RESEARCH STATEMENT

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My area of research is commutative algebra, particularly the research of multiplicity, integral closure and reduction of ideals. The concept of multiplicity originated as a geometric invariant. Given a field k and polynomials $f_1, ..., f_k$ in n variables (elements of the ring $k[x_1, ..., x_n]$) let $(a_1, ..., a_n$ be a point in k^n such that each f_i satisfies $f_i(a_1, ..., a_n) = 0$. Let R be the ring $(k[x_1, ..., x_n])/(f_1, ..., f_n)$ and $\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$. Consider the length $\lambda(R/\mathfrak{m}^n)$. This is becomes polynomial for large values of n of degree d, which is equal to the dimension of the ring R. The degree d coefficient is $\frac{e(R)}{d!}$ where $e(R)$ is the multiplicity at that point. My reseach is focused on studying generalizations of this multiplicity to determine what properties these may characterize.

For a commutative ring R and an ideal I, the integral closure of I is defined to be the ideal

$$\overline{I} = \left\{ r \in R \mid r^n + a_1 r^{n-1} + ... + a_{n-1} r + a_n = 0 \text{ for some } n \text{ and for } a_i \in I^i \right\}.$$

For $I \subseteq J$ ideals in R, I is said to be a reduction of J if $IJ^n = J^{n+1}$ for some n. Reduction ideals were introduced by Northcott and Rees as a tool to study the integral closure of an ideal. We have the following relation between integral closure and reductions.

Theorem 1. *Let R be a commutative noetherian ring and $I \subseteq J$ ideals. The following are equivalent:*

(1) $I \subseteq J$ is a reduction;
(2) $\overline{I} = \overline{J}$;
(3) $J \subseteq \overline{I}$.

There are many algebraic objects that are of interest when studying the integral closure of an ideal. The algebra $R[It] = \bigoplus_{n=0}^{\infty} I^n t^n$ is called the Rees algebra and $R[It, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I^n t^n$

where $I^n = R$ when $n \leq 0$ is the extended Rees algebra. The associated graded ring is defined by

$$G_I(R) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1} \cong R[It]/IR[It] \cong R[It,t^{-1}]/(t^{-1}).$$

If $(R, \mathfrak{m})$ is a local ring of dimension d and I is $\mathfrak{m}$-primary, then the lengths $\lambda(I^i/I^{i+1})$ are finite for all i. The Hilbert function

$$h(n) = \sum_{i=0}^{n} \lambda(I^i/I^{i+1})$$

is eventually a polynomial of degree d with rational coefficients. The leading coefficient of this polynomialis given by $\frac{e(I)}{d!}$, where $e(I)$ is an integer which is called the Samuel multiplicity of I. It is fairly straightforward to prove that if $I \subseteq J$ is a reduction, then $e(I) = e(J)$.

In 1961, Rees proved the following theorem.

Theorem 2. [7, Theorem 11.3.1] *Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq J$ be $\mathfrak{m}$-primary ideals. If I is a reduction of J, then $e(I) = e(J)$. Moreover, if R is formally equidimensional, then the converse also holds.*

The constraint that R be formally equidimensional is a rather weak assumption; it means that all minimal prime ideals of the completion of R have the same dimension.

There have been many attempts to generalize Theorem 2 for ideals that are not $\mathfrak{m}$-primary, a case in which the classical Samuel multiplicity is no longer defined. The analytic spread of an ideal is defined by

$$\ell(I) = \dim(G_I(R)/\mathfrak{m}G_I(R)).$$

It is known that $\ell(I)$ is between $\operatorname{ht}(I)$ and $\dim(R)$. In the case that $\ell(I) = \dim(R)$, I is said to have maximal analytic spread. In 1993, Achilles and Manaresi defined the j-multiplicity for ideals of maximal analytic spread

$$j(I) = e(H^0_{\mathfrak{m}}(G_I(M))).$$

Further, it was shown that

$$j(I) = \sum_{\mathfrak{p} \in \operatorname{Assh}_{G_I(R)}(G_I(R)/\mathfrak{m}G_I(R))} e(G_I(R)/\mathfrak{p})\lambda((G_I(R))_{\mathfrak{p}}).$$

One can prove that if I is $\mathfrak{m}$-primary, then $j(I) = e(I)$.

In 2001, Flenner and Manaresi proved the following theorems.

Theorem 3. [6, Lemma 3.2] *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $I \subseteq J$ ideals. If $j(I_{\mathfrak{p}}) = j(J_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, then $j(I) \geq j(J)$.*

Theorem 4. [6, Theorem 3.3] *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $I \subseteq J$ ideals. Then the following are equivalent:*

(1) $I \subseteq J$ is a reduction;
(2) $j(I_{\mathfrak{p}}) = j(J_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$;
(3) $j(I_{\mathfrak{p}}) = j(J_{\mathfrak{p}})$ for all $\mathfrak{p} \in \bigcup_n \operatorname{Ass}(R/\overline{I^n})$

McAdam [8] proved that the set $\operatorname{Ass}(R/\overline{I^n})$ is an ascending set of primes which eventually stabilizes and that these primes are exactly the primes in which the analytic spread of $I_{\mathfrak{p}}$ is maximal. Further, he proved that this set of primes is finite. Thus the equality $j(I_{\mathfrak{p}}) = j(J_{\mathfrak{p}})$ only needs to be checked at finitely many prime ideals. A better characterization would be one that involves invariants that can be computed by considering only the ring R and not all of its localizations.

Another generalization of the classical Samuel multiplicity comes from the bigraded ring

$$G_{\mathfrak{m}}(G_I(R)) = \bigoplus_{i=0}^{\infty} \bigoplus_{j=0}^{\infty} \frac{I^i\mathfrak{m}^j + I^{i+1}}{I^i\mathfrak{m}^{j+1} + I^{i+1}}.$$

The Hilbert function

$$h(u, v) = \sum_{i=0}^{u} \sum_{j=0}^{v} \lambda\Big(\frac{I^i\mathfrak{m}^j + I^{i+1}}{I^i\mathfrak{m}^{j+1} + I^{i+1}}\Big)$$

is polynomial for $u, v \gg 0$ of degree $d = \dim R$. The degree d part of this polynomial is given by

$$\sum_{k+n=d} \frac{c_k(I)}{k!n!} u^k v^n,$$

where $c_k(I)$ are integers. Originally defined by Achilles and Manaresi in [2], the integer $c_k(I)$ is called the k-th generalized Hilbert-Samuel multiplicity. They also proved that $c_0(I) = j(I)$. Further, the following was proved.

Theorem 5. [2, Theorem 2.3] *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* d *and* I *an ideal of* R*. Denote* $\ell = \ell(I)$ *and* $q = \dim(R/I)$*. Then:*

(1) $c_i(I) = 0$ *for* $i < d - \ell$ *or* $i > q$*;*

(2) $c_{d-\ell}(I) = \sum_\beta e(\mathfrak{m}G_I(R)_\beta)e(G_I(R)/\beta)$ *where* β *runs through all the highest dimensional associated primes of* $G_I(R)/\mathfrak{m}G_I(R)$ *such that* $\dim(G_I(R)/\beta)+\dim(G_I(R)_\beta) = \dim G_I(R)$*;*

(3) $c_q(I) = \sum_\mathfrak{p} e(I_\mathfrak{p})e(R/\mathfrak{p})$ *where* $\mathfrak{p}$ *runs through all the highest dimensional associated primes of* R/I *such that* $\dim(R/\mathfrak{p}) + \dim R_\mathfrak{p} = \dim R$*.*

In 2001, Ciupercă proved the following.

Theorem 6. [3, Theorem 2.7] *Let* $(R, \mathfrak{m})$ *be a local ring,* $I \subseteq J$ *ideal. If* I *is a reduction of* J*, then* $c_i(I) = c_i(J)$ *for all* $i = 0, ..., d$*.*

Several other proofs of this result were given later. See for example [9, Corollary 11.5]. My research has been motivated by finding a suitable converse of this theorem.

To do so, we begin by proving the following result that expresses each generalized multiplicity $c_k(I, M)$ which is defined in the more general case of R-modules, as a linear combination of certain local j-multiplicities.

Theorem 7. *Let* $(R, \mathfrak{m})$ *be a local ring,* M *a finitely generated* R*-module of dimension* d*,* $I \subseteq R$ *an ideal with* $\ell_M(I) = \ell$*, and denote*

$$\Lambda_k = \Lambda_k(I, M) = \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}(M/IM), \dim(R/\mathfrak{p}) = k, \dim(R/\mathfrak{p}) + \dim M_\mathfrak{p} = \dim M\}.$$

Further, let $\{x_2, ..., x_\ell\}$ *be a sequence of elements of* I *such that* x_i *is a superficial element for* $(I, (M/(x_2, ..., x_{i-1})M)_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M/IM)$ *such that* $\dim((M/(x_2, ..., x_{i-1})M)_\mathfrak{p}) = \ell_{(M/(x_2,...,x_{i-1})M)_\mathfrak{p}}(IR_\mathfrak{p}) \geq 2$*.*

Assume that the following condition is satisfied:

$$(*) \dim M_\mathfrak{p} = \dim(I^n M_\mathfrak{p}) \text{ for all } n \geq 0 \text{ and for all } \mathfrak{p} \in \operatorname{Supp}(M/IM)$$

Then for all k=$d - \ell$,...,d *we have*

$$c_k(I, M) = \sum_{\mathfrak{p} \in \Lambda_k} c_0(IR_\mathfrak{p}, (M/(x_2, ..., x_{d-k})M)_\mathfrak{p})e(R/\mathfrak{p}).$$

The technical assumption $(*)$ is very weak and is always satisfied when $\text{ht}_M(I) > 0$ or whenever $c_i(I, M) \neq 0$ for some $i < d$.

Again, note that the sum involved is finite by a result of McAdam [8], which states that the only primes for which $\ell(I_{\mathfrak{p}})$ is maximal are in $\bigcup_{n\geq 0} \text{Ass}(R/\overline{I^n}$ and this is a finite set. Therefore $c_0(I_{\mathfrak{p}})$ is nonzero for finitely many primes. We start by showing this equality is true for c_q where $q = \dim M/IM$. This is a module version of the Achilles and Manaresi result (Theorem 5(3)). Then the condition $(*)$ allows us to replace M with I^nM without significantly changing the multiplicity sequence, $c_k(I, M) = c_k(I, I^nM)$ for $k = 0, ..., d-1$. Since $\text{depth}_I(I^nM) > 0$ for $n \gg 0$, we can then find a superficial element which is a nonzero divisor so that $c_k(I, M) = c_k(I, M/xM)$ for $k = 0, ..., d-2$. From this point we can use induction on the dimension of the module M to finish the proof.

By the result of Flenner and Manaresi (Theorem 3) relating the local j-multiplicities and reduction, I am able to prove a partial converse of Theorem 6:

Theorem 8. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated formally equidimensional module, and $I \subseteq J$ ideals with $\bigcup_{n\geq 0} \text{Ass}(R/\overline{I^n}) = \bigcup_{n\geq 0} \text{Ass}(R/\overline{J^n})$. Assume $c_k(I, M) = c_k(J, M)$ for all $k \leq d = \dim(M)$. Then I is a reduction of (J, M).*

In the future, I would like to investigate the following questions.

Question. *Is the assumption $\bigcup_{n\geq 0} \text{Ass}(R/\overline{I^n}) = \bigcup_{n\geq 0} \text{Ass}(R/\overline{J^n})$ necessary in 8?*

The previous question may be considerably difficult. However, in the case that $\dim R = 2$, the assumption is not necessary. This leads to the question:

Question. *Are there some conditions on the ideals I and J and the ring R where the assumption $\bigcup_{n\geq 0} \text{Ass}(R/\overline{I^n}) = \bigcup_{n\geq 0} \text{Ass}(R/\overline{J^n})$ is not necessary?*

In considering the recent work of Jefferies and Montaño [5] and the relationship between the j-multiplicity and the multiplicity sequence, I have considered the following question.

Question. *Can the multiplicity sequence for a monomial ideal be calculated by considering some volume?*

Considering that $c_0(I) = j(I) = e(H^0_{\mathfrak{m}}(G_I(R))$, I have considered the following question.

Question. *Can the multiplicities $c_i(I, M)$ be calculated easier by considering the multiplicity of some other module, such as a local cohomology module?*

References

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