HOMOMESY IN PRODUCTS OF THREE CHAINS AND MULTIDIMENSIONAL RECOMBINATION

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Abstract. We generalize the notion of recombination defined by D. Einstein and J. Propp in order to study homomesy on 3-dimensional posets under rowmotion and promotion. We have two main results. We state and prove conditions under which recombination can be performed in $n$-dimensions. We also apply recombination to show a homomesy result on the product of chains $[2] \times [a] \times [b]$ under rowmotion and promotion. Additionally, we determine that this homomesy result does not generalize to arbitrary products of 3-chains.

1. Introduction

The promotion and rowmotion actions on order ideals of posets have generated significant interest in recent algebraic combinatorics research, giving rise to many beautiful results. Striker and Williams showed that there is an equivariant bijection between order ideals of any ranked poset under promotion and rowmotion [12]. This means that the orbit structure is the same under rowmotion and promotion, so if we want to study the orbits of rowmotion, we could instead study the orbits of promotion, or vice versa. This is particularly useful for many 2-dimensional posets, as orbits under promotion are easier to study than under rowmotion. Further studying orbits of posets under rowmotion and promotion, Propp and Roby found homomesy on $J([a] \times [b])$ under both actions [8], with Einstein and Propp discovering a way to relate homomesy results between the two using a technique called recombination [6]. Homomesy means that under an action, a statistic on an object has the same average over every orbit of the action; this suggests that the orbits are “nice”. Finally, Dilks, Pechenik, and Striker generalized the idea of promotion for multi-dimensional posets by sweeping through the poset in a particular direction with a hyperplane [4]. Furthermore, they showed that for a given poset, there is an equivariant bijection between any of the multidimensional promotions that they defined. In other words, as with the 2-dimensional case, the orbit structure is the same for each promotion. With the generalization of promotion to higher dimensions, it is natural to explore the concepts of homomesy and recombination in higher dimensions. We will show in Theorem 4.1 that $J([2] \times [a] \times [b])$ exhibits homomesy under promotion with the cardinality statistic. One might hope that an arbitrary product of chains $[a_1] \times \cdots \times [a_n]$ would also exhibit homomesy under rowmotion/promotion, as a product of chains is a “nice” poset. However, we show in Remark 5.1 that Theorem 4.1 does not extend to an arbitrary product of three chains. Additionally, the equivariant bijection result of [12] along with the 2-dimensional recombination result of [6] reveals an intimate relationship between rowmotion and promotion. Our $n$-dimensional recombination result, Theorem 4.5, further displays this relationship.

2. Rowmotion and Promotion Background

We begin by recalling some definitions regarding posets, rowmotion, and promotion.

Definition 2.1. A poset $P$ is a set with a binary relation, denoted $\leq$, that is reflexive, antisymmetric, and transitive. A subset $I$ of $P$ is called an order ideal if for any $t \in I$ and $s \leq t$ in $P$, then $s \in I$. 
Definition 2.2. Let $P$ be a poset and $J(P)$ its set of order ideals. For any $e \in P$, the toggle $t_e : J(P) \to J(P)$ is defined as follows:

$$t_e(I) = \begin{cases} I \cup \{e\} & \text{if } e \notin I \text{ and } I \cup \{e\} \in J(P) \\ I \setminus \{e\} & \text{if } e \in I \text{ and } I \setminus \{e\} \in J(P) \\ I & \text{otherwise.} \end{cases}$$

Remark 2.3. The toggles $t_e$ and $t_f$ commute whenever neither $e$ nor $f$ covers the other.

The classical definition of rowmotion, denoted Row, can be stated as follows.

Definition 2.4. Let $P$ be a poset and $I \in J(P)$. $\text{Row}(I)$ is the order ideal generated by the minimal elements of $P$ not in $I$.

However, this is not the only way to view rowmotion. Cameron and Fon-der-Flaass proved that we may instead toggle elements from top to bottom.

Theorem 2.5 ([2]). Let $L$ be a linear extension of a poset $P$. Then $t_L^{-1}(1)t_L^{-1}(2)\cdots t_L^{-1}(n)$ acts as rowmotion.

The benefit of the toggle perspective is that we can study other actions that are closely related to rowmotion. In [12], Striker and Williams defined another action, which they called promotion, on ranked posets under a projection to a two-dimensional lattice. By defining columns on ranked posets, promotion is the action toggling columns left to right. If we denote promotion as Pro, we may see Row and Pro are linked in the following way.

Theorem 2.6 ([12]). For any ranked poset $P$, there is an equivariant bijection between $J(P)$ under Pro and $J(P)$ under Row.

Moreover, Striker and Williams found that in many cases, it was easier to prove the orbit sizes of Pro compared to Row. The reason for this is that in many cases, the action of Pro on $J(P)$ is in equivariant bijection with the rotation of another object. As a result, in order to study the orbits of Row, it is often useful to study Pro and apply Theorem 2.6.

Dilks, Pechenik, and Striker further generalized the notion of promotion for higher dimensional posets. They defined promotion for ranked posets in higher dimensions with respect to an $n$-dimensional lattice projection by sweeping through with an affine hyperplane in a particular direction [4]. In our case, we will not need the notion of lattice projections, but will rather use the natural embedding of the product of $n$ chains into $\mathbb{N}^n$. More specifically, we’ll use the following definition.

Definition 2.7 ([4]). Let $P = [a_1] \times \cdots \times [a_n]$ be the product of $n$ chains poset where we consider the elements in the standard $n$-dimensional embedding as vectors in $\mathbb{N}^n$, and let $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$. Let $T_v^i$ be the product of toggles $t_x$ for all elements $x$ of $P$ that lie on the affine hyperplane $\langle x, v \rangle = i$. If there is no such $x$, then this is the empty product, considered to be the identity. Define promotion with respect to $v$ as the toggle product $\text{Pro}_v = \cdots T_v^{-2}T_v^{-1}T_v^0T_v^1T_v^2\cdots$

By Remark 2.3, elements of the poset commute if there is no covering relation between them. So we show the previous definition is well-defined in the following way.

Remark 2.8 ([4]). Two elements of the poset that lie on the same affine hyperplane $\langle x, v \rangle = i$ cannot be part of a covering relation.

Now that we’ve established $\text{Pro}_v$ and verified it is well-defined, we can relate it to the previously established Row.

Proposition 2.9 ([4]). For a finite ranked poset $P$, $\text{Pro}_{(1,1,\ldots,1)} = \text{Row}$.
The orbit structure of order ideals of certain posets under rowmotion and promotion has been well-studied. Another phenomenon, isolated by Propp and Roby, appears frequently among many of these posets and will be the subject of the next section.

3. The homomesy phenomenon and recombination

In this section, we state known results in 2-dimensions; we will later generalize these results to $n$-dimensions.

**Definition 3.1.** Given a finite set $S$, an action $\tau : S \to S$, and a statistic $f : S \to K$ where $K$ is a field of characteristic zero, then $(S, \tau, f)$ exhibits **homomesy** if there exists $c \in K$ such that for every $\tau$-orbit $O$

$$\frac{1}{|O|} \sum_{x \in O} f(x) = c$$

If such a $c$ exists, we will say the triple is $c$-mesic.

Homomesy results have been observed in many well known combinatorial objects. A few examples are given by rowmotion and promotion on certain posets [8] [10], birational rowmotion on a product of two chains [6], various actions on certain tableaux [1] [8], various actions on binary strings [9], rotations on permutation matrices [9], certain toggles on noncrossing partitions [5], Suter’s action on Young diagrams [8] (with proof due to D. Einstein), and others. To expound on one of these examples, Propp and Roby proved the following results on a product of chains.

**Theorem 3.2** ([8]). Let $f$ be the cardinality statistic. Then $(J([a] \times [b]), \text{Pro}, f)$ is $c$-mesic with $c = ab/2$.

**Theorem 3.3** ([8]). Let $f$ be the cardinality statistic. Then $(J([a] \times [b]), \text{Row}, f)$ is $c$-mesic with $c = ab/2$.

It is beneficial to study $J([a] \times [b])$ under Pro rather than Row, as $J([a] \times [b])$ under Pro is in bijection with an object that rotates. This fact makes the proof of Theorem 3.2 fairly straightforward. Propp and Roby also have a direct proof of Theorem 3.3 in [8]; however, it is much more technical than in the promotion case. Einstein and Propp found a more elegant way to prove Theorem 3.3 in [6] by using a technique they called recombination. The idea behind recombination is that we may start with an orbit from $J([a] \times [b])$ under Row and take sequential layers from order ideals to form a new orbit. More precisely, we will define our layers in the following way.

**Definition 3.4.** Define the $j$th $\gamma$-layer of $P = [a_1] \times \cdots \times [a_n]$:

$$L^j_\gamma = \{(i_1, i_2, \ldots, i_n) \in P \mid i_\gamma = j\}$$

and the $j$th $\gamma$-layer of $I \in J(P)$:

$$L^j_\gamma(I) = \{(i_1, i_2, \ldots, i_n) \in I \mid i_\gamma = j\}.$$

Additionally, given $L^j_\gamma$ and $L^j_\gamma(I)$, define

$$(L^j_\gamma)^* = \{(i_1, i_2, \ldots, i_{\gamma-1}, i_{\gamma+1}, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \in P \text{ and } i_\gamma = j\},$$

$$(L^j_\gamma(I))^* = \{(i_1, i_2, \ldots, i_{\gamma-1}, i_{\gamma+1}, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \in I \text{ and } i_\gamma = j\}.$$

Einstein and Propp referred to each $L^j_\gamma$ as a negative fiber of $P$; we use the notation $L^j_\gamma$ and $L^j_\gamma(I)$ as it more naturally describes our layers when we generalize to higher dimensions. Furthermore, we define $(L^j_\gamma)^*$ and $(L^j_\gamma(I))^*$ as it will be useful to view these layers in the $(n - 1)$-dimensional setting. Using $L^j_\gamma(I)$, we define a new orbit from $J([a] \times [b])$ that is an orbit under Pro. More specifically, Einstein and Propp proved the following, which we restate in the above notation. See Figure 1 for an example.
Proposition 3.5 ([6]). Let \( I \in J([a] \times [b]). \) Define \( \Delta I = \bigcup_j L^j_1 (\text{Row}^{j-1}(I)) \). Then \( \text{Pro}(\Delta I) = \bigcup_j L^j_1 (\text{Row}^j(I)) \).

In Theorem 4.5, we will generalize this notion to higher dimensional products of chains. Before doing so, however, we observe important properties of Row and Pro and how their toggles commute in the \([a] \times [b]\) case. In order to state this observation, we introduce an additional definition. This definition will also prove useful when discussing commuting toggles in \(n\)-dimensions.

**Definition 3.6.** Let \( P = [a_1] \times \cdots \times [a_n], (v_1, v_2, \ldots, v_n) \) where \( v_j \in \{\pm 1\} \), and \( \gamma \in \{1, 2, \ldots, n\}. \) Furthermore, let \( v^* = (v_1, v_2, \ldots, v_{\gamma-1}, v_{\gamma+1}, \ldots, v_n) \). Define \( T^j_{\text{Pro}^*} \), as the toggle product of \( \text{Pro}^* \) on \((L^j_1)^*\).

The following result is discussed in [12] in Theorem 5.4 and in [6] in Section 8.

**Proposition 3.7 ([6][12]).** Let \( I \in J([a] \times [b]). \) \( \text{Row} = \text{Pro}_{(1,1)} = \prod_{j=1}^a T^j_{\text{Pro}_{(1,1)}}, \) and \( \text{Pro} = \prod_{j=1}^b T^{a+1-j}_{\text{Pro}_{(1,1)}}, \). In other words, we can commute the toggles of Row so we toggle \( L^1_1, \) followed by \( L^1_1, \) and so on, toggling each layer from top to bottom. To see why we can do this, we’ll look at an example. In Figure 2a, we can commute the red toggle with both green toggles as the red element does not have a covering relation with either green element. Therefore, when performing Row we can toggle both green elements before the red element, and hence all of \( L^1_1 \) before the red element. Similar reasoning applies for each \( L^j_1, \) and as a result we can toggle Row in the order denoted in Figure 2b, where the green layer is first, the red layer is second, and the white layer third. Additionally, the toggle order in each layer is denoted with an arrow. Note that for Pro, we would have a similar picture except we would toggle the white layer first, then red, then green.

4. Homomesy on \( J([2] \times [a] \times [b]) \) and Higher Dimensional Recombination

Having explored the homomesy results of Theorem 3.2 and 3.3 along with the notion of recombination, we state our first main result.

**Theorem 4.1.** Let \( f \) be the cardinality statistic. Then for any \( v = (\pm 1, \pm 1, \pm 1), \) the triple \( (J([2] \times [a] \times [b]), \text{Pro}_v, f) \) is \(c\)-mesic with \( c = ab.\)

In order to prove Theorem 4.1, we will define the notion of recombination for a product of chains in full generality.
(a) We can commute the toggle of either green element with the red element, as there is no covering relation between them.

(b) We toggle the layer 1, then layer 2, then layer 3, with arrows denoting toggle order in each layer. This toggle order is the same as Row on this poset.

**Figure 2**

**Definition 4.2.** Let \( P = [a_1] \times \cdots \times [a_n], I \in J(P) \) and \( v = (v_1, v_2, \ldots, v_n) \) where \( v_j = \pm 1 \). Define \( \Delta^\gamma I = \bigcup_j L^j_\gamma (\text{Pro}^{-1}_\gamma(I)) \) where \( \gamma \in \{1, \ldots, n\} \). We will call \( \Delta^\gamma I \) the \((v, \gamma)\)-recombination of \( I \). When context is clear, we will suppress the \((v, \gamma)\).

The idea behind recombination is the same as in the 2-dimensional case: we take one layer from each order ideal in a sequence of order ideals from a promotion orbit to form the layers of a new order ideal. See Figure 3 for an example. In addition to generalizing recombination to \( n \)-dimensions, we will also generalize Proposition 3.7 to \( n \)-dimensions.

**Proposition 4.3.** Let \( I \in J([a_1] \times \cdots \times [a_n]), v = (v_1, v_2, \ldots, v_n) \) where \( v_j = \pm 1 \), and \( \gamma \in \{1, 2, \ldots, n\} \). Then \( \text{Pro}_v = \prod_{j=1}^n T_{\text{Pro}_v}^\alpha \) where \( \alpha = (a_\gamma + 1 - j \text{ if } v_\gamma = -1, j \text{ if } v_\gamma = 1) \).

**Proof.** Suppose \( x := (x_1, \ldots, x_n), y := (y_1, \ldots, y_n) \in I \) with \( x \in L^j_\gamma(I) \) and \( y \in L^k_\gamma(I) \) for some \( j \) and \( k \). We want to show that \( x \) and \( y \) are toggled in the same order in \( \text{Pro}_v \) and \( \prod_{j=1}^n T_{\text{Pro}_v}^\alpha \).

**Case** \( j \neq k \): Without loss of generality, \( j > k \). Furthermore, we can assume \( x_\gamma = y_\gamma + 1 \) and \( x_i = y_i \) for \( i \neq \gamma \). If this was not the case, \( x \) and \( y \) could not have a covering relation and we could commute the toggles.

If \( v_\gamma = 1 \): In \( \prod_{j=1}^n T_{\text{Pro}_v}^\alpha \), \( x \) is toggled before \( y \) by definition. Additionally,

\[
\langle x, v \rangle = v_1 x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_n x_n > v_1 y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_n y_n = \langle y, v \rangle
\]

and so \( x \) is toggled before \( y \) in \( \text{Pro}_v \).

If \( v_\gamma = -1 \): In \( \prod_{j=1}^n T_{\text{Pro}_v}^\alpha \), \( y \) is toggled before \( x \) by definition. Additionally,

\[
\langle x, v \rangle = v_1 x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_n x_n < v_1 y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_n y_n = \langle y, v \rangle
\]

and so \( y \) is toggled before \( x \) in \( \text{Pro}_v \).

**Case** \( j = k \): In other words, \( x_\gamma = y_\gamma \). Therefore,

\[
\langle x, v \rangle > \langle y, v \rangle \iff v_1 x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_n x_n > v_1 y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_n y_n
\]

\[
\iff v_1 x_1 + \cdots + v_\gamma-1 x_\gamma-1 + v_\gamma+1 x_\gamma+1 + \cdots + v_n x_n > v_1 y_1 + \cdots + v_\gamma-1 y_\gamma-1 + v_\gamma+1 y_\gamma+1 + \cdots + v_n y_n
\]

\[
\iff \langle x^*, v^* \rangle > \langle y^*, v^* \rangle
\]

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where \( x^*, y^* \) are \( x \) and \( y \) with \( x_\gamma \) and \( y_\gamma \) deleted, respectively. Therefore, \( x \) can be toggled before \( y \) in \( \text{Pro}_v \) if and only if \( x \) can be toggled before \( y \) in \( \prod_{j=1}^{a_\gamma} T_{\text{Pro}_v}^{a_j} \).

In other words, if we want to take \( \text{Pro}_v \), we can commute our toggles to toggle by layers of the form \( L_\gamma^j \) instead of using Definition 2.7. More specifically, if \( v_\gamma = 1 \), we toggle in the order of \( L_\gamma^0, L_\gamma^1, \ldots, L_\gamma^1 \). If \( v_\gamma = -1 \), we toggle in the order of \( L_\gamma^1, L_\gamma^2, \ldots, L_\gamma^0 \).

Now that we have established \( n \)-dimensional recombination and toggle commutation, we will determine under what conditions recombination results in an order ideal.

Figure 3. Performing \( \text{Pro}_{(1,1,-1)} \) on the red order ideal results in the blue order ideal.

Lemma 4.4. Let \( I \in \mathcal{J}([a_1] \times \cdots \times [a_n]) \). Suppose we have \( v \) and \( \gamma \) such that \( v_\gamma = 1 \). Then \( \Delta_\gamma^1 I \) is an order ideal of \( P \).

Proof. Suppose \( (i_1, \ldots, i_n) \in \Delta_\gamma^1 I \). By definition, \( (i_1, \ldots, i_j - 1, \ldots, i_n) \in \Delta_\gamma^0 I \) for \( j \neq \gamma \). To show that \( \Delta_\gamma^1 I \) is an order ideal, it suffices to show \( (i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in \Delta_\gamma^0 I \) for \( i_\gamma \geq 2 \); if \( i_\gamma = 1 \) there is nothing to show. Because \( (i_1, \ldots, i_n) \in \Delta_\gamma^1 I \), we have \( (i_1, \ldots, i_n) \in L_\gamma^1 \text{Pro}_v, \text{Pro}_v + 1 \). By Proposition 4.3, \( \text{Pro}_v = \prod_{j=1}^{a_\gamma} T_{\text{Pro}_v}^{a_j} \), which implies we can commute the toggle relations in \( \text{Pro}_v \) so that \( L_\gamma^1 \) is toggled before \( L_\gamma^{i_\gamma - 1} \). As a result, we must have \( (i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in L_\gamma^{i_\gamma - 1} \text{Pro}_v \text{Pro}_v^{-1} \). Therefore, \( (i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in \Delta_\gamma^0 I \).

We can now state our second main result, which gives conditions under which we can perform \( n \)-dimensional recombination. This result will allow us to prove Theorem 4.1.

Theorem 4.5. Let \( I \in \mathcal{J}([a_1] \times \cdots \times [a_n]) \). Suppose we have \( v = (v_1, v_2, \ldots, v_n) \) where \( v_j = \pm 1, u = (u_1, u_2, \ldots, u_n) \) where \( u_j = \pm 1, v^* = (v_1, \ldots, v_{\gamma - 1}, v_{\gamma + 1}, \ldots, v_n), u^* = (u_1, \ldots, u_{\gamma - 1}, u_{\gamma + 1}, \ldots, u_n) \) and \( \gamma \) such that \( v_\gamma = 1, u_\gamma = -1, \) and \( v^* = u^* \). Then \( \text{Pro}_v(\Delta_\gamma^1 I) = \Delta_\gamma^0 (\text{Pro}_v(I)) \).

Proof. First, note that \( \Delta_\gamma^1 I \) is an order ideal by Lemma 4.4. Also note that \( \text{Pro}_v = \prod_{j=1}^{a_\gamma} T_{\text{Pro}_v}^{a_j} \) and \( \text{Pro}_v = \prod_{j=1}^{a_\gamma} T_{\text{Pro}_v}^{a_{\gamma + 1} - j} \) by Proposition 4.3. We will show \( \text{Pro}_v(\Delta_\gamma^1 I) = \Delta_\gamma^0 (\text{Pro}_v(I)) \) by showing \( L_\gamma^k (\text{Pro}_v(\Delta_\gamma^1 I)) = L_\gamma^k (\Delta_\gamma^0 (\text{Pro}_v(I))) \) for each \( k \in \{1, 2, \ldots, a_\gamma \} \). There are three cases.
Case $1 < k < a_\gamma$: Let $J = \text{Pro}^{k-1}(I)$. We can commute the toggles of $\text{Pro}_v$ so that $L^{k+1}_\gamma$ of $J$ is toggled before $L^k_\gamma$ of $J$, which is toggled before $L^{k-1}_\gamma$ of $J$. Thus, when applying the toggles of $\text{Pro}_v$ to $L^k_\gamma$ of $J$, the layer above is $L^{k+1}_\gamma(\text{Pro}_v(J))$ whereas the layer below is $L^{k-1}_\gamma(J)$. Additionally, we can also commute the toggles of $\text{Pro}_u$ so $L^{k-1}_\gamma$ of $\Delta^k I$ is toggled before $L^k_\gamma$ of $\Delta^k I$, which is toggled before $L^{k+1}_\gamma$ of $\Delta^k I$. Therefore, when applying the toggles of $\text{Pro}_u$ to $L^k_\gamma$ of $\Delta^k I$, the layer below is $L^{k-1}_\gamma(\text{Pro}_u(\Delta^k I))$, whereas the layer above is $L^{k+1}_\gamma(\Delta^k I)$. However, $L^{k-1}_\gamma(\text{Pro}_u(\Delta^k I)) = L^{k-1}_\gamma(J)$, $L^{k}_\gamma(\Delta^k I) = L^{k}_\gamma(J)$, and $L^{k+1}_\gamma(\Delta^k I) = L^{k+1}_\gamma(\text{Pro}_v(J))$. Therefore, when applying $\text{Pro}_v$ to $L^k_\gamma$ of $J$ and $\text{Pro}_u$ to $L^k_\gamma$ of $\Delta^k I$, both layers are the same and have the same layers above and below them. Because $u^* = v^*$, we have $\text{Pro}_u^* = \text{Pro}_v^*$ and so the result of toggling this layer is $L^k_\gamma(\text{Pro}_u(\Delta^k I))$, which is the same as $L^k_\gamma(\text{Pro}_v(J)) = L^k_\gamma(\text{Pro}_v^*(I)) = L^k_\gamma(\Delta^k I(\text{Pro}_v(I)))$.

Case $k = 1$: As above, when applying $\text{Pro}_v$ to $L^1_\gamma$ of $I$ and $\text{Pro}_u$ to $L^1_\gamma$ of $\Delta^1 I$, both of these layers are the same, along with the layers above them. Because $k = 1$, there is not a layer below. As above, $\text{Pro}_u^* = \text{Pro}_v^*$ and so we again get $L^1_\gamma(\text{Pro}_u(\Delta^1 I)) = L^1_\gamma(\Delta^1 I(\text{Pro}_v(I)))$.

Case $k = a_\gamma$: Again, as above, when applying $\text{Pro}_v$ to $L^{a_\gamma}_\gamma$ of $I_{\ell + a_\gamma - 1}$ and $\text{Pro}_u$ to $L^{a_\gamma}_\gamma$ of $I_{\ell}$, both of these layers are the same along with the layers below them. Because $k = a_\gamma$ there is not a layer above. Again, $\text{Pro}_u^* = \text{Pro}_v^*$ and so $L^{a_\gamma}_\gamma(\text{Pro}_u(\Delta^2 I)) = L^{a_\gamma}_\gamma(\Delta^2 I(\text{Pro}_v(I)))$. □

![Diagram](image)

**Figure 4.** The purple layers correspond under recombination. In Example 4.6, we will demonstrate the idea of the proof using the order ideals in the blue and red boxes.

**Example 4.6.** To see an example of the proof technique, we will refer to Figures 4, 5, and 6. We begin with the same orbit under Row as in Figure 1. Let $I$ denote the first order ideal in this orbit; using recombination we form the order ideal $\Delta^1_{(1,1)} I$. We want to verify $\Delta^1_{(1,1)} I$ is an orbit under $\text{Pro}$ by showing that corresponding layers in the Row and $\Delta^1_{(1,1)} I$ orbit result in the same layer after performing Row and $\text{Pro}$, respectively. The purple layers $L^1_\gamma(I)$ in both orbits of Figure 4 correspond under recombination. We can commute the toggles of Row as we did in Figure 2b. We can also commute the toggles of $\text{Pro}$ so we toggle layer 3, then layer 2, then layer 1 in Figure 2b. This means when performing Row, we first would toggle the layer indicated by the green arrow in
Figure 5. When performing Row to the left figure, $L_3^1(I)$ is toggled first in the direction indicated. When performing Pro to the right figure, $L_1^1(I)$ is toggled first in the direction indicated.

Figure 6. After performing the toggles from Figure 5, the order ideal in the left figure now has $L_3^1(I)$ from the order ideal that follows it in the orbit of Row. Similarly, the order ideal in the left figure has $L_1^1(I)$ from the order ideal that follows it in the orbit of Pro. When performing toggles on the purple layer, the three layers are the same.

We have two immediate corollaries that will be useful in the proof of Theorem 4.1

**Corollary 4.7.** $\text{Pro}_{(1,1,-1)}(\Delta_{(1,1,1)}^3(I)) = \Delta_{(1,1,1)}^3(\text{Pro}_{(1,1,1)}(I))$.

*Proof.* $v = (1,1,1)$, $u = (1,1,-1)$, and $\gamma = 3$ satisfy the assumptions of Theorem 4.5. □

**Corollary 4.8.** $\text{Pro}_{(-1,1,-1)}(\Delta_{(1,1,-1)}^1(I)) = \Delta_{(1,1,-1)}^1(\text{Pro}_{(1,1,-1)}(I))$.

*Proof.* $v = (1,1,-1)$, $u = (-1,1,-1)$, and $\gamma = 1$ satisfy Theorem 4.5. □
Note that we may invert recombination. For example, if we start with an orbit of \( J([2] \times [a] \times [b]) \) under \( \text{Pro}_{(1,1,1)} \), we can use this theorem to acquire an orbit of \( J([2] \times [a] \times [b]) \) under \( \text{Pro}_{(-1,1,1)} \). Although the details aren’t shown explicitly below, this observation is important in proving \( J([2] \times [a] \times [b]) \) exhibits homomesy under \( \text{Pro}_{(-1,1,1)} \).

To show our desired homomesy result, we will relate the order ideals of our posets to increasing tableaux. To do so, we first need a map from \( J([a] \times [b] \times [c]) \) to increasing tableaux defined by Dilks, Pechenik, and Striker.

**Definition 4.9.** An increasing tableau of shape \( \lambda \) is a filling of boxes in shape \( \lambda \) with positive integers such that the entries strictly increase from left to right across rows and strictly increase from top to bottom along columns. We will use \( \text{Inc}^q(\lambda) \) to indicate the set of increasing tableaux of shape \( \lambda \) with entries at most \( q \).

**Definition 4.10** ([4]). Define a map \( \Psi : J([a] \times [b] \times [c]) \to \text{Inc}^{a+b+c-1}(a \times b) \) in the following way. Let \( I \in J([a] \times [b] \times [c]) \). If we think of \( I \) as a pile of cubes in an \( a \times b \times c \) box, project onto the \( a \times b \) face. Record in position \((i, j)\) the number of boxes of \( I \) with coordinate \((i, j, k)\) for some \( 0 \leq k \leq c - 1 \). This results in a filling of a Young diagram of shape \( a \times b \) with nonnegative entries that weakly decrease from left to right and top to bottom. By rotating the diagram \( 180^\circ \), our Young diagram is now weakly increasing in rows and columns. Now increase each label by one more than the distance to the upper left corner box. This results in an increasing tableau, which we denote \( \Psi(I) \).

Along with defining the map, Dilks, Pechenik, and Striker also proved the following result for \( \Psi \), where \( K\text{-Pro} \) denotes \( K \)-theoretic promotion.

**Theorem 4.11** ([4]). \( \Psi \) is an equivariant bijection between \( J([a] \times [b] \times [c]) \) under \( \text{Pro}_{(1,1,1)} \) and \( \text{Inc}^{a+b+c-1}(a \times b) \) under \( K\text{-Pro} \).

Furthermore, we can relate the cardinality of \( I \) to the sum of the entries in \( \Psi(I) \).

**Lemma 4.12.** If \( I \in J([2] \times [a] \times [b]) \), the sum of the boxes in \( \Psi(I) \) is equal to \( f(I) + a(a + 2) \) where \( f \) is the cardinality statistic.

**Proof.** This follows from the definition of \( \Psi \) and the shape of \( \Psi(I) \).

As a result of this remark, if we can find an appropriate homomesy result on increasing tableaux, we can transfer the result over to \( J([2] \times [a] \times [b]) \) under \( \text{Pro}_{(1,1,1)} \) using \( \Psi \), then to \( J([2] \times [a] \times [b]) \) under \( \text{Pro}_{(-1,1,1)} \) using Corollary 4.7. As it turns out, the appropriate homomesy result has already been discovered by Bloom, Pechenik, and Saracino.

**Theorem 4.13** ([1]). Let \( \lambda \) be a \( 2 \times n \) rectangle for any \( n \), let \( \mu \subseteq \lambda \) be a set of elements fixed under \( 180^\circ \) rotation, and let \( \sigma_\mu \) be the statistic of summing the entries in the boxes of \( \mu \). Then for any \( q \), \( \text{Inc}^q(\lambda) \), \( K\text{-Pro} \), \( \sigma_\mu \) is homomesic.

Note that the entire \( 2 \times n \) rectangle is fixed under \( 180^\circ \) rotation. Moreover, for \( I \in J([2] \times [a] \times [b]) \), \( \Psi(I) \) is an increasing tableau of shape \( 2 \times a \). With this theorem, we now have sufficient machinery to prove Theorem 4.1.

**Proof of Theorem 4.1.** Using Theorem 4.13, Lemma 4.12, and the map \( \Psi \), we may already conclude \( J([2] \times [a] \times [b]) \) exhibits homomesy under \( \text{Pro}_{(1,1,1)} \). Moreover, \( \text{Pro}_{(-1,1,1)} \) reverses the direction that our hyperplanes sweep through our poset, which merely reverses our orbits of order ideals. As a result, we may conclude that \( J([2] \times [a] \times [b]) \) also exhibits homomesy under \( \text{Pro}_{(-1,1,1)} \). To prove Theorem 4.1 for the remaining \( v \), we’ll begin with \( v = (1, 1, 1) \), which is Row.

Let \( O_1, O_2 \) be orbits of \( J([2] \times [a] \times [b]) \) under Row. Additionally, let \( R_1 = \{ \Delta^3_{(1,1,1)} I : I \in O_1 \} \) and \( R_2 = \{ \Delta^3_{(1,1,1)} I : I \in O_2 \} \) be the corresponding recombination orbits. Moreover, because \( R_1 \)
and $R_2$ are orbits under $\text{Pro}_{(1,1,-1)}$, by Corollary 4.7 the average of the cardinality over $R_1$ and $R_2$ must be equal. As a result, the average of the cardinality over $O_1$ and $O_2$ must be equal. Hence, $J([2] \times [a] \times [b])$ is homomesic under Row. Again, because $\text{Pro}_{(-1,-1,-1)}$ merely reverses the direction of hyperplane toggles, we conclude that $J([2] \times [a] \times [b])$ is homomesic under $\text{Pro}_{(-1,-1,-1)}$.

We now turn our attention to $\text{Pro}_{(-1,1,-1)}$ and $\text{Pro}_{(1,-1,-1)}$. Using Corollary 4.8 and similar arguments as above, we see $J([2] \times [a] \times [b])$ is homomesic under both $\text{Pro}_{(-1,1,-1)}$ and $\text{Pro}_{(1,-1,-1)}$.

We have shown the desired triples are homomesic, but we still must show the orbit average is $ab$. Due to rotational symmetry, the filters of $J$ must be equal. As a result, the average of the cardinality over $J$ must be equal. Hence, $c$ must also be $ab$.

We immediately obtain the following corollaries by symmetry.

**Corollary 4.14.** Let $f$ be the cardinality statistic. Then for any $v = (\pm 1, \pm 1, \pm 1)$, the triple $(J([a] \times [2] \times [b]), \text{Pro}_v, f)$ is $c$-mesic with $c = ab$.

**Corollary 4.15.** Let $f$ be the cardinality statistic. Then for any $v = (\pm 1, \pm 1, \pm 1)$, the triple $(J([a] \times [b] \times [2]), \text{Pro}_v, f)$ is $c$-mesic with $c = ab$.

**Proof of Corollaries 4.14 and 4.15.** Given an orbit $O$ of $(J([a] \times [2] \times [b])$ under $\text{Pro}_v$ for some $v = \{\pm 1, \pm 1, \pm 1\}$, we can use a cyclic rotation of coordinates and appropriate choice of $v'$ to obtain an orbit $O'$ of $J([2] \times [a] \times [b])$ under $\text{Pro}_{v'}$ such that $O$ and $O'$ are in bijection. A similar argument applies to $J([a] \times [b] \times [2])$.

5. **Related results**

In this section, we prove several related results and corollaries. This includes showing that our homomesy result does not hold in an arbitrary product of three chains, but finding a subset of the poset that does exhibit homomesy. We also show that if we increase the number of chains further, a similar homomesy result does not hold, even if we restrict ourselves to just chains of length two. Additionally, we use our main homomesy result to obtain a new homomesy result on increasing tableaux. Finally, we use refined homomesy results on increasing tableaux to state more refined homomesy results on order ideals.

To show that Theorem 4.1 does not generalize for arbitrary products of chains $[a] \times [b] \times [c]$, we use $[3] \times [3] \times [4]$ as a counterexample.

**Remark 5.1.** For $v = \{\pm 1, \pm 1, \pm 1\}$, the triple $(J([3] \times [3] \times [4]), \text{Pro}_v, f)$ is not homomesic.

**Proof.** A calculation using Sage Math [11] shows that $J([3] \times [3] \times [4])$ under Row with statistic $f$ has 456 orbits with average 18, 2 orbits with average 161/9, and 2 orbits with average 163/9. Using recombination, we get the result for any $v = \{\pm 1, \pm 1, \pm 1\}$.

We can further inquire about homomesy in higher dimensions using only chains of length two. We discover homomesy in $[2] \times [2] \times [2] \times [2]$, but a negative result in higher dimensions.

**Remark 5.2.** For $v = \{\pm 1, \pm 1, \pm 1\}$, the triple $(J([2] \times [2] \times [2] \times [2]), \text{Pro}_v, f)$ is $c$-mesic with $c = 8$. However, the triple $(J([2] \times [2] \times [2] \times [2]), \text{Pro}_v, f)$ is not homomesic.

**Proof.** A calculation using Sage Math [11] shows that $(J([2] \times [2] \times [2] \times [2])$ under Row with statistic $f$ has 36 orbits, all with average 8. However, $(J([2] \times [2] \times [2] \times [2] \times [2])$ has 771 orbits with average 16, 60 orbits with average 115/7, 60 orbits with average 109/7, 30 orbits with average 61/4, 30 orbits with average 67/4, 6 orbits with average 11, and 6 orbits with average 21. Using recombination, we once again get the result for any $v = \{\pm 1, \pm 1, \pm 1\}$. 


For our main homomesy result, we used a bijection $\Psi$ to translate a homomesy result on increasing tableaux to a product of chains. After rotation on our product of chains to obtain Corollary 4.15, we can translate back to increasing tableaux using $\Psi$ to obtain an additional homomesy result on increasing tableaux. This is in the same spirit as the tri-fold symmetry used by Dilks, Pechenik, and Striker [4].

**Corollary 5.3.** Let $\lambda$ be an $a \times b$ rectangle for any $n$ and let $\sigma_\lambda$ be the statistic of summing the entries in the boxes of $\lambda$. Then $(\text{Inc}^{a+b+1}(\lambda), K\text{-Pro}, \sigma_\lambda)$ is $c$-mesic with $c = ab + \frac{ab(a+b)}{2} = \frac{ab(2+a+b)}{2}$.

**Proof.** Each orbit of $\text{Inc}^{a+b+1}(\lambda)$ under $K\text{-Pro}$ corresponds to an orbit of $J([a] \times [b] \times [2])$ under $\text{Pro}_{(1,1,-1)}$. For each $I \in J([a] \times [b] \times [2]), \sigma_\lambda(\Psi(I)) = f(I) + \frac{ab(a+b)}{2}$ where $f$ is the cardinality statistic. Applying Corollary 4.15, the result follows. \hfill $\square$

Additionally, we have a more refined homomesy result of the Theorem 4.1. We obtain this using the rotational symmetry condition of Theorem 4.13. Define the columns $L_{1,2}^{j_1,k_1}, L_{1,2}^{j_2,k_2}, \ldots, L_{1,2}^{j_n,k_n}$ be such that $(j_1, k_1), (j_2, k_2), \ldots, (j_n, k_n)$ are rotationally symmetric. If $f_L(I)$ denotes the cardinality of $I$ on $L_{1,2}^{j_1,k_1}, L_{1,2}^{j_2,k_2}, \ldots, L_{1,2}^{j_n,k_n}$, then $J([2] \times [a] \times [b]), \text{Pro}_{(1,1,-1)}, f_L$ is $c$-mesic with $c = \frac{na}{2}$.

**Corollary 5.4.** Let $I_{1,2}^{j_1,k_1}, L_{1,2}^{j_2,k_2}, \ldots, L_{1,2}^{j_n,k_n}$ be such that $(j_1, k_1), (j_2, k_2), \ldots, (j_n, k_n)$ are rotationally symmetric. If $f_L(I)$ denotes the cardinality of $I$ on $L_{1,2}^{j_1,k_1}, L_{1,2}^{j_2,k_2}, \ldots, L_{1,2}^{j_n,k_n}$, then $(J([2] \times [a] \times [b]), \text{Pro}_{(1,1,-1)}, f_L)$ is $c$-mesic with $c = \frac{nb}{2}$. \hfill $\square$

Pechenik further generalized the results of [1] and the result stated in Theorem 4.13. From this, we get a more general statement of Corollary 5.4. We summarize the relevant definition and theorem below.

**Definition 5.5 ([7]).** The frame of a partition $\lambda$ is the set $\text{Frame}(\lambda)$ of all boxes in the first or last row, or in the first or last column.

**Theorem 5.6 ([7]).** Let $S$ be a subset of $\text{Frame}(n \times n)$ that is fixed under $180^\circ$ rotation. Then $(\text{Inc}^g(m \times n), K\text{-Pro}, \sigma_S)$ is $c$-mesic with $c = \frac{(g+1)|S|}{2}$.

The following corollary follows directly from this result.

**Corollary 5.7.** Let $P = [a_1] \times [a_2] \times [a_3]$. Additionally let $L_{1,2}^{j_1,k_1}, L_{1,2}^{j_2,k_2}, \ldots, L_{1,2}^{j_n,k_n}$ be rotationally symmetric in $P$ where each $j_i$ is 1 or $a_1$ and each $k_i$ is 1 or $a_2$. If $f_L(I)$ denotes the cardinality of $I$ on $L_{1,2}^{j_1,k_1}, L_{1,2}^{j_2,k_2}, \ldots, L_{1,2}^{j_n,k_n}$, then $(J([a_1] \times [a_2] \times [a_3]), \text{Pro}_{(1,1,-1)}, f_L)$ is $c$-mesic with $c = \frac{na_3}{2}$.

**Proof.** We know the triple is $c$-mesic by Theorem 5.6; we must now show that $c = \frac{na_3}{2}$. Using the same reasoning as Corollary 5.4, the global average of $f_L(I) = \frac{na_3}{2}$ and as a result, $c = \frac{na_3}{2}$. \hfill $\square$

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References


