RADO NUMBERS FOR SOME NON-HOMOGENOUS INEQUALITIES

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Abstract. Given a linear equation, inequality, or system of equations/inequalities L, the t-color Rado number is the least integer n such that for every t-coloring of the ordered set $(1, 2, \ldots, n)$ there is a monochromatic solution to L. If there is not such an integer, we will say the t-color Rado number is infinite. Bialostocki and Schaal [2] determined the 2-color Rado numbers for the system $x_1 + x_2 + \cdots + x_{m-1} < x_m$ where $x_1 < x_2 < \cdots < x_m$, $m \geq 3$. In this paper, we will add an integer constant to the first inequality in this system. In other words, we will find the 2-color Rado numbers for the system $x_1 + x_2 + \cdots + x_{m-1} + c < x_m$ where $x_1 < x_2 < \cdots < x_m$ and $m \geq 3$.

1. Introduction

In 1916, it was proved by Schur [1] that for every positive integer t there exists a smallest integer $n = S(t)$ such that every t-coloring of $\{1, 2, ..., n\}$, $\Delta : \{1, 2, ..., n\} \to \{0, 2, ..., t-1\}$ there exists a monochromatic solution to the equation $x_1 + x_2 = x_3$. In other words, there exist x_1, x_2, x_3 that satisfy $x_1 + x_2 = x_3$ and $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$. Further work was done to find particular values of n for specific values of t for a variety of equations and inequalities. In 2000, Bialostocki and Schaal [2] found values of n for 2-colorings for the inequality $x_1 + x_2 + \cdots + x_{m-1} < x_m$ and the system of inequalities

$$x_1 + x_2 + \cdots + x_{m-1} < x_m$$

$$x_1 < x_2 < \cdots < x_m.$$

The focus of this paper is 2-colorings for the system of inequalities

$$x_1 + x_2 + \cdots + x_{m-1} + c < x_m$$

$$x_1 < x_2 < \cdots < x_m$$

where c is an integer. More specifically, **our main theorem**, Theorem 2.6 gives a function for the Rado number of this system for all $m \geq 3$ and $c \in \mathbb{Z}$. To do this, we establish four different types of colorings, then determine the structure that yields the longest coloring that avoids a monochromatic solution.

In Section 2, we have several introductory definitions and the statement of our main theorem. In Section 3, we begin the proof of our main theorem with the condition $c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3$. In Section 4, we continue the proof when $c > \frac{-3m^2}{2} + \frac{9m}{2} - 3$.

2. Background and Statement of the Main Theorem

Before we can state the main theorem, we must establish a few definitions.

Definition 2.1. Let $t \geq 1$ be given where $t \in \mathbb{N}$. Also, let $n \in \mathbb{N}$ and define $[a, b]$ as the set $\{x \in \mathbb{N} \mid a \leq x \leq b\}$. We will define a *t-coloring* as a function $\Delta : [1, n] \to [0, t-1]$.

Definition 2.2. Given a coloring Δ, an ordered set $(x_1, x_2, \ldots, x_m)$ is *monochromatic* in the coloring Δ if and only if $\Delta(x_i) = \Delta(x_j)$ for all i, j such that $1 \leq i, j \leq m$.

Definition 2.3. Let S represent an equation, inequality, or system of equations/inequalities. An ordered set $(x_1, x_2, \ldots, x_m)$ is a *monochromatic solution to S* if and only if $(x_1, x_2, \ldots, x_m)$ is monochromatic and satisfies S.

Definition 2.4. The *t-color Rado number* of an equation, inequality, or system of equations or inequalities L is the least integer $n = R_t(L)$, if it exists, such that every t-coloring $\Delta : [1, n] \to [0, t-1]$ has a monochromatic solution that satisfies L. If such an integer does not exist, we will say the t-color Rado number of L is infinite.

Definition 2.5. For any $m, c \in \mathbb{Z}$ where $m \geq 3$, let $L(m, c)$ be the system

$$L(m, c) : x_1 + x_2 + \cdots + x_{m-1} + c < x_m$$

$$x_1 < x_2 < \cdots < x_m.$$

We will also establish the following notation. Since this paper only concerns 2-colorings, let $R(L(m,c)) := R_2(L(m,c))$. We can now state the main theorem.

Theorem 2.6. *For every $m \geq 3$ and $c \in \mathbb{Z}$,*

$$R(L(m,c)) = \begin{cases} 2m-1 \\ \qquad \text{if } c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3 \\ \\ max\left\{\frac{3m^2}{2} - \frac{5m}{2} + c + 2, r_0^2(2-m) + r_0(\frac{m^2}{2} - \frac{5m}{2} + 2 - c) + \frac{m^3}{2} - \frac{m^2}{2} + mc + 1\right\} \\ \qquad \text{if } \frac{-3m^2}{2} + \frac{9m}{2} - 3 < c \leq \frac{m^2}{2} - \frac{7m}{2} + 4 \\ \\ \frac{m^3}{2} - \frac{m^2}{2} + mc + 1 \\ \qquad \text{if } c > \frac{m^2}{2} - \frac{7m}{2} + 4 \end{cases}$$

where $r_0 = \left\lfloor \frac{m^2-3m-2c}{4m-8} \right\rfloor$

Before we begin the proof, we introduce two addition definitions.

Definition 2.7. A coloring is *mc-good* if it avoids a monochromatic solution to $L(m,c)$.

Definition 2.8. A *maximal coloring* is any 2-coloring $\Delta : [1, R(L(m,c)) - 1] \to [0,1]$ such that Δ avoids a monochromatic solution to $L(m,c)$.

Note that a maximal coloring is the longest possible 2-coloring that avoids a monochromatic solution to $L(m,c)$; in other words, it is the longest mc-good coloring.

In order to find $R(L(m,c))$ for any given m and c, we will be looking for maximal colorings. It is apparent that a maximal coloring has at least $m-1$ integers colored 0 and at least $m-1$ integers colored 1. Thus, we will only consider colorings where $|\Delta^{-1}(0)| \geq m-1$ and $|\Delta^{-1}(1)| \geq m-1$. For a given mc-good coloring Δ the first $m-1$ integers colored 0 will be designated by $a_1 < a_2 < \cdots < a_{m-1}$. Similarly, the first $m-1$ integers colored 1 will be designated by $b_1 < b_2 < \cdots < b_{m-1}$. If they exist, define the mth integer colored 0 as a_m and define the mth integer colored 1 as b_m. Without loss of generality, we may assume

$a_{m-1} < b_{m-1}$ for any coloring. Define the integer r to be the number of b_i's in the interval $[1, a_{m-1}]$, i.e.,

$$r = \left|[1, a_{m-1}] \cap \Delta^{-1}(1)\right|$$

Moreover, since $a_{m-1} < b_{m-1}$, we know $r \in [0, m-2]$

The sum of the a_i's for $i \in [1, m-1]$ and the sum of the b_i's for $i \in [1, m-1]$ play an important role in determining the length of a coloring. We will define

$$S_0 = \sum_{i=1}^{m-1} a_i + c \quad \text{and} \quad S_1 = \sum_{i=1}^{m-1} b_i + c.$$

We should note that if a coloring $\Delta : [1, n] \to [0, 1]$ is mc-good and a_m exists, then $a_m \leq S_0$. Similarly, if b_m exists, then $b_m \leq S_1$. Later we will need an additional coloring Δ'; we will define r' and $a'_1, \ldots, a'_{m-1}, b'_1, \ldots, b'_{m-1}, S'_0$, and S'_1 in the obvious way.

We now need a definition to classify each coloring into one of four possibilities.

Definition 2.9. An mc-good coloring Δ is of *Type A* if $S_0 \leq a_{m-1}$ and $S_1 \leq b_{m-1}$.
An mc-good coloring Δ is of *Type B* if $S_0 \leq a_{m-1}$ and $S_1 > b_{m-1}$.
An mc-good coloring Δ is of *Type C* if $S_0 > a_{m-1}$ and $S_1 \leq b_{m-1}$.
An mc-good coloring Δ is of *Type D* if $S_0 > a_{m-1}$ and $S_1 > b_{m-1}$.

If $S_0 \leq a_{m-1}$, then an mc-good coloring can only have $m-1$ integers colored 0, for otherwise $(a_1, a_2, \ldots, a_{m-1}, a_m)$ would be a monochromatic solution to $L(m, c)$. Similarly, if $S_1 \leq b_{m-1}$, an mc-good coloring can only have $m-1$ integers colored 1. The distinction between these four types of colorings is necessary as each will behave differently when we are looking for maximal colorings. Note: If $c = 0$, Δ is of Type D. [2]

3. When $c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3$

Proof of Theorem 2.6. Let $m \geq 3$ and $c \in \mathbb{Z}$ be given. We will first find the values of $R(L(m, c))$ for $c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3$. Assume $c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3$. Any coloring $\Delta : [1, 2m-2] \to [0, 1]$ with $|\Delta^{-1}(0)| = |\Delta^{-1}(1)| = m-1$ avoids a monochromatic solution to $L(m, c)$, so $R(L(m, c)) \geq 2m-1$. To show the upper bound, let $\Delta : [1, 2m-1] \to [0, 1]$ be an

arbitrary coloring. We will show there must exist a monochromatic solution to $L(m,c)$. Let $a_1,\ldots,a_{m-1},b_1,\ldots,b_{m-1},S_0$, and S_1 be as previously defined. By the Pigeon Hole Principle, $|\Delta^{-1}(0)|\geq m$ or $|\Delta^{-1}(1)|\geq m$. First, assume $|\Delta^{-1}(0)|\geq m$. Therefore,

$$\begin{aligned}
S_0 &= \sum_{i=1}^{m-1} a_i + c \\
&= \sum_{i=1}^{m-2} a_i + c + a_{m-1} \\
&\leq \sum_{i=1}^{m-2} (m-1+i) + c + a_{m-1} \\
&= \frac{3m^2}{2} - \frac{9m}{2} + 3 + c + a_{m-1} \\
&\leq \frac{3m^2}{2} - \frac{9m}{2} + 3 + \left(\frac{-3m^2}{2} + \frac{9m}{2} - 3\right) + a_{m-1} \\
&= a_{m-1} \\
&< a_m.
\end{aligned}$$

A similar technique can be used to show that if $|\Delta^{-1}(1)|\geq m$, then $S_1 < b_m$. Thus, Δ contains a monochromatic solution to $L(m,c)$, so $R(L(m,c))\leq 2m-1$. Hence, when $c\leq \frac{-3m^2}{2}+\frac{9m}{2}-3$ we have $R(L(m,c))=2m-1$.

Before moving on to other values of c, we will prove a lemma stating that these are the only values of c that can have Type A colorings as maximal colorings.

Lemma 3.1. *If a maximal coloring is of Type A, then* $c\leq \frac{-3m^2}{2}+\frac{9m}{2}-3$.

Proof. Suppose $\Delta:[1,n]\to[0,1]$ is a maximal coloring of Type A. Since Δ is Type A, we know $S_0\leq a_{m-1}$ and $S_1\leq b_{m-1}$. Therefore $|\Delta^{-1}(0)|=m-1$ and $|\Delta^{-1}(1)|=m-1$, so $n=2m-2$. Because $a_{m-1}<b_{m-1}$, we have

$$a_{m-1}=m-1+r \quad \text{and} \quad b_{m-1}=2m-2.$$

Therefore, we need to find values of c such that $S_0 \leq m-1+r$ and $S_1 \leq 2m-2$. We see that

$$\begin{aligned} S_0 &\leq \sum_{i=1}^{m-1}(r+i)+c \\ &= (m+2r)\left(\frac{m-1}{2}\right)+c \\ &= \frac{m^2}{2}-\frac{m}{2}+rm-r+c. \end{aligned}$$

This means that the largest value S_0 can be is $\frac{m^2}{2}-\frac{m}{2}+rm-r+c$. Because we want $S_0 \leq m-1+r$, if we set

$$\frac{m^2}{2}-\frac{m}{2}+rm-r+c \leq m-1+r$$

we can find all c values that guarantee the inequality holds. Solving for c we get

$$c \leq \frac{-m^2}{2}+\frac{3m}{2}+r(2-m)-1.$$

Also, because we want to find the c values that satisfy this for any $r \in [0, m-2]$ and because $(2-m) \leq 0$, we want the inequality

$$\begin{aligned} c &\leq \frac{-m^2}{2}+\frac{3m}{2}+(m-2)(2-m)-1 \\ &= \frac{-3m^2}{2}+\frac{11m}{2}-5. \end{aligned}$$

Now, we will do a similar process for S_1. We find that

$$\begin{aligned} S_1 &\leq \sum_{i=1}^{m-1}(m-1+i)+c \\ &= \frac{3m^2}{2}-\frac{5m}{2}+1+c. \end{aligned}$$

This means that the largest value S_1 can be is $\frac{3m^2}{2} - \frac{5m}{2} + 1 + c$. Because we want $S_1 \leq 2m-2$, if we set

$$\frac{3m^2}{2} - \frac{5m}{2} + 1 + c \leq 2m - 2$$

we can find all c values that guarantee the inequality holds. Solving for c we find

$$c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3.$$

Thus, to ensure we satisfy $S_0 \leq m - 1 + r$ and $S_1 \leq 2m - 2$, we want the inequality

$$c \leq min\left\{\frac{-3m^2}{2} + \frac{11m}{2} - 5, \frac{-3m^2}{2} + \frac{9m}{2} - 3\right\}$$

to be satisfied. However,

$$\frac{-3m^2}{2} + \frac{9m}{2} - 3 \leq \frac{-3m^2}{2} + \frac{11m}{2} - 5$$

if $m \geq 2$. Because $m \geq 3$, we know this is the case. Therefore, we have shown that if Δ is a Type A coloring, then $c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3$. $\square$

4. When $c > \frac{-3m^2}{2} + \frac{9m}{2} - 3$

By Lemma 3.1, when considering maximal colorings for $c > \frac{-3m^2}{2} + \frac{9m}{2} - 3$, we only need to consider Type B, Type C, and Type D colorings. We must look for the longest mc-good colorings of these types. Recall that we assumed

$$\left|\Delta^{-1}(0)\right| \geq m - 1 \quad \text{and} \quad \left|\Delta^{-1}(1)\right| \geq m - 1.$$

For a given m and c, the mc-good Type B, Type C, and Type D colorings can be partitioned into equivalence classes based on how the first $m - 1 + r$ integers are colored. We will formalize this with the following definition.

Definition 4.1. Two mc-good colorings Δ_1 and Δ_2 are in the same equivalence class if and only if Δ_1 and Δ_2 have the same value of r and $\Delta_1(x) = \Delta_2(x)$ for all $x \in [1, m - 1 + r]$.

If we can find the longest coloring in each equivalence class, then we will only need to consider these when searching for maximal colorings. We will now show how to derive the longest coloring of an equivalence class from an arbitrary coloring in that equivalence class.

Let an equivalence class E of mc-good colorings be given. Let $\Delta|_{[1,m-1+r]}$ be the coloring produced when every coloring in E is restricted to the domain $[1, m-1+r]$. We will use $\Delta|_{[1,m-1+r]}$ to find Δ', which is the unique longest coloring in E. We will call Δ' the *end-optimized* coloring of E. For $\Delta|_{[1,m-1+r]}$, let r and $a_1, \ldots, a_{m-1}$, $b_1, \ldots, b_r$ and S_0 be as previously defined. Note that $b_{r+1}, \ldots, b_{m-1}$ and S_1 do not exist for $\Delta|_{[1,m-1+r]}$.

Lemma 4.2. *There is a unique longest coloring Δ' in E.*

Proof. We will begin by setting up three different cases. First, let

$$T = \sum_{i=1}^{r} b_i + \sum_{i=1}^{m-1-r} (S_0 + i) + c.$$

We will use this definition in two of the cases.

Case i: Assume $S_0 \leq a_{m-1}$. Define n' as

$$n' = \sum_{i=1}^{r} b_i + \sum_{i=1}^{m-1-r} (m - 1 + r + i) + c.$$

Define $\Delta' : [1, n'] \to [0, 1]$ as

$$\Delta'(x) = \begin{cases} \Delta(x) & \text{if } 1 \leq x \leq m-1+r \\ 1 & \text{if } m+r \leq x \leq n' \end{cases}$$

This coloring Δ' is mc-good and an element of E. For Δ', let $a'_1, \ldots, a'_{m-1}, b'_1, \ldots, b'_{m-1}, S'_0$, and S'_1 be as previously defined. Note that $S_0 = S'_0$ and $S'_1 = n'$, $S'_0 < S'_1$ and that Δ' is a Type B coloring. More specifically, Δ' cannot be Type A by Lemma 3.1.

Case *ii*: Assume $S_0 > a_{m-1}$ and $S_0 + (m-1-r) \geq T$. Let $n' = S_0 + (m-1-r)$ and define $\Delta' : [1, n'] \to [0, 1]$ as

$$\Delta'(x) = \begin{cases} \Delta(x) & \text{if } 1 \leq x \leq m-1+r \\ 0 & \text{if } m+r \leq x \leq S_0 \\ 1 & \text{if } S_0+1 \leq x \leq n' \end{cases}$$

This coloring Δ' is *mc*-good and an element of E. Again for Δ', let $a'_1, \ldots, a'_{m-1}, b'_1, \ldots, b'_{m-1}, S'_0$, and S'_1 be as previously defined. Note that $S_0 = S'_0$ and $S'_1 = T \leq n' = S_0 + (m-1-r) = b_{m-1}$ and Δ' is a Type C coloring.

Case *iii*: Assume $S_0 > a_{m-1}$ and $S_0 + (m-1-r) < T$. Let $n' = T$ and define $\Delta' : [1, n'] \to [0, 1]$ by

$$\Delta'(x) = \begin{cases} \Delta(x) & \text{if } 1 \leq x \leq m-1+r \\ 0 & \text{if } m+r \leq x \leq S_0 \\ 1 & \text{if } S_0+1 \leq x \leq n' \end{cases}$$

This coloring Δ' is *mc*-good and an element of E. Again, for Δ' let $a'_1, \ldots, a'_{m-1}, b'_1, \ldots, b'_{m-1}, S'_0$, and S'_1 be as previously defined. Note that $S_0 = S'_0$ and $S'_1 = n' = T > b_{m-1}$ and $S'_0 < S'_1$, and Δ' is a Type D coloring.

For all three cases, every coloring in E has the same values for $a_1, \ldots, a_{m-1}$. Hence, every coloring in E has the same values for S_0. As a result, any coloring $\Delta : [1, n] \to [0, 1]$ in E with $\Delta(x) = 1$ for $x \in [m+r, S_0]$ must be shorter than Δ'. This is because $n = max\{S_1, S_0 + (m-2-r)\}$ where $S_1 < S'_1$, and so $n < max\{S'_1, S_0 + (m-1-r)\} = n'$. Therefore, Δ' is the unique longest coloring in the equivalence class E. $\square$

From now on, we do not need to consider arbitrary colorings when searching for maximal colorings, we only need to consider end-optimized colorings. In order to optimize the beginning of the colorings, we will now define Type B, Type C, and Type D swaps.

Definition 4.3. Suppose $\Delta : [1, n] \to [0, 1]$ is an end-optimized Type B coloring, and let $a_1, \ldots, a_{m-1}, b_1, \ldots, b_{m-1}, r, S_0$, and S_1 be as previously defined. Assume $r \geq 1$. Since $r \geq 1$ and $\Delta(m-1+r) = 0$, there exists a $\rho \in [1, m-2+r]$ such that $\Delta(\rho) = 1$ and $\Delta(\rho+1) = 0$. We will define a *Type B swap* $\Delta' : [1, n+1] \to [0, 1]$ by

$$\Delta'(x) = \begin{cases} 0 & \text{if } x = \rho \\ 1 & \text{if } x = \rho + 1 \\ \Delta(x) & \text{if } x \in [1, m-1+r] - \{\rho, \rho+1\} \\ 1 & \text{if } m + r \leq x \leq n+1 \end{cases}$$

We note that Δ' is longer than Δ. Also, by construction, Δ' is an end-optimized coloring. We now must verify that the coloring Δ' produced by a Type B swap is Type B.

Let $a'_1, \ldots, a'_{m-1}, b'_1, \ldots, b'_{m-1}, r', S'_0$, and S'_1 be as previously defined. Note that there exists an $s \in [1, m-1]$ and $t \in [1, r]$ such that $b_t = \rho$ and $a_s = \rho + 1$. We see that $a'_i = a_i$ for $i \in [1, m-1] - \{s\}$ and $a'_s = a_s - 1$ and $b'_i = b_i$ for $i \in [1, r] - \{t\}$ and $b'_t = b_t + 1$. Also, if $\rho \in [1, m-3+r]$, then $r' = r$ and if $\rho = [m-2+r]$, then $r' = r - 1$.

Lemma 4.4. *The coloring Δ' produced by a Type B swap is a Type B coloring.*

Proof. We know $S_0 \leq a_{m-1}$ and $S_1 > b_{m-1}$. We also know that $a'_{m-1} = a_{m-1}$ or $a'_{m-1} = a_{m-1} - 1$, and that $b'_{m-1} = b_{m-1}$. Therefore, $S'_0 = S_0 + \rho - (\rho+1) = S_0 - 1 \leq a_{m-1} - 1 \leq a'_{m-1}$ and $S'_1 = S_1 + (\rho+1) - \rho = S_1 + 1 > S_1 > b_{m-1} = b'_{m-1}$. Thus, we can conclude $S'_0 \leq a'_{m-1}$ and $S'_1 > b'_{m-1}$. This implies Δ' is Type B. $\square$

We have verified that after performing a Type B swap, we have a Type B and end-optimized coloring. Therefore, we can continue to perform Type B swaps until $r = 0$. Once we reach $r = 0$, we will call the coloring *Type B complete*. A Type B complete coloring is of

the form $\Delta : [1, n] \to [0, 1]$ where

$$\Delta(x) = \begin{cases} 0 & \text{if } 1 \leq x \leq m-1 \\ 1 & \text{if } m \leq x \leq n \end{cases}$$

We will define this more formally later in the proof with a calculation of n.

Definition 4.5. Now suppose $\Delta : [1, n] \to [0, 1]$ is an end-optimized Type C coloring, and let $a_1, \ldots, a_{m-1}, b_1, \ldots, b_{m-1}, r, S_0$, and S_1 be as previously defined. Assume there exists $\rho \in [1, m-2+r]$ such that $\Delta(\rho) = 0$ and $\Delta(\rho+1) = 1$. We will define an operation called a *Type C swap* that can be performed to produce a new coloring $\Delta' : [1, n'] \to [0, 1]$ as follows. Let $n' = max\{S_0 + (m-r), S_1 + (m-2-r)\}$ and $\Delta' : [1, n'] \to [0, 1]$ be defined by

$$\Delta'(x) = \begin{cases} 1 & \text{if } x = \rho \\ 0 & \text{if } x = \rho + 1 \\ \Delta(x) & \text{if } x \in [1, m-1+r] - \{\rho, \rho+1\} \\ 0 & \text{if } m + r \leq x \leq S_0 + 1 \\ 1 & \text{if } S_0 + 2 \leq x \leq n' \end{cases}$$

Because $n = S_0 + (m-1-r) < S_0 + (m-r) \leq n'$ we see that Δ' is longer than Δ. Also, by construction, Δ' is an end-optimized coloring.

Let $a'_1, \ldots, a'_{m-1}, b'_1, \ldots, b'_{m-1}, S'_0$, and S'_1 be as previously defined. Note that there exists an $s \in [1, m-2]$ and $t \in [1, r]$ such that $a_s = \rho$ and $b_t = \rho + 1$. We see that $a'_i = a_i$ for $i \in [1, m-1] - \{s\}$ and $a'_s = a_s + 1$ and $b'_i = b_i$ for $i \in [1, r] - \{t\}$ and $b'_t = b_t - 1$. We now must verify that the coloring Δ' produced by a Type C swap is either Type C or Type D.

Lemma 4.6. *The coloring Δ' produced by a Type C swap is either a Type C or Type D coloring.*

Proof. We know $S_0 > a_{m-1} = m - 1 + r$. We also know that $a'_{m-1} = a_{m-1}$. Therefore, $S'_0 = S_0 + (\rho + 1) - \rho = S_0 + 1 > a_{m-1} + 1 > a_{m-1} = a'_{m-1}$. Thus, Δ' is Type C or Type D. $\square$

Note that we cannot show that $S'_1 \leq b'_{m-1}$, i.e. that Δ' is Type C. Because $S'_0 = S_0 + 1$, we know $b'_k = b_k + 1$ for $k \in [r+1, m-1]$. Therefore,

$$\begin{aligned} S'_1 &= S_1 + (m - 1 - r) + \rho - (\rho + 1) \\ &= S_1 + (m - 2 - r) \\ &\leq b_{m-1} + (m - 2 - r) \end{aligned}$$

If $S_1 = b_{m-1}$, then $S'_1 > b'_{m-1}$ when $m - 2 - r > 1$ as we get that $S'_1 = b_{m-1} + (m - 2 - r) > b_{m-1} + 1 = b'_{m-1}$. This shows that a Type C swap can result in Δ' being a Type D coloring.

One thing to note is that if $r = 0$ we cannot have a Type C coloring as $S'_1 = \sum_{i=1}^{m-1} S'_0 + i > S'_0 + (m-1) = b'_{m-1}$. Thus, $r \geq 1$. We can perform Type C swaps until one of two conditions are met. The first is that we get a coloring that is Type D. For this case, we will then begin performing Type D swaps, which we will define below. If this is not the case, we will perform Type C swaps until all of the integers colored 1 in $[1, m - 1 + r]$ have been swapped to the front of the coloring. We will call this coloring *Type C complete*. A Type C complete coloring is of the form $\Delta : [1, n] \to [0, 1]$ where

$$\Delta(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq r \\ 0 & \text{if } r + 1 \leq x \leq S_0 \\ 1 & \text{if } S_0 + 1 \leq x \leq n \end{cases}$$

We will define this coloring more formally later in the proof with a calculation of n.

Definition 4.7. Now suppose $\Delta : [1, n] \to [0, 1]$ is an end-optimized Type D coloring, and let $a_1, \ldots, a_{m-1}, b_1, \ldots, b_{m-1}, r, S_0$, and S_1 be as previously defined. We will define

an operation called a *Type D swap* on Δ to produce a new coloring Δ' as follows. Assume there exists a $\rho \in [1, m-2+r]$ such that $\Delta(\rho) = 0$ and $\Delta(\rho+1) = 1$. Let $n' = max\{S_0 + (m-r), S_1 + (m-2-r)\}$ and $\Delta' : [1, n'] \to [0, 1]$ be defined by

$$\Delta'(x) = \begin{cases} 1 & \text{if } x = \rho \\ 0 & \text{if } x = \rho + 1 \\ \Delta(x) & \text{if } x \in [1, m-1+r] - \{\rho, \rho+1\} \\ 0 & \text{if } m + r \leq x \leq S_0 + 1 \\ 1 & \text{if } S_0 + 2 \leq x \leq n' \end{cases}$$

Because $n = S_1 \leq S_1 + (m-2-r) \leq n'$ we see that Δ' is as long or longer than Δ. Also, Δ' is an end-optimized coloring by construction.

Let $a'_1, \ldots, a'_{m-1}, b'_1, \ldots, b'_{m-1}, S'_0$, and S'_1 be as previously defined. Note that there exists an $s \in [1, m-2]$ and $t \in [1, r]$ such that $a_s = \rho$ and $b_t = \rho + 1$. We see that $a'_i = a_i$ for $i \in [1, m-1] - \{s\}$ and $a'_s = a_s + 1$ and $b'_i = b_i$ for $i \in [1, r] - \{t\}$ and $b'_t = b_t - 1$. We now must verify that the coloring Δ' produced by a Type D swap is either Type C or Type D.

Lemma 4.8. *The coloring Δ' produced by a Type D swap is either a Type C or Type D coloring.*

Proof. We know $S_0 > a_{m-1} = m - 1 + r$. We also know that $a'_{m-1} = a_{m-1}$. Therefore, $S'_0 = S_0 + (\rho+1) - \rho = S_0 + 1 > a_{m-1} + 1 = a'_{m-1}$. As a result, Δ' is either Type C or Type D. □

Note that we cannot show that $S'_1 > b'_{m-1}$, i.e. that Δ' is Type D. Because $S'_0 = S_0 + 1$, we know $b'_k = b_k + 1$ for $k \in [r+1, m-1]$. Therefore,

$$\begin{aligned} S'_1 &= S_1 + (m-1-r) + \rho - (\rho+1) \\ &= S_1 + (m-2-r). \end{aligned}$$

If $S_1 = b_{m-1} + 1$ and $r = m - 2$, then

$$\begin{aligned} S_1' &= S_1 + (m - 2 - r) \\ &= b_{m-1} + 1 + (m - 2 - (m - 2)) \\ &= b_{m-1}'. \end{aligned}$$

Therefore, $S_1' \leq b_{m-1}'$ and so Δ' is Type C. This shows that a Type D Swap can result in a Type C coloring. We should note that if $r = 0$ in a Type D coloring, we cannot perform a Type D swap and we are done. If $r \geq 1$, we can perform Type D swaps until one of two conditions are met. The first is that we get a coloring that is Type C. For this case, we will then begin performing Type C swaps until a Type C complete coloring is produced. If this is not the case, we will perform Type D swaps until all of the integers colored 1 in $[1, m-1+r]$ have been swapped to the front of the coloring. We will call this coloring *Type D complete*. A Type D complete coloring is of the form $\Delta : [1, n] \to [0, 1]$ where

$$\Delta(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq r \\ 0 & \text{if } r + 1 \leq x \leq S_0 \\ 1 & \text{if } S_0 + 1 \leq x \leq n \end{cases}$$

Note that if $r = 0$ then there will not be any integers colored 1 at the beginning of the coloring. We will define this more formally later in the proof with a calculation of n.

Now we will establish six definitions that will help us determine the length of a maximal coloring.

Definition 4.9. Define $f_B(m, c)$ as

$$\begin{aligned} f_B(m, c) &= \sum_{i=1}^{m-1} (m - 1 + i) + c \\ &= \frac{3m^2}{2} - \frac{5m}{2} + 1 + c, \end{aligned}$$

define $f_C(m,c,r)$ as

$$\begin{aligned} f_C(m,c,r) &= \sum_{i=1}^{m-1}(r+i)+c+(m-1-r) \\ &= r(m-2)+\frac{m^2}{2}+\frac{m}{2}-1+c, \end{aligned}$$

and define $f_D(m,c,r)$ as

$$\begin{aligned} f_D(m,c,r) &= \sum_{i=1}^{r} i + \sum_{i=1}^{m-1-r}\left(\frac{m^2}{2}-\frac{m}{2}+rm-r+c+i\right)+c \\ &= r^2\left(2-m\right)+r\left(\frac{m^2}{2}-\frac{5m}{2}+2-c\right)+\frac{m^3}{2}-\frac{m^2}{2}+mc. \end{aligned}$$

Using m, c, and r, define

$$\Delta_{B:m,c}(x) = \begin{cases} 0 & \text{if } 1 \le x \le m-1 \\ 1 & \text{if } m \le x \le f_B(m,c), \end{cases}$$

define

$$\Delta_{C:m,c,r}(x) = \begin{cases} 1 & \text{if } 1 \le x \le r \\ 0 & \text{if } r+1 \le x \le \frac{m^2}{2}-\frac{m}{2}+rm-r+c \\ 1 & \text{if } \frac{m^2}{2}-\frac{m}{2}+rm-r+c+1 \le x \le f_C(m,c,r), \end{cases}$$

and define

$$\Delta_{D:m,c,r}(x) = \begin{cases} 1 & \text{if } 1 \le x \le r \\ 0 & \text{if } r+1 \le x \le \frac{m^2}{2}-\frac{m}{2}+rm-r+c \\ 1 & \text{if } \frac{m^2}{2}-\frac{m}{2}+rm-r+c+1 \le x \le f_D(m,c,r). \end{cases}$$

Note that if $r = 0$, $\Delta_{D:m,c,r}$ will not have any integers colored 1 at the beginning of the coloring. We can see that if m and c are such that $\Delta_{B:m,c}$ is a Type B coloring, then $\Delta_{B:m,c}$ is a Type B complete coloring. If m, c and r are such that $\Delta_{C:m,c,r}$ is a Type C coloring, then $\Delta_{C:m,c,r}$ is a Type C complete coloring. Finally, if m, c and r are such that $\Delta_{D:m,c,r}$ is a Type D coloring, then $\Delta_{D:m,c,r}$ is a Type D complete coloring. We will have a definition to clarify when this is the case.

Definition 4.10. If $\Delta_{B:m,c}$ is a well-defined, mc-good, Type B coloring, we will call it $type - good$. Similarly, if $\Delta_{C:m,c,r}$ is a well-defined, mc-good coloring, Type C coloring, or if $\Delta_{D:m,c,r}$ is a well-defined, mc-good, Type D coloring we will refer to these as $type - good$ as well.

In other words, we want to emphasize when the subscript on the coloring corresponds to the type of the coloring. It is possible to have colorings that are not type-good. For example, $\Delta_{C:m,c,0}$ will never be a type-good coloring.

We can see that because Type B and Type C swaps produce longer colorings, and because Type D swaps produce a coloring that is as long or longer than the original, the type-good colorings $\Delta_{B:m,c}$ and $\Delta_{C:m,c,r}$ and $\Delta_{D:m,c,r}$ will be candidates for maximal colorings. We should note that for a given m and c there is a unique Type B complete coloring, but the Type C complete and Type D complete colorings are functions of r. Therefore, to find the longest Type C complete coloring for a given m and c, we will look at

$$\max_{r \in [0, m-2]} f_C(m, c, r).$$

Note that $\Delta_{C:m,c,r}$ might not be type-good for a given r. Similarly, to find the longest Type D complete coloring for a given m and c, we will look at

$$\max_{r \in [0, m-2]} f_D(m, c, r).$$

Again, note that $\Delta_{D:m,c,r}$ might not be type-good for a given r. We should also note that $\Delta_{B:m,c}$ is not type-good if $\Delta_{D:m,c,0}$ is type-good, and vice versa. However, since we assumed

$c > \frac{-3m^2}{2} + \frac{9m}{2} - 3$, we know that exactly one of $\Delta_{B:m,c}$ or $\Delta_{D:m,c,0}$ must be type-good as it cannot be a Type A or Type C coloring.

By taking the maximum of the lengths of the complete colorings, we will find a candidate for $R(L(m,c))$:

$$M(L(m,c)) = max\left\{f_B(m,c), \max_{r\in[0,m-2]} f_C(m,c,r), \max_{r\in[0,m-2]} f_D(m,c,r)\right\} + 1.$$

However, we cannot yet state that $M(L(m,c)) = R(L(m,c))$ as it is not clear that $M(L(m,c))$ will correspond to an appropriate type-good coloring. This is because $f_B(m,c), f_C(m,c,r)$, and $f_D(m,c,r)$ are defined even if $\Delta_{B:m,c}, \Delta_{C:m,c,r}$, or $\Delta_{D:m,c,r}$ are not type-good. However, if $\Delta_{B:m,c}$ is type-good, then $f_B(m,c)$ gives us the length of $\Delta_{B:m,c}$. Similarly for $\Delta_{C:m,c,r}$ and $\Delta_{D:m,c,r}$. We will now show that $M(L(m,c))$ is never given by $\max_{r\in[0,m-2]} f_C(m,c,r)$.

Lemma 4.11. $M(L(m,c)) = max\left\{f_B(m,c), \max_{r\in[0,m-2]} f_D(m,c,r)\right\} + 1.$

Proof. To verify the claim, we will show that $\max_{r\in[0,m-2]} f_C(m,c,r) < f_B(m,c)$ when $\Delta_{B:m,c}$ is type-good and $\max_{r\in[0,m-2]} f_C(m,c,r) < f_D(m,c,0)$ when $\Delta_{D:m,c,0}$ is type-good. To do this, we will start by showing $f_B(m,c) - f_C(m,c,r) > 0$ for any r. We know

$$\begin{aligned} f_B(m,c) - f_C(m,c,r) &= \left(\frac{3m^2}{2} - \frac{5m}{2} + 1 + c\right) - \left(r(m-2) + \frac{m^2}{2} + \frac{m}{2} - 1 + c\right) \\ &= m^2 - 3m + 2 - r(m-2). \end{aligned}$$

Because $r \leq m-2$, we have $r(m-2) \leq (m-2)(m-2)$, which implies $-r(m-2) \geq -(m-2)(m-2)$. Therefore,

$$\begin{aligned} m^2 - 3m + 2 - r(m-2) &\geq m^2 - 3m + 2 - (m-2)(m-2) \\ &= m^2 - 3m + 2 - (m^2 - 4m + 4) \\ &= m - 2. \end{aligned}$$

Because $m \geq 3$, we know $m-2 > 0$. Therefore, we have shown that $f_B(m,c) - f_C(m,c,r) > 0$ for any r and so $\underset{r\in[0,m-2]}{max} f_C(m,c,r) < f_B(m,c)$. Now, we will show $f_D(m,c,0) > f_C(m,c,r)$. Because

$$f_B(m,c) = \sum_{i=1}^{m-1}(m-1+i) + c$$

and

$$f_D(m,c,0) = \sum_{i=1}^{m-1}(S_0+i) + c$$

and $S_0 > m-1$ for $\Delta_{D:m,c,0}$, we know that

$$f_D(m,c,0) > f_B(m,c) > f_C(m,c,r).$$

Therefore, $\underset{r\in[0,m-2]}{max} f_C(m,c,r) < f_D(m,c,0)$. Because either $\Delta_{B:m,c}$ or $\Delta_{D:m,c,0}$ must be type-good, we have shown that $M(L(m,c)) = max\left\{f_B(m,c), \underset{r\in[0,m-2]}{max} f_D(m,c,r)\right\} + 1$. □

Now we must justify that $M(L(m,c))$ corresponds to an appropriate type-good coloring.

Lemma 4.12. *If $f_B(m,c) \geq \underset{r\in[0,m-2]}{max} f_D(m,c,r)$ then $\Delta_{B:m,c}$ is type-good.*

Proof. Suppose $\Delta_{B:m,c}$ is not type-good. Then $\Delta_{D:m,c,0}$ is type-good and has length $f_D(m,c,0)$. For $\Delta_{D:m,c,0}$ we know $S_0 > a_{m-1}$. Therefore,

$$S_0 = \frac{m^2}{2} - \frac{m}{2} + c > m-1$$

and so

$$f_D(m,c,0) = \sum_{i=1}^{m-1}\left(\frac{m^2}{2} - \frac{m}{2} + c + i\right) + c > f_B(m,c).$$

Hence, $f_B(m,c) < f_D(m,c,0) \leq \underset{r\in[0,m-2]}{max} f_D(m,c,r)$ which is a contradiction. □

Define r_0 be a value of r that maximizes the length of $\Delta_{D:m,c,r}$.

Lemma 4.13. *If $f_B(m,c) < \underset{r\in[0,m-2]}{max} f_D(m,c,r)$ then $\Delta_{D:m,c,r_0}$ is type-good.*

Proof. If $\Delta_{B:m,c}$ is not type-good then $\Delta_{D:m,c,0}$ is, and so $\Delta_{D:m,c,r_0}$ is also type-good and we are done. Therefore, we can assume $\Delta_{B:m,c}$ is type-good. Suppose $\Delta_{D:m,c,r_0}$ is not type-good.

Let Δ be the end-optimized coloring beginning with r_0 integers colored 1 followed by $m-1$ integers colored 0. Let S_0 and S_1 be as previously defined for Δ and let n be the length of Δ. We have three cases.

Case *i*: Assume Δ is Type A. Then $n = b_{m-1} = 2m-2$. Also, because $f_D(m,c,r_0)$ calculates S_1 and $S_1 \leq b_{m-1}$, we have $f_D(m,c,r_0) \leq b_{m-1}$. Since $\Delta_{B:m,c}$ is type-good and the length of $\Delta_{B:m,c} \geq 2m-2$, we have $f_B(m,c) \geq f_D(m,c,r_0)$, a contradiction.

Case *ii*: Assume Δ is Type B. Then $S_0 \leq a_{m-1}$. Because $S_0 = \frac{m^2}{2} - \frac{m}{2} + rm - r + c$, we know $\frac{m^2}{2} - \frac{m}{2} + rm - r + c + i \leq b_{r+i}$. Therefore, $f_D(m,c,r_0) \leq n$. Since $\Delta_{B:m,c}$ is the longest type B coloring, $n \leq f_B(m,c)$ and so $f_D(m,c,r_0) \leq f_B(m,c)$, a contradiction.

Case *iii*: Assume Δ is Type C. Then $S_1 \leq b_{m-1} = n$. Because $f_D(m,c,r_0)$ calculates S_1, we know $f_D(m,c,r_0) \leq n$. However, because Δ is Type C, $n \leq \underset{r\in[0,m-2]}{max} f_C(m,c,r) \leq f_B(m,c)$ by Lemma 4.10. Thus, $f_D(m,c,r_0) \leq f_B(m,c)$, a contradiction.

Every case resulted in a contradiction. Thus, we know $\Delta_{D:m,c,r_0}$ must be type-good. $\square$

We get $R(L(m,c)) = M(L(m,c)) = max\left\{f_B(m,c), \underset{r\in[0,m-2]}{max} f_D(m,c,r)\right\} + 1$ as a result of Lemmas 4.11, 4.12, and 4.13. Our next goal to improve this result is to determine r_0 as a function of m and c.

Recall that the length of $\Delta_{D:m,c,r}$ is given by

$$f_D(m,c,r) = r^2(2-m) + r\left(\frac{m^2}{2} - \frac{5m}{2} + 2 - c\right) + \frac{m^3}{2} - \frac{m^2}{2} + mc$$

which is a quadratic in r that opens down. Therefore, the length of $\Delta_{D:m,c,r}$ is maximized at

$$\begin{aligned} r &= \frac{\frac{-m^2}{2} + \frac{5m}{2} - 2 + c}{2(2-m)} \\ &= \frac{m^2 - 5m + 4 - 2c}{4m-8}. \end{aligned}$$

We know r must be an integer value, however, so adding $\frac{1}{2}$ and using the floor function rounds r to the nearest integer. Thus will define r_0 as

$$r_0 = \left\lfloor \frac{m^2 - 3m - 2c}{4m-8} \right\rfloor.$$

However, we must ensure $0 \leq r_0 \leq m-2$. The term $\frac{m^2-3m-2c}{4m-8}$ decreases as c increases, so we need to find when $\left\lfloor \frac{m^2-3m-2c}{4m-8} \right\rfloor$ becomes 0. This occurs when

$$\frac{m^2-3m-2c}{4m-8} < 1.$$

If we solve for c, we find

$$c > \frac{m^2}{2} - \frac{7m}{2} + 4.$$

In the case where $r_0 \leq 0$, the value $r = 0$ will give us the longest coloring of $\Delta_{D:m,c,r}$. We should also verify $r_0 \leq m-2$. If this wasn't the case, we would have

$$\frac{m^2-3m-2c}{4m-8} \geq m-1.$$

However, solving for c yields

$$c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 4.$$

Because we assumed $c > \frac{-3m^2}{2} + \frac{9m}{2} - 3$ for this case, we must have $r_0 \leq m-2$.

So to summarize, $f_D(m,c,r_0)$ gives the length of the longest Type D coloring when $\frac{-3m^2}{2} + \frac{9m}{2} - 3 < c \leq \frac{m^2}{2} - \frac{7m}{2} + 4$ and $f_D(m,c,0)$ gives the length of the longest Type D coloring when $c > \frac{m^2}{2} - \frac{7m}{2} + 4$. When we combine everything we have shown above, we get

$$R(L(m,c)) = \begin{cases} 2m-1 \\ \qquad \text{if } c \leq \frac{-3m^2}{2} + \frac{9m}{2} - 3 \\ \\ max\left\{\frac{3m^2}{2} - \frac{5m}{2} + c + 2, r_0^2(2-m) + r_0(\frac{m^2}{2} - \frac{5m}{2} + 2 - c) + \frac{m^3}{2} - \frac{m^2}{2} + mc + 1\right\} \\ \qquad \text{if } \frac{-3m^2}{2} + \frac{9m}{2} - 3 < c \leq \frac{m^2}{2} - \frac{7m}{2} + 4 \\ \\ \frac{m^3}{2} - \frac{m^2}{2} + mc + 1 \\ \qquad \text{if } c > \frac{m^2}{2} - \frac{7m}{2} + 4 \end{cases}$$

where $r_0 = \left\lfloor \frac{m^2-3m-2c}{4m-8} \right\rfloor$. $\square$

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