbf{I}(\mathbb{V})$ share the same graded Betti numbers where $\mathbb{V}$ was chosen to be a special reduced set of points called a **partial intersection**. Partial intersections are useful to work with since their graded Betti numbers are known [37].

Question for Current & Future Research. For a given family of non-homogeneous fat point schemes, can we find reduced point sets in special position (such as partial intersections) which share the same graded Betti numbers? If so, use these reduced sets to relate the distance and other coding invariants with homological data of the fat point scheme. I have made some progress on this problem in joint work in progress with E. Guardo [15].

One way to show that a fat point scheme $\mathbb{Y} \subset \mathbb{P}^2$ and a partial intersection $\mathbb{V} \subset \mathbb{P}^2$ have the same graded Betti numbers is to show: (1) the schemes have the same Hilbert function; and (2) $\mathbf{I}(\mathbb{Y})$ and $\mathbf{I}(\mathbb{V})$ have the same generating degrees. However, characterizing the Hilbert functions of fat points is a very challenging problem in itself (even in $\mathbb{P}^2$) to which we dedicate the next section.

4. Hilbert Functions, Graded Betti Numbers, and Initial Degrees of Fat Points

4.1. **Underlying Motivation.** Hilbert functions have been extensively studied and have played a central role in commutative algebra and algebraic geometry. Perhaps the most celebrated result in the algebraic setting is Macaulay's Theorem which involves lex ideals (described below).

Given two natural numbers a and d we can define a unique natural number, denoted $a^{<d>}$, in terms of the **d-binomial expansion of a** (for details, see pages 159 - 161 of [6]). The function $\_^{<d>}$ turns out to describe the growth of lex ideals from one degree to the next. Macaulay's Theorem says that this behaviour holds for all homogeneous ideals.

Theorem 4.1. [6, 36, 40] **(Macaulay's Theorem)** Let $I \subseteq R = k[x_0, \ldots, x_n]$ be a homogeneous ideal.

(a) There exists a lex ideal $L \subseteq R$ such that $H(R/I) = H(R/L)$.

(b) For any lex ideal $L \subseteq R$ we have the bounds $H(R/L, d+1) \leq H(R/L, d)^{<d>}$ for $d \geq 0$. Thus, $H(R/I, d+1) \leq H(R/I, d)^{<d>}$ for $d \geq 0$.

Many people have tried to characterize the possible Hilbert functions of families of homogeneous ideals I whose elements share special properties. Approaches have included determining the ideals L which play analogous roles to the lex ideals in part (a) of Macaulay's Theorem and describing the Hilbert functions combinatorially by giving bounds on each $H(R/I, d)$ as in part (b) of Macaulay's Theorem. These characterizations have developed in a variety of different directions (for some examples see [2, 21, 34, 35, 38]). On the geometric side, we can consider Hilbert functions of points.

Definition 4.2. If $\mathbb{X} \subseteq \mathbb{P}^n$ is a set of points, then the **Hilbert function of** $\mathbb{X}$ is $H(\mathbb{X}) := H(R/\mathbf{I}(\mathbb{X}))$.

Geramita–Maroscia–Roberts [24] characterize the functions which arise as Hilbert functions of finite sets of distinct, reduced points in $\mathbb{P}^n$. Empowered with such a characterization, the Hilbert function of a set of points $\mathbb{X}$ can then be exploited to obtain both algebraic data about $\mathbf{I}(\mathbb{X})$ and geometric information about $\mathbb{X}$. For example, E. D. Davis [17] and Bigatti–Geramita–Migliore [3] use extremal behaviour of Hilbert functions (described by part (b) of Theorem 4.1) to guarantee that some subsets have special properties.

4.2. **The Fat Points Case.** Whereas for points in $\mathbb{P}^n$ with $n = 1$ the Hilbert function depends only on the multiplicities, for $n > 1$ the Hilbert function depends also on the position of the points, and in a very subtle way.

Question 4.3. Given positive integers $m_1, \ldots, m_r$, is there a characterization of the functions that occur as Hilbert functions of fat points in $\mathbb{P}^n$ whose multiplicities are $m_1, \ldots, m_r$? What are all the possible graded Betti numbers?

Question 4.3 is open in $\mathbb{P}^2$ with each $m_i = 2$ (called **double point schemes**). However, if $r \leq 8$ then we know all the possible Hilbert functions for any fat point scheme in $\mathbb{P}^2$ for any given set of multiplicities [22, 27]. In addition, one of the main results of Geramita–Migliore–Sabourin in [25] is to give a criterion characterizing a family of functions, all of which occur as Hilbert functions of double point schemes in $\mathbb{P}^2$.

In [12], B. Harbourne, Z. Teitler and I study the Hilbert function of a fat point scheme $\mathbb{Y} \subset \mathbb{P}^n$ given the multiplicities and data about which subsets of the support are collinear. We define a reduction vector d based on collinearity data and use d to give upper and lower bounds on $H(\mathbb{Y})$ in each degree. Here is a brief description of our approach for $\mathbb{P}^2$.

Definition 4.4. [12] Let $\mathbb{Y} = m_1P_1 + m_2P_2 + \cdots + m_rP_r \subset \mathbb{P}^2$.

(1) Let $\mathbb{L}$ be a line in $\mathbb{P}^2$. We define $\deg(\mathbb{L} \cap \mathbb{Y}) := \sum_{P_i \in \mathbb{L}} m_i$. The **subscheme of $\mathbb{Y}$ residual to $\mathbb{L}$** is $\mathbb{Y}' := b_1P_1 + b_2P_2 + \cdots + b_rP_r$, where $b_i = m_i$ if $P_i \notin \mathbb{L}$ and $b_i = \max(m_i - 1, 0)$ if $P_i \in \mathbb{L}$.

(2) We say that a sequence $\mathbb{L}_1, \ldots, \mathbb{L}_{n+1}$ of lines **totally reduces** $\mathbb{Y}$ if at least m_j of the lines $\mathbb{L}_i$ contain P_j for each j. To such a sequence of lines we associate the **reduction vector** $d = (d_1, \ldots, d_{n+1})$ with $d_i = \deg(\mathbb{L}_i \cap \mathbb{Y}_{i-1})$, where $\mathbb{Y} = \mathbb{Y}_0$ and $\mathbb{Y}_i$ is the residual of $\mathbb{Y}_{i-1}$ with respect to $\mathbb{L}_i$.

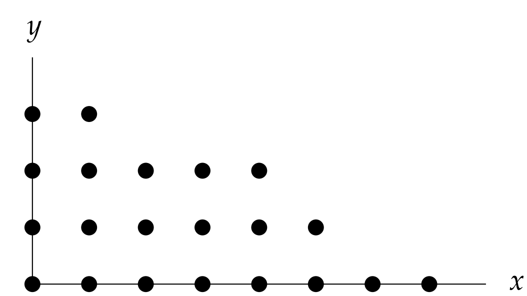
The lower bound given in [12] on the Hilbert function of a fat point subscheme in $\mathbb{P}^2$ can be described combinatorially via a simple diagram.

Definition 4.5. Let $d = (d_1, \ldots, d_{n+1})$ be a non-negative integer vector.

(1) The **standard configuration** S_d determined by d consists of the d_j leftmost first quadrant lattice points of $\mathbf{Z} \times \mathbf{Z}$ on each horizontal line with second coordinate $j - 1$ for $1 \leq j \leq n + 1$.

(2) We define the function f_d where $f_d(t)$ for $t \geq 0$ is the number of points in S_d which are in or on the triangle whose vertices are $(0, 0)$, $(t, 0)$ and $(0, t)$.

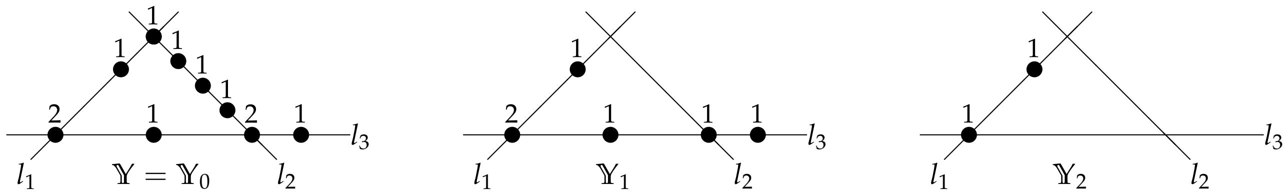
Example 4.6. Let $d = (8, 6, 5, 2)$. Then S_d is the set of lattice points in $\mathbf{Z} \times \mathbf{Z}$ shown in the figure below.



In this example, $f_d = (1, 3, 6, 10, 14, 17, 20, 21, 21, 21, \ldots)$.

In a similar, but somewhat more complicated way, one can define a function F_d which has the property that $F_d(t) \geq f_d(t)$ for all $t \geq 0$. We also have a simple criterion to determine exactly when $f_d = F_d$. In particular, if $d = (d_1, \ldots, d_{n+1})$ such that $d_i > d_{i+1}$ then $f_d = F_d$.

Example 4.7. Let $\mathbb{Y} = \mathbb{Y}_0 \subset \mathbb{P}^2$ be the fat point scheme described in the following figure, where the numbers associated to the points represent the multiplicities.



Given the sequence $\mathbb{L}_1 = l_2, \mathbb{L}_2 = l_3, \mathbb{L}_3 = l_1$ of lines $\mathbb{L}_i$, $\mathbb{Y}_i$ is obtained from $\mathbb{Y}_{i-1}$ by reducing by 1 the multiplicities of all the points on the line $\mathbb{L}_i$ (and dropping points whose multiplicity becomes 0). We say that the sequence of lines $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ **totally reduces** $\mathbb{Y}$ since $\mathbb{Y}_3 = \emptyset$. The output of this reduction procedure is the reduction vector $d = (6, 5, 2)$: counting with multiplicity, $\mathbb{Y}_0$ has 6 points on $\mathbb{L}_1$, $\mathbb{Y}_1$ has 5 points on $\mathbb{L}_2$, and $\mathbb{Y}_2$ has 2 points on $\mathbb{L}_3$.

Theorem 4.8. [12] If $\mathbb{L}_1, \ldots, \mathbb{L}_{n+1}$ is any sequence of lines which totally reduces a fat point scheme $\mathbb{Y} \subset \mathbb{P}^2$ with reduction vector d, then

$$f_d(t) \leq H(\mathbb{Y}, t) \leq F_d(t)$$

for all $t \geq 0$. Moreover, for example, if the entries of d are strictly decreasing, then $H(\mathbb{Y}) = f_d$.

Using similar methods we also give upper and lower bounds on the graded Betti numbers for $\mathbf{I}(\mathbb{Y})$.

For future related projects, we are naturally led to the following questions.

Questions for Current & Future Research. Suppose $\mathbb{Y} = m_1P_1 + \cdots + m_rP_r$ is a fat point scheme in $\mathbb{P}^2$ with an associated reduction vector d for some totally reducing sequence of lines.

- Different choices of lines will produce different upper and lower bounds f_d and F_d. How do the bounds change with respect to these choices?
- How sharp are the bounds f_d and F_d?
- What can be said about $H(\mathbb{Y})$ if we apply residuation using higher degree curves?
- If $f_d \neq F_d$, can we nonetheless find a formula for $H(\mathbb{Y})$?
- Can we use the bounds f_d and F_d to obtain non-trivial geometric information about $\mathbb{Y}$?

Ideally, we would like to fine-tune the bounds f_d and F_d in general:

Question for Current & Future Research. Suppose $\mathbb{Y} = m_1P_1 + \cdots + m_rP_r$ is a fat point scheme in $\mathbb{P}^n$. We can apply the methods from [12] to obtain upper and lower bounds for $H(\mathbb{Y})$. Can we find nice combinatorial descriptions of f_d and F_d? In special cases?

I also plan to apply the technique from [12] to study codes:

Question 4.9. The minimum distance of a reduced linear code is the block-length of the code minus the maximum number of the points dual to the columns of a generating matrix contained in some hyperplane. In $\mathbb{P}^2$, this number is the maximum number of collinear points of a given set of points. What information about a linear code can be revealed by applying the approach of residuation?

We conclude this section with an application to initial degree. Related to the Hilbert function and graded Betti numbers, there are several numerical characters that have gained the attention of many researchers over the years. One of the more important such characters is α.

Definition 4.10. Let $\mathbb{Y} = m_1P_1 + \cdots + m_rP_r$ be a fat point scheme of $\mathbb{P}^n$. Then we define the **initial degree of** $\mathbb{Y}$, denoted $\alpha(\mathbb{Y})$, to be the least degree t such that $\mathbf{I}(\mathbb{Y})_t \neq 0$.

That is, $\alpha(\mathbb{Y})$ is simply the initial degree of $I(\mathbb{Y})$ defined in Section 2. B. Harbourne and C. Huneke [30] conjecture the following relationship:

Conjecture 4.11. Let $R = k[\mathbb{P}^2] = k[x_0, x_1, x_2]$ and consider $\mathbb{X} = P_1 + \cdots + P_r$ and $\mathbb{Y} = \sum_{i=1}^{r}(2s-1)P_i$. Then $\mathbf{I}(\mathbb{Y}) \subset (x_0, x_1, x_2)^{s-1}\mathbf{I}(\mathbb{X})^s$.

Conjecture 4.11 would imply the following conjecture.

Conjecture 4.12. Let $\mathbb{X} = P_1 + \cdots + P_r \subset \mathbb{P}^2$ and $\mathbb{Y} = \sum_{i=1}^{r}(2s-1)P_i \subset \mathbb{P}^2$. Then

$$\alpha(\mathbb{Y}) \geq s\alpha(\mathbb{X}) + (s-1).$$

There are families of examples where this does not hold (see [5, 18, 31]). That being said, there is also evidence that the conjecture holds for large families of fat point schemes. One natural family to consider is that of **line count configurations** since it has been shown that this family exhibits extremal behaviour in various settings. We say $\mathbb{X} = \mathbb{X}_1 + \cdots + \mathbb{X}_t \subset \mathbb{P}^2$ is a *line count configuration of type* $c = (c_1, \ldots, c_t)$ if each $\mathbb{X}_i$ consists of c_i points on a line $\mathbb{L}_i$ where the lines $\mathbb{L}_1, \ldots, \mathbb{L}_t$ are distinct and no point of $\mathbb{X}$ occurs where two of the lines $\mathbb{L}_i$ meet. After re-indexing, we assume that $c_i \leq c_{i+1}$ for $1 \leq i \leq t-1$. S. G. Hartke and I have investigated Conjecture 4.12 for line count configurations.

Theorem 4.13. [16] *If $\mathbb{W} \subset \mathbb{P}^2$ is a line count configuration of type $c = (c_1, c_2, \ldots, c_t)$ where $c_i \geq i$ for $1 \leq i \leq t$, then for all integers $r \geq 1$ we have:*

(a) $\alpha((2r-1)\mathbb{W}) \geq r\alpha(\mathbb{W}) + (r-1)$;
(b) $\alpha(2r\mathbb{W}) \geq r\alpha(\mathbb{W}) + r$.

The proof of Theorem 4.13 highly uses the residuating algorithm from [12] discussed above along with a careful analysis of the combinatorics underlying the relevant inequalities.

Question for Current & Future Research. Can we further apply the residuating bounds to verify Conjecture 4.12 and the other Harbourne–Huneke conjectures for fat point schemes in a variety of configurations?

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